INEQUALITIES AND COMMON FIXED POINTS FOR MÖBIUS GROUPS IN $\overline{R}^n$

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ABSTRACT. In this paper we prove some inequalities for Möbius transformations and discrete groups in $\overline{R}^n$. We also get several sufficient and necessary conditions that two elements of Möbius group M($\overline{R}^n$) have a common fixed point.

1. INTRODUCTION. For each f and g of Möbius group M($\overline{R}^n$) we let $[f,g]$ denote $fgf^{-1}g^{-1}$, (f,g) denote $fgfg^{-1}$ and $< f,g >$ denote the group generated by f and g.

In 1976[1], T.Jorgensen proved the following famous inequality which is basic for discrete groups:

Theorem A Suppose that f and $g \in M(\overline{R}^2)$ generate a discrete non-elementary group. Then

$$|tr^2(f) - 4| + |tr[f,g] - 2| \geq 1 \tag{1}$$

The lower bound is best possible.

There has been strong activity in the area of Möbius transformations in several dimension since L.V.Ahlfors published his two famous papers[5][6]. Gilman thinks it is important to establish the Jorgensen's inequality and it's similar forms in $\overline{\mathbf{R}}^n$([7]). In recent years many Jorgensen type inequalities in $\overline{\mathbf{R}}^2$ have been established[2][3][4]. However, to study the Möbius groups in $\overline{\mathbf{R}}^n$ is very difficult. It is well known that M($\overline{\mathbf{R}}^2$) and M($\overline{\mathbf{R}}^n$), Where $n > 2$, have many essential distinctions.

In this paper we study the Jorgensen's inequality and its similar forms. We obtain the following inequality which is equivalent to Jorgensen's inequality:

Theorem B Suppose that f and $g \in M(\overline{R}^2)$ generate a non-elementary discrete group. Then

$$|tr^2(f) - 4| + |tr(f,g) - 2| \geq 1 \tag{2}$$

The lower bound is best possible.

We also prove some inequality for discrete Möbius groups in $\overline{R}^n$ and study (f,g) futher, we obtain several sufficient and necessary conditions that two elements of M($\overline{R}^n$) have common fixed points.

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2. SOME BASIC CONCEPTS. The Clifford algebra A^n shall be the associative algebra over the real numbers generated by n - 1 elements $e_1, e_2, \ldots, e_{n-1}$ subject to the relation $e_h^2 = -1, e_h e_k = -e_k e_h (h \neq k)$,and no others. Every $a \in A_n$ has a unique representation in the form $a = a_0 + \sum a_v E_v$, where a_0 and a_v are real and the sum ranges over all multi-indices $v = (v_1, v_2, \ldots, v_p)$ with $0 < v_1 < v_2 < \cdots < v_p \leq n-1$, and $E_v = e_{v_1} e_{v_2} \cdots e_{v_p}$. The Clifford numbers of the special form $x = x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1}$ are called vectors. They form an n-dimensional subspace V^n which we shall usually identity with $\mathbf{R^n}$.

Definition 2.1 $|a|^2 = a_0^2 + \sum_v a_v^2$ for each $a \in A_n$

The algebra A_n has three important involutions. The main conjugation consists in replacing every e_h by $-e_h$. We shall denote the main conjugation of a by $a^{'}$. It is an automorphism in the sense that $(a+b)^{'} = a^{'} + b^{'}$ and $(ab)^{'} = a^{'}b^{'}$. Next by reversing the order of the factor in each $E_v = e_{v_1} \cdots e_{v_p}$, we obtain a conjugation $a \to a^*$. Obviously, $(ab)^* = b^* a^*$. These conjugation can be combined to a third, $\bar{a} = (a^{'})^* = (a^*)^{'}$.

Definition 2.2 The center $\Im_n$ of A_n consists of all $a \in A_n$ which commutes with every element of A_n.

Definition 2.3 The Clifford group Γ_n consists of all $a \in A_n$ which can be written as products of non-zero vectors in $\mathbf{R^n}$.

Definition 2.4 The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Clifford matrix in dimension n if the following conditions are fulfilled:

(1) $a, b, c, d \in \Gamma_n \cup \{0\}$;

(2) $ad^* - bc^* = 1$;

(3) ac^{-1} and $c^{-1}d \in \mathbf{R^n}$ if $c \neq 0$;

(4) db^{-1} and $b^{-1}a \in \mathbf{R^n}$ if $b \neq 0$.

We let $SL(2, \Gamma_n)$ denote the set of Clifford matrices. For every Möbius transformation $g \in M(\overline{\mathbf{R}}^n)$ we have the expression $g(x) = (ax+b)(cx+d)^{-1}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \Gamma_n)$.

We classify the elements of Möbius group M($\overline{\mathbf{R}}^n$)/{Id} as the following:

Definition 2.5 (1)If g is conjugate to $\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$, Where $\gamma \in \mathbf{R}/\{\pm 1, 0\}$, then g is called hyperbolic;

(2)If g is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{'} \end{pmatrix}$, where $|\lambda|$=1,$\lambda \neq \pm 1$, then g is called elliptic;

(3)If g is conjugate to $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, where $t \in \mathbf{R}^n/\{0\}$, then g is called strictly parabolic;

(4)If g is conjugate to $\begin{pmatrix} \eta t & 0 \\ t & t\eta \end{pmatrix}$, where $\eta \in \mathbf{R}^n/\mathbf{R}$,t$\neq$ t*, and either $t \notin \Im_n$($\Im_n$ is the center of A^n) or t$\eta \notin \Im_n$, then g is called qusi-parbolic;

(5)If g is conjugate to $\begin{pmatrix} \gamma\lambda & 0 \\ 0 & \gamma^{-1}\lambda^{'} \end{pmatrix}$, where $|\lambda| = 1, \lambda \neq \pm 1, \gamma \in \mathbf{R}/\{\pm 1, 0\}$, then g is called loxodromic;

(6)If g is conjugate to $\begin{pmatrix} \lambda & -\gamma^2 t^{'} \\ t & \lambda^{'} \end{pmatrix}$, where $|\lambda| < 1, \gamma \in \mathbf{R}, t \neq 0, t \in \mathbf{R}^n$ and$|(\lambda^* + \lambda^{'})^2 - Re(\lambda^* + \lambda^{'})^2|^2 = -[(\lambda^* + \lambda^{'})^2 - Re(\lambda^* + \lambda^{'})^2]^2$ can not hold at the same time, and for $\forall u \in \mathbf{R}^n/0$, μt $\neq -($t$\mu)^{'}$, then g is called motion.

Remark 2.6 It is easy to prove that g is strictly parabolic if and only if g conjugates to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Definition 2.7 $tr(g) = a + d^*$ is the trace of $g, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \Gamma_n)$

Definition 2.8. A subgroup G of M($\overline{\mathbf{R}}^n$) is said to be elementary if and only if G has a finite G-orbit in $\overline{\mathbf{R}}^{n+1}$.

3. INEQUALITIES.

Now we establish a inequality in $\overline{\mathbf{R}}^2$:

Theorem 3.1(Theorem B). Suppose that f and $g \in M(\overline{\mathbf{R}}^2)$ generate a non-elementary group. Then

$$|tr^2(f) - 4| + |tr(f, g) - 2| \geq 1$$

The lower bound is best possible.

Proof. Case 1: f is parabolic. As the trace is invariant under conjugation we may assume that

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $c \neq 0$. We are assuming that the inequality fails and we have $|c| < 1$
Let $g_0 = g, g_{n+1} = g_n f g_n^{-1}$. Then

$$g_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a_n c_n & a_n^2 \\ -c_n^2 & 1 + a_n c_n \end{pmatrix}$$

From this we deduce that $c_n = -(-c)^{2^n}$, so $c_n \to 0$, $|a_n| \leq n + |a_0|$. Thus $a_n c_n \to 0$.
Then $g_{n+1} \to f$ As $< f, g >$ is discrete, the inequality holds.

Case 2: f is loxodromic or elliptic. Whithout loss of generality, Set

$$f = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix}, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Where $bc \neq 0$, We assume that the inequality fails, then

$$\mu = |tr^2(f) - 4| + |tr(fgfg^{-1}) - 2| = (1 + |ad|)|u - 1/u|^2 < 1$$

Let $g_0 = g$, $g_{n+1} = g_n f g_n^{-1}$

$$g_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} a_n d_n u - b_n c_n/u & a_n b_n(1/u - u) \\ c_n d_n(u - 1/u) & a_n d_n/u - b_n c_n u \end{pmatrix}$$

So $b_{n+1}c_{n+1} = (b_n c_n)(a_n d_n)(u - 1/u)^2$. We obtain $|b_n c_n| \leq \mu^n |bc|$, $|b_n c_n| \leq \mu^n |ad|$.
So $b_n c_n \to 0$, $a_n d_n \to 1$ and $a_{n+1} \to u, d_{n+1} \to 1/u$.
Also, we obtain $|b_{n+1}/b_n| \leq \mu^{1/2}|\mu|$. So $|b_{n+1}/u^{n+1}| < (1 + \mu^{1/2}/2)|b_n/u^n|$
Thus $b_n/u^n \to 0$ and $c_n u^n \to 0$. It follows that

$$f^{-n} g_{2n} f^n = \begin{pmatrix} a_{2n} & b_{2n}/u^{2n} \\ u^{2n} c_{2n} & d_{2n} \end{pmatrix} \to f$$

As $< f, g >$ is discrete, the inequality holds.

To show that the lower bound in the theorem 3.1 is best possible, consider the group generated by $f(z) = z + 1$ and $g(z) = -1/z$.

Remark 3.2 Using the Lie-product of f and g([3]) we can prove that the inequality in Theorem 3.1 is equivalent to the Jorgensen's inequality in the theorem A. The following is the brief proof:

Let $\phi = fg - gf$ be the Lie-product of f and g, then ϕ is elliptic of order 2 and conjugates to f and g to their inverses. Applying theorem 3.1 to f and $g\phi$ yields theorem A. Applying theorem A to f and $g\phi$ yields theorem 3.1.

When $n > 2$, we have the following theorems:

Theorem 3.3 When $n \neq 4l(l = 0, 1, 2, \cdots,)$, f is loxodromic or elliptic, $f = \begin{pmatrix} \gamma\lambda & 0 \\ 0 & \gamma^{-1}\lambda' \end{pmatrix}$, where $\gamma \in R/\{0, 1\}$, $|\lambda| = 1$, $\lambda \neq \pm 1$, $\lambda \in \Im_n$, $< f, g >$ is a discrete subgroup, f and g have no common fixed point, $g(F_f) \neq F_f$. Then

$$|tr^2(f) - 4| + |tr(f, g) - 2| \geq 1$$

Proof. As $n \neq 4l$, $\lambda \in \Im_n$, we have $\lambda = \lambda^*, \lambda' = \overline{\lambda}$, Write g= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If f is elliptic of order two, then the inequality holds. When f is elliptic, we may assume that f is not of order two.

We obtain $a \neq 0, d \neq 0, b \neq 0, c \neq 0$.

Set $g_o = g$, $g_{m+1} = g_m f g_n^{-1}$, Write $g_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}$, We have

$$a_{m+1} = a_m d_m^* \gamma\lambda - b_m c_m^* \gamma^{-1}\lambda'.$$

$$b_{m+1} = a_m b_m^* (\gamma^{-1}\lambda' - \gamma\lambda).$$

$$c_{m+1} = c_m d_m^* (\gamma\lambda - \gamma^{-1}\lambda').$$

$$d_{m+1} = a_m d_m^* \gamma^{-1}\lambda' - b_m c_m^* \gamma\lambda$$

$$fgfg^{-1} = \begin{pmatrix} \gamma^2\lambda\lambda^* ad^* - bc^* & ba^* + \gamma^2\lambda\lambda^* ab^* \\ cd^* - \gamma^{-2}\lambda'\overline{\lambda} dc^* & -cb^* + \gamma^{-2}\lambda'\overline{\lambda} da^* \end{pmatrix}$$

So

$$|tr(fgfg^{-1})-2| = |ad^*||\gamma\lambda - \gamma^{-1}\lambda^{'}|^2$$

Set

$$\alpha = |tr^2(f)-4| + |tr(f,g)-2| = (1+|ad^*|)|\gamma\lambda - \gamma^{-1}\lambda^{'}|^2$$

We assume $\alpha < 1$. We have $a_m \neq 0, b_m \neq 0, c_m \neq o, d_m \neq 0$. We deduce (by induction): $|b_m c_m| \leq \alpha^m |bc|$, and $|b_m c_m| \leq \alpha^m |ad|$,
hence $|b_m c_m| \to 0$, $a_m \to \gamma\lambda$, $d_m \to \gamma^{-1}\lambda'$, $b_m/\gamma^m \to 0$, $c_m \gamma^m \to 0$
So $f^{-m} g_{2m} f^m \to f$ As $< f, g >$ is discrete, we have $\alpha \geq 1$.

Similarly, we can prove the following theorems:

Theorem 3.4. When $n \neq 4l (l = 0, 1, 2 \cdots)$, f is loxodromic or elliptic, f conjugates to $f_0 = \begin{pmatrix} \gamma\lambda & 0 \\ 0 & \gamma^{-1}\lambda^{'} \end{pmatrix}$, Where $\gamma \in R/\{0\}$, $\lambda \neq \pm 1$, $|\lambda| = 1$, $\lambda \in \Im_n$, $< f, g >$ is a discrete subgroup. If $f(0) = 0$ or $f(\infty) = \infty$, f and g have no common fixed point, $g(F_f) \neq F_f$, then

$$|tr^2(f)-4| + |tr(f,g)-2| \geq 1$$

Theorem 3.5. When $n \neq 4l (l = 0, 1, 2, \cdots)$, f is loxodromic or elliptic, f coujugates to $f_0 = \begin{pmatrix} \gamma\lambda & 0 \\ 0 & \gamma^{-1}\lambda^{'} \end{pmatrix}$, where $\gamma \in R/\{0\}$, $\lambda \neq \pm 1$, $|\lambda| = 1$, $\lambda \in \Im_n$. There exists $h = \begin{pmatrix} \alpha & \beta \\ \rho & \delta \end{pmatrix} \in SL(2, \Gamma_n)$, such that $f = hf_0h^{-1}$, $< f, g >$ is a discrete subgroup, f and g have no common fixed point, $g(F_f) \neq F_f$. Write $h^{-1}gh = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad^* \in \Im_n$, then

$$|tr^2(f)-4| + |tr(f,g)-2| \geq 1$$

Remark 3.6 When n=4l, if we add the condition $\lambda \in \mathbf{R}$ and $ad^* \in \mathbf{R}$ in Theorem 3.4 and Theorem 3.5, then the inequality is still holds.

Similar to the proof of theorem 3.1 and theorem 3.3, we can prove:

Theorem 3.7 If $< f, g >$ is a discrete non-elementary subgroup. $f = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, where $t \in \mathbf{R}^n/0$, or $f = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, where $t \in \mathbf{R}$, $t \neq \pm 1$, then

$$|tr^2(f)-4| + |tr(f,g)-2| \geq 1$$

Theorem 3.8 $< f, g >$ is discrete, f is strictly parabolic, f conjugates to $f_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, there exists $h \in SL(2, \Gamma_n)$, such that $f = hf_0h^{-1}$, f and g have no common fixed point, write $h^{-1}gh = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $ad^* \in \mathbf{R}$, then

$$|tr^2(f)-4| + |tr(f,g)-2| \geq 1$$

Remark 3.9 When $n \neq 4l (l = 0, 1, 2, \cdots)$, we can replace the condition $ad^* \in \mathbf{R}$ by $ad^* \in \Im_n$

Using theorem 3.5, we can obtain the following theorem(similar to [9])

Theorem 3.10. Assume that $f = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in SL(2,\Gamma_n)$ is hyperbolic, where $|a_1| \neq |d_1|$, and $g = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in SL(2,\Gamma_n)$.If $< f, g >$ is discrete non-elementary group, then there is $m \in \mathbf{R}, 0 < m \leq 1$ such that

$$|tr^2(f) - 4| + |tr(f,g) - 2|/m \geq 1$$

4. COMMON FIXED POINTS. When n=2, we have the following theorem.

Theorem 4.1. Let $f, g \in M(\overline{\mathbf{R}}^2)$, then f and g have a common fixed point in $\overline{\mathbf{R}}^2$ if and only if

$$tr(f,g) = tr^2(f) - 2$$

Proof. Without loss of generality, we let

$$f = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

we have

$$tr(f,g) = a\alpha a\delta + d\delta d\alpha = a^2 + d^2 = tr^2(f) - 2.$$

Now suppose that $tr(f,g) = tr^2 f - 2$.

If f is parabolic we can take $a = d = b = 1, g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

So

$$2 = tr(f,g) = 2 - \gamma^2,$$

Then $\gamma = 0$, thus both f and g fix ∞ ;

If f is not parabolic, we can assume that

$$f = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} (a \neq d), g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

So $tr(f,g) = (a^2 + d^2)\alpha\delta - 2\gamma\beta = tr^2(f) - 2$

Then $\gamma\beta = 0$, So g fixes one of 0 and ∞.

When $n \geq 2$, we have the following theorem:

Theorem 4.2 Suppose that $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $f = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ (neither is the identity) or $f = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is parabolic. If f and g have two fixed points at most and one fixed point at least, then f and g have a common fixed point x=0 if and only if

$$(f,g) \sim \begin{pmatrix} a\alpha a\alpha^{-1} & 0 \\ * & \frac{1}{|a|^4} a' \alpha' a' (\alpha')^{-1} \end{pmatrix}, (\alpha \neq 0)$$

Where $*$ is a certain Clifford number and $\sim$ denotes the conjugation.

Proof. The necessity is clear. Now we prove the sufficiency.
Let $f = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We have

$$(f,g) = \begin{pmatrix} a\alpha a\delta^* - a\beta d\gamma* & a\beta d\alpha^* - a\alpha a\beta^* \\ d\gamma a\delta^* - d\delta d\gamma^* & d\delta d\alpha^* - d\gamma a\beta^* \end{pmatrix}$$

Let $(f,g)_{ij}$ represent the element of (f,g) in the i-th row and j-th column. We consider the folowing two cases: $|a| = 1$ and $|a| \neq 1$.

If $|a| = 1$, then $d = a^{'}$ and g is elliptic. As $\beta^*(\alpha^*)^{-1} = \alpha^{-1}\beta$, $a\bar{a} = 1$ and $(a^*)^{-1} = a^{'}$, we obtain that $(f,g)_{12} = a\beta a^{'}\alpha^* - a\alpha a\beta^* = 0$ if and only if $x = \alpha^{-1}\beta$ is a solution of the equation.

$$x = axa^{'-1} \tag{3}$$

Let $x = \alpha^{-1}\beta$ be a solution of equation (3), then $\beta = \alpha a\alpha^{-1}\beta$. Therefore

$$(f,g)_{11} = a\alpha a\alpha^{-1}$$

In addition, let $x = \alpha^{-1}\beta$ be a solution of the equation (3), then $a\beta^* = \beta^*(\alpha^*)^{-1}d\alpha^*$. Thus

$$(f,g)_{22} = a^{'}\alpha^{'}a^{'}(\alpha^{'})^{-1}.$$

Now we study the solvablity of (3). The equation (3) is

$$x = ax(a^{'})^{-1} \equiv Ax \tag{4}$$

Where

$$A = Q\begin{pmatrix} A_1 & & & & \\ & \ddots & & & 0 \\ & & A_r & & \\ & & & -E_{2s} & \\ & & & & E_t \end{pmatrix} Q^{-1},\ A_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}, i = 1, \cdots, r$$

and Q is an orthogonal matrix. Similar to the proof of theorem 2.5 in [9], We know that there do not exists non-zero solutions of (4). Since $(f,g)_{12} = 0$, we have $\alpha^{-1}\beta = 0$, So $\beta = 0$. Thus f and g have a common fixed point x=0.

If $|a| \neq 1$, then $f = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is loxodromic. Similarly, for the case that f is loxodromic or parabolic, we can prove that f and g have a common fixed point $x = 0$.

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