

ITERATIVE SCHEMES FOR APPROXIMATING SOLUTIONS OF ACCRETIVE OPERATORS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce two iterative schemes for approximating solutions of the equation $0 \in Av$, where A is an accretive operator which satisfies the range condition. Our methods are motivated by Halpern's type iteration and Mann's type iteration.

1. INTRODUCTION

Let E be a real Banach space, let $A \subset E \times E$ be an m -accretive operator and let $J_r = (I + rA)^{-1}$ for $r > 0$. Our paper is concerned with iterative schemes for solving the equation $0 \in Av$. One well-known method is the following: $x_0 = x \in E$,

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where $\{r_n\}$ is a sequence of positive real numbers. The convergence of (1.1) in Hilbert spaces has been studied by Rockafellar [16], Brézis and Lions [1], Lions [7] and Pazy [11]. The results in Banach spaces have been studied by Bruck and Reich [4], Reich [12, 13, 14], Nevanlinna and Reich [9], Bruck and Passty [3] and Jung and Takahashi [6]. On the other hand, Halpern [5] and Mann [8] introduced the following iterative schemes for approximating a fixed point of a nonexpansive mapping T of E into itself:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \dots \quad (1.2)$$

and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \dots, \quad (1.3)$$

respectively, where $x_0 = x \in E$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, the iterative schemes (1.2) and (1.3) have been studied extensively. See, for example, Takahashi [19, 20] and the references mentioned there.

In this paper, motivated by (1.1), (1.2) and (1.3), we introduce two iterative schemes to solve $0 \in Av$: one is Halpern's type and the other is Mann's type. Our methods will be defined for accretive operators which satisfy the range condition.

2. PRELIMINARIES

Throughout this paper, we denote the set of all nonnegative integers by \mathbb{N} . Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E . We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E , we denote strong convergence

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of $\{x_n\}$ to $x \in E$ by $x_n \rightarrow x$ and weak convergence by $x_n \rightharpoonup x$. The modulus of convexity of E is defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\}$$

for every ε with $0 \leq \varepsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. Further, δ satisfies that

$$\left\| \frac{x+y}{2} \right\| \leq r \left(1 - \delta \left(\frac{\varepsilon}{r} \right) \right)$$

for every $x, y \in E$ with $\|x\| \leq r, \|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. We also know that if C is a closed convex subset of a uniformly convex Banach space E , then for each $x \in E$, there exists a unique element $u = Px \in C$ with $\|x-u\| = \inf\{\|x-y\| : y \in C\}$. Such a P is called the metric projection of E onto C . Let $U = \{x \in E : \|x\| = 1\}$. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for every $x \in E$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2.1)$$

is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E . A Banach space E is said to satisfy Opial's condition [10] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n \rightarrow \infty} \|x_n - y\| < \liminf_{n \rightarrow \infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$.

Let C be a closed convex subset of E . A mapping $T: C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D . Let D be a subset of C . A mapping P of C into D is said to be sunny if $P(Px + t(x - Px)) = Px$ whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into itself is said to be a retraction if $P^2 = P$. We denote the closure of the convex hull of D by $\overline{\text{co}}D$.

Let I denote the identity operator on E . An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. If A is accretive, then we have $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $x_i \in D(A), y_i \in Ax_i, i = 1, 2$ and $r > 0$. An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r: R(I + rA) \rightarrow D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A . We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in A J_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an accretive operator A which satisfies the range condition, we have $A^{-1}0 = F(J_r)$ for all $r > 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$.

In the sequel, unless otherwise stated, we assume that $A \subset E \times E$ is an accretive operator which satisfies the range condition and J_r is the resolvent of A for $r > 0$.

3. HALPERN'S TYPE ITERATIVE SCHEME

In this section, we study the strong convergence of Halpern's type iteration. We employ the methods of Wittmann [23] and Shioji and Takahashi [17] for the proof of the following theorem.

Theorem 1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Assume that $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} r_n = \infty$. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, \quad n \in \mathbb{N}.$$

If $A^{-1}0 \neq \emptyset$ and $\{J_t x\}$ converges strongly to $z \in A^{-1}0$ as $t \rightarrow \infty$, then $\{x_n\}$ converges strongly to $z \in A^{-1}0$.

Proof. Let $y_n = J_{r_n} x_n$ and $u \in A^{-1}0$. Then we have

$$\begin{aligned} \|x_1 - u\| &= \|\alpha_0 x + (1 - \alpha_0) y_0 - u\| \\ &\leq \alpha_0 \|x - u\| + (1 - \alpha_0) \|y_0 - u\| \\ &\leq \alpha_0 \|x - u\| + (1 - \alpha_0) \|x_0 - u\| \\ &= \|x - u\|. \end{aligned}$$

If $\|x_n - u\| \leq \|x - u\|$ for some $n \in \mathbb{N} \setminus \{0\}$, then we can show that $\|x_{n+1} - u\| \leq \|x - u\|$. Then, by induction, $\{x_n\}$ is bounded. Therefore $\{y_n\}$ is also bounded. Next we shall show that

$$\limsup_{n \rightarrow \infty} \langle x - z, J(x_n - z) \rangle \leq 0. \tag{3.1}$$

We know $(x - J_t x)/t \in AJ_t x$ and $A_{r_n} x_n \in Ay_n$. Since A is accretive, we have

$$\langle A_{r_n} x_n - \frac{x - J_t x}{t}, J(y_n - J_t x) \rangle \geq 0$$

and hence

$$\langle x - J_t x, J(y_n - J_t x) \rangle \leq t \langle A_{r_n} x_n, J(y_n - J_t x) \rangle$$

for all $n \in \mathbb{N}$ and $t > 0$. Then, since $A_{r_n} x_n = (x_n - y_n)/r_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} \langle x - J_t x, J(y_n - J_t x) \rangle \leq 0 \tag{3.2}$$

for all $t > 0$. Since $J_t x \rightarrow z$ as $t \rightarrow \infty$ and the norm of E is uniformly Gâteaux differentiable, for any $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$|\langle z - J_t x, J(y_n - J_t x) \rangle| \leq \frac{\varepsilon}{2}$$

and

$$|\langle x - z, J(y_n - J_t x) - J(y_n - z) \rangle| \leq \frac{\varepsilon}{2}$$

for all $t \geq t_0$ and $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
& |\langle x - J_t x, J(y_n - J_t x) \rangle - \langle x - z, J(y_n - z) \rangle| \\
& \leq |\langle x - J_t x, J(y_n - J_t x) \rangle - \langle x - z, J(y_n - J_t x) \rangle| \\
& \quad + |\langle x - z, J(y_n - J_t x) \rangle - \langle x - z, J(y_n - z) \rangle| \\
& = |\langle z - J_t x, J(y_n - J_t x) \rangle| + |\langle x - z, J(y_n - J_t x) - J(y_n - z) \rangle| \\
& \leq \varepsilon
\end{aligned} \tag{3.3}$$

for all $t \geq t_0$ and $n \in \mathbb{N}$. Therefore, from (3.2) and (3.3), we have

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \langle x - z, J(y_n - z) \rangle & \leq \limsup_{n \rightarrow \infty} \langle x - J_t x, J(y_n - J_t x) \rangle + \varepsilon \\
& \leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{n \rightarrow \infty} \langle x - z, J(y_n - z) \rangle \leq 0. \tag{3.4}$$

On the other hand, since $x_{n+1} - y_n = \alpha_n(x - y_n) \rightarrow 0$ as $n \rightarrow \infty$ and the norm of E is uniformly Gâteaux differentiable, we have

$$\lim_{n \rightarrow \infty} |\langle x - z, J(x_{n+1} - z) \rangle - \langle x - z, J(y_n - z) \rangle| = 0. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain (3.1).

From $(1 - \alpha_n)(y_n - z) = (x_{n+1} - z) - \alpha_n(x - z)$, we have

$$(1 - \alpha_n)^2 \|y_n - z\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n \langle x - z, J(x_{n+1} - z) \rangle$$

and hence

$$\begin{aligned}
\|x_{n+1} - z\|^2 & \leq (1 - \alpha_n)^2 \|y_n - z\|^2 + 2\alpha_n \langle x - z, J(x_{n+1} - z) \rangle \\
& \leq (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n \langle x - z, J(x_{n+1} - z) \rangle
\end{aligned}$$

for all $n \in \mathbb{N}$. By (3.1), for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\langle x - z, J(x_{n+1} - z) \rangle \leq \frac{\varepsilon}{2}$$

for all $n \geq m$. Hence we have

$$\|x_{n+m+1} - z\|^2 \leq (1 - \alpha_{n+m}) \|x_{n+m} - z\|^2 + \alpha_{n+m} \varepsilon$$

for all $n \in \mathbb{N}$. By induction, we obtain

$$\begin{aligned}
\|x_{n+m+1} - z\|^2 & \leq \|x_m - z\|^2 \prod_{i=m}^{n+m} (1 - \alpha_i) + \left\{ 1 - \prod_{i=m}^{n+m} (1 - \alpha_i) \right\} \varepsilon \\
& \leq \|x_m - z\|^2 \exp\left(-\sum_{i=m}^{n+m} \alpha_i\right) + \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, from $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have

$$\limsup_{n \rightarrow \infty} \|x_n - z\|^2 = \limsup_{n \rightarrow \infty} \|x_{n+m+1} - z\|^2 \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\{x_n\}$ converges strongly to z . \square

The convergence of $\{J_t x\}$ as $t \rightarrow \infty$ was discussed by Takahashi and Ueda [22]. See also Reich [15].

Theorem 2. *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I+rA)$. If $A^{-1}0 \neq \emptyset$, then the strong $\lim_{t \rightarrow \infty} J_t x$ exists and belongs to $A^{-1}0$ for all $x \in C$. Further, if $Px = \lim_{t \rightarrow \infty} J_t x$ for each $x \in C$, then P is a sunny nonexpansive retraction of C onto $A^{-1}0$.*

Let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself. Then $A = I - T$ is an accretive operator which satisfies $C = \overline{D(A)} \subset \bigcap_{r>0} R(I+rA)$; see Takahashi [18]. Then, putting $A = I - T$ in Theorem 1, we obtain the following result by using Theorem 2.

Corollary 3. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let T be a nonexpansive mapping from C into itself. Assume that $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} r_n = \infty$. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \frac{1}{1+r_n}x_n + \frac{r_n}{1+r_n}Ty_n, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)y_n, \quad n \in \mathbb{N}. \end{cases}$$

If $F(T) \neq \emptyset$ and $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \downarrow 0$, then $\{x_n\}$ converges strongly to $z \in F(T)$, where z_t is a unique element of C which satisfies $z_t = tx + (1-t)Tz_t$.

In the case where A is an m -accretive operator, we obtain the following result by using Theorem 2.

Corollary 4. *Let E be a Banach space with a uniformly Gâteaux differentiable norm and let $A \subset E \times E$ be an m -accretive operator. Assume that $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} r_n = \infty$. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)J_{r_n}x_n, \quad n \in \mathbb{N}. \tag{3.6}$$

If $A^{-1}0 \neq \emptyset$ and $\{J_t x\}$ converges strongly to $z \in A^{-1}0$ as $t \rightarrow \infty$, then $\{x_n\}$ converges strongly to $z \in A^{-1}0$.

4. MANN'S TYPE ITERATIVE SCHEME

In this section, we prove a weak convergence theorem for Mann's type iteration. Before proving the theorem, we need the following two lemmas.

Lemma 5 (Browder [2]). *Let C be a closed bounded convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $Tz = z$.*

Lemma 6 (Reich [14]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable norm, let C be a nonempty closed convex subset of E and let $\{T_0, T_1, T_2, \dots\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \dots T_0$ for all $n \in \mathbb{N}$. Then the set $\bigcap_{n=0}^{\infty} \overline{\text{co}}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=0}^{\infty} F(T_n)$.*

For the proof of Lemma 6, see Takahashi and Kim [21]. Now we can prove the following weak convergence theorem.

Theorem 7. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I+rA)$. Assume that $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\liminf_{n \rightarrow \infty} r_n > 0$. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n \in \mathbb{N}. \quad (4.1)$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Proof. Let u be an element of $A^{-1}0$ and $y_n = J_{r_n} x_n$. Then for $l = \|x - u\|$, the set $D = \{z \in E : \|z - u\| \leq l\}$ is a nonempty closed bounded convex subset of E which is invariant under J_s for $s > 0$. So $\{x_n\} \subset D$ is bounded. From

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n x_n + (1 - \alpha_n) y_n - u\| \\ &\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \|y_n - u\| \\ &\leq \|x_n - u\|, \end{aligned}$$

$\lim_{n \rightarrow \infty} \|x_n - u\|$ exists. Without loss of generality, we may assume that $\lim_{n \rightarrow \infty} \|x_n - u\| \neq 0$. Since A is accretive, we have

$$\begin{aligned} \|y_n - u\| &\leq \left\| y_n - u + \frac{r_n}{2} (A_{r_n} x_n - 0) \right\| \\ &= \left\| y_n - u + \frac{1}{2} (x_n - y_n) \right\| \\ &= \left\| \frac{x_n + y_n}{2} - u \right\| \\ &\leq \|x_n - u\| \left\{ 1 - \delta \left(\frac{\|x_n - y_n\|}{\|x - u\|} \right) \right\} \end{aligned}$$

and hence

$$\begin{aligned} (1 - \alpha_n) \|x_n - u\| \delta \left(\frac{\|x_n - y_n\|}{\|x - u\|} \right) &\leq (1 - \alpha_n) \{ \|x_n - u\| - \|y_n - u\| \} \\ &= \|x_n - u\| - \alpha_n \|x_n - u\| - (1 - \alpha_n) \|y_n - u\| \\ &\leq \|x_n - u\| - \|x_{n+1} - u\|. \end{aligned}$$

Then, by $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\lim_{n \rightarrow \infty} \|x_n - u\| \neq 0$, we obtain $\delta(\|x_n - y_n\|/\|x - u\|) \rightarrow 0$. This implies $x_n - y_n \rightarrow 0$. So, from

$$\begin{aligned} \|y_n - J_1 y_n\| &= \|(I - J_1) y_n\| = \|A_1 y_n\| \leq \inf\{\|z\| : z \in Ay_n\} \\ &\leq \|A_{r_n} x_n\| = \left\| \frac{x_n - y_n}{r_n} \right\| \end{aligned}$$

and $\liminf_{n \rightarrow \infty} r_n > 0$, we have $y_n - J_1 y_n \rightarrow 0$. Further, letting $v \in E$ be a weak subsequential limit of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$, we get $y_{n_i} \rightharpoonup v$. Then it follows from Lemma 5 that $v \in F(J_1) = A^{-1}0$.

We assume that E has a Fréchet differentiable norm. Putting $T_n = \alpha_n I + (1 - \alpha_n) J_{r_n}$ and $S_n = T_n T_{n-1} \cdots T_0$, we have $\bigcap_{n=0}^{\infty} F(T_n) = A^{-1}0$ and $\{v\} = \bigcap_{n=0}^{\infty} \overline{\text{co}}\{x_m : m \geq n\} \cap A^{-1}0$ by Lemma 6. Therefore $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Next we assume that E satisfies Opial's condition. Let v_1 and v_2 be two weak subsequential limits of the sequence $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. As above, we have

$v_1, v_2 \in A^{-1}0$. We claim that $v_1 = v_2$. If not, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - v_1\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - v_1\| < \lim_{i \rightarrow \infty} \|x_{n_i} - v_2\| = \lim_{n \rightarrow \infty} \|x_n - v_2\| \\ &= \lim_{j \rightarrow \infty} \|x_{n_j} - v_2\| < \lim_{j \rightarrow \infty} \|x_{n_j} - v_1\| = \lim_{n \rightarrow \infty} \|x_n - v_1\|. \end{aligned}$$

This is a contradiction. Hence we have $v_1 = v_2$. This implies that $\{x_n\}$ converges weakly to an element of $A^{-1}0$. \square

We can also study the strong convergence of (4.1) by using the metric projection.

Proposition 8. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Assume $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by (4.1). If $A^{-1}0 \neq \emptyset$ and P is the metric projection of E onto $A^{-1}0$, then $\{Px_n\}$ converges strongly to an element of $A^{-1}0$.*

Proof. We have

$$\begin{aligned} \|Px_{n+1} - x_{n+1}\| &\leq \|Px_n - x_{n+1}\| \\ &\leq \alpha_n \|Px_n - x_n\| + (1 - \alpha_n) \|Px_n - J_{r_n} x_n\| \\ &\leq \|Px_n - x_n\| \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \|Px_n - x_n\|$ exists. We shall show that $\{Px_n\}$ is a Cauchy sequence. Let $a = \lim_{n \rightarrow \infty} \|Px_n - x_n\|$. If $a = 0$, then for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\|Px_n - x_n\| \leq \varepsilon/4$ for all $n \geq N_0$. If $n, m \geq N_0$, then we have

$$\begin{aligned} \|Px_n - Px_m\| &\leq \|Px_n - x_n\| + \|x_n - Px_{N_0}\| \\ &\quad + \|Px_{N_0} - x_m\| + \|x_m - Px_m\| \\ &\leq \|Px_n - x_n\| + \|x_{N_0} - Px_{N_0}\| \\ &\quad + \|Px_{N_0} - x_{N_0}\| + \|x_m - Px_m\| \\ &\leq \varepsilon. \end{aligned}$$

Then $\{Px_n\}$ is Cauchy. Let $a > 0$. If $\{Px_n\}$ is not Cauchy, then there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, there are $n, m \geq N$ with $\|Px_n - Px_m\| \geq \varepsilon$. Choose $d > 0$ such that

$$\frac{a}{a+d} > 1 - \delta \left(\frac{\varepsilon}{a+d} \right)$$

and $N_1 \in \mathbb{N}$ such that $\|Px_n - x_n\| < a + d$ for all $n \geq N_1$. For this $N_1 \in \mathbb{N}$, there exist $n, m \geq N_1$ such that $\|Px_n - Px_m\| \geq \varepsilon$. For all $l \in \mathbb{N}$ with $l \geq n$ and $l \geq m$, we have

$$\|Px_n - x_l\| \leq \|Px_n - x_n\| < a + d$$

and

$$\|Px_m - x_l\| \leq \|Px_m - x_m\| < a + d.$$

Since E is uniformly convex, we obtain

$$\begin{aligned} a &= \lim_{l \rightarrow \infty} \|Px_l - x_l\| \\ &\leq \limsup_{l \rightarrow \infty} \left\| \frac{Px_n + Px_m}{2} - x_l \right\| \\ &\leq (a + d) \left(1 - \delta \left(\frac{\varepsilon}{a + d} \right) \right) \\ &< a, \end{aligned}$$

which is a contradiction. Thus the proof is completed. \square

As direct consequences of Theorem 7, we obtain the following two results.

Corollary 9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable or which satisfies Opial's condition and let T be a nonexpansive mapping of C into itself. Assume that $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\liminf_{n \rightarrow \infty} r_n > 0$. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \frac{1}{1+r_n}x_n + \frac{r_n}{1+r_n}Ty_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \quad n \in \mathbb{N}. \end{cases}$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $F(T)$.

Corollary 10. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let $A \subset E \times E$ be an m -accretive operator. Assume that $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\liminf_{n \rightarrow \infty} r_n > 0$. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, \quad n \in \mathbb{N}.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

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