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ON RINGS WHOSE PRIME FACTORS ARE SIMPLE

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ABSTRACT. The connection between the π-regularity of an associative ring with identity and the simplicity of all of its prime factors has long been investigated by many authors. We prove, in this note, that for a ring R whose maximal right (or left) ideals are two-sided, the following conditions are equivalent: (1) Every prime ideal of R is maximal; (2) Every prime ideal of R is primitive; (3) Every prime factor of R is Artinian; (4) Every prime factor of R is von Neumann regular; (5) R is π-regular and $R/P(R)$ is strongly regular; (6) R is strongly π-regular and $R/P(R)$ is strongly regular; (7) $R/P(R)$ is strongly regular. This generalizes and sharpens all the known results in [8], [2], [6], [9] and [10].

Throughout, R denotes an associative ring with identity, $J(R)$ and $P(R)$ stand for the Jacobson radical and the prime radical of R, respectively. R is called π-regular if for every $x \in R$, there exists a number n (depending on x) and an element $a \in R$ such that $x^n a x^n = x^n$. R is called strongly π-regular if for every $x \in R$, there exists a number n (depending on x) and an element $a \in R$ such that $x^n = ax^{n+1}$. It is well-known that the latter definition is left-right symmetric ([3]) and that strongly π-regular rings are π-regular ([1]). We call a ring R strongly regular if for every $x \in R$, there exists $y \in R$ such that $x = x^2 y$. Strongly regular rings are exactly those von Neumann regular rings with all idempotents central, or without nonzero nilpotent element (cf. Goodearl [5], Ch.3).

The connection between the π-regularity of a ring R and the simplicity of all of its prime factors has been investigated by many authors. The earliest result of this type was proved by Storrer [8]: For a commutative ring R, the following conditions are equivalent: (1) R is π-regular; (2) $R/P(R)$ is von Neumann regular; (3) Every prime ideal of R is maximal. Fisher and Snider extended this result to P.I. rings ([4], Theorem 2.3). In another direction, Chandran ([2], Theorem 3) generalized Storrer's result to duo rings (i.e. every one-sided ideal is two-sided). Later, Chandran's result was further generalized to right duo rings (every right ideal is two-sided) by Hirano ([6], Corollary 1) and to bounded weakly right duo rings (see below for definition) by Yao ([9], Theorem 3). Recently, the author proved that for a right quasi-duo ring R, if every prime ideal is right primitive, then R is strongly π-regular and $R/J(R)$ is strongly regular (Yu [10], Theorem 2.5).

The main purpose of this note is to prove the following theorem. Call a ring right (left) quasi-duo if every maximal right (left) ideal is two-sided.

Theorem 1. *Let R be a right quasi-duo ring, then the following conditions are equivalent:*
(1) Every prime ideal of R is maximal;
(2) Every prime ideal of R is right primitive;
(3) Every prime factor of R is Artinian;
(4) Every prime factor of R is von Neumann regular;
(5) R is strongly π-regular and $R/P(R)$ is strongly regular;

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(6) R is π-regular and $R/P(R)$ is strongly regular;
(7) $R/P(R)$ is strongly regular.

Proof. (1) $\Longrightarrow$ (2): Trivial.

(2) $\Longrightarrow$ (3): This follows from the fact that right primitive right quasi-duo rings are division rings (see [10] or [11]).

(3) $\Longrightarrow$ (4): $J(R)$ is nilpotent for Artinian rings,therefore prime factors of R are semisimple Artinian, which are trivially von Neumann regular.

(4) $\Longrightarrow$ (1): Right quasi-duo rings have the property that all primitive factors are Artinian and the property that all primitive factors are Artinian is inherited by factor rings. Let Q be a prime ideal of R, then R/Q is a prime von Neumann regular ring with primitive factors Artinian. Appealing to Theorem 6.2 of Goodearl [5], we have that R/Q is Artinian, hence Q is maximal.

(1) $\Longrightarrow$ (5): Since we have proved that the first four statements are all equivalent for a right quasi-duo ring, we freely use any one of the four to get (5).

For an arbitrary ring R, R is strongly π-regular if and only if every prime factor of R is strongly π-regular ([4], Theorem 2.1). Artinian rings are trivially strongly π-regular, so R is strongly π-regular by (3). $R/P(R)$ is also strongly π-regular, since strongly π-regularity is preserved by homomorphic images. To show that $R/P(R)$ is strongly regular, we note that (2) implies $J(R) = P(R)$. Yu [10] proved that for a right quasi-duo ring, all nilpotent elements are contained in the Jacobson radical ([10], Lemma 2.3), so $R/P(R)$ contains no nonzero nilpotent elements. Take any $a \in R/P(R)$, then there exists $x \in R/P(R)$ such that $a^k = a^{k+1}x$ for some k, by the strongly π-regularity of $R/P(R)$. Azumaya ([1], Theorem 3) guarantees that we may further assume $ax = xa$, hence $(a^k x - a^{k-1})^2 = 0$. Inductively, we have $a = a^2 x$, which completes the proof for this direction.

(5) $\Longrightarrow$ (6) $\Longrightarrow$ (7): Trivial.

(7) $\Longrightarrow$ (1): Let Q be a prime ideal of R, then $Q \supseteq P(R)$. But prime ideals of a strongly regular ring are maximal ([5], Theorem 3.2 (b).), so Q is maximal. □

Several corollaries are immediate.

Corollary 2 (Yu [10], Theorem 2.5). *For a right quasi-duo ring R, if every prime ideal of R is right primitive, then R is strongly π-regular and $R/J(R)$ is strongly regular.* □

In [9], a ring R is called weakly right duo if for every $a \in R$ there exists a natural number $n(a)$ such that the right ideal $a^{n(a)}R$ is actually two-sided and R is called bounded weakly right duo if R is weakly right duo and there is a natural number N such that $n(a) \leq N$ for all $a \in R$. Weakly right duo rings are right quasi-duo by Yu ([10], Proposition 2.2) and *there exist right quasi-duo rings which are not weakly right duo* (see [10]). Theorem 3 of Yao [9] says that for a bounded weakly right duo ring R, R is π-regular if and only if every prime ideal of R is maximal, which extended the more older results of Chandran ([2], Theorem 3) for duo rings (every one-sided ideal is two-sided), and of Hirano ([6], Corollary 1) for right duo rings (every right ideal is two-sided). We now generalize and sharpen all these to the following

Corollary 3. *For a weakly right duo ring R, the following conditions are all equivalent:*
(1) Every prime ideal of R is maximal;
(2) Every prime ideal of R is right primitive;
(3) Every prime factor of R is Artinian;
(4) Every prime factor of R is von Neumann regular;
(5) R is strongly π-regular and $R/P(R)$ is strongly regular;

(6) R is π-regular and R/P(R) is strongly regular;
(7) R/P(R) is strongly regular. □

Actually, conditions (1) through (4) are equivalent for a larger class of rings, i. e. rings with primitive factors Artinian, which contains all P. I. rings as a subclass. Fisher and Snider ([4], Theorem 2.3) showed that for P. I. rings, conditions (1) through (4) are all equivalent and further equivalent to strongly π-regularity. Similar result was established for rings with bounded index of nilpotence (i. e. there exists a fixed number n such that $x^n = 0$ for every nilpotent element) by Hirano ([7], Theorem 2). It is attempting to ask if our Theorem 1 can be established for P.I. rings, rings with bounded index of nilpotence or even for rings with primitive factors Artinian. The following example shows that the answer is no in all three cases. It is a modification of an example of Kaplansky ([4]).

Example 4. *Let F be a field, $M_2(F)$ denotes the 2×2 matrices over F.*

$$R = \{(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

$$\cdots,)|n \in N; a_i, b_i, c_i, d_i, a, b \in F\}$$

It is easy to check that any ideal of R is one of the following three types:

I. $Q_A = \{(I_1, I_2, \cdots, I_n, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \cdots)|n \in N; I_i \in M_2(F);$

$$I_i = 0, i \in A\},$$

II. $Q'_A = \{(I_1, I_2, \cdots, I_n, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \cdots)|n \in N; I_i \in M_2(F);$

$$I_i = 0, i \in A\},$$

III. $Q''_B = \{(I_1, I_2, \cdots, I_n, 0, 0, 0, \cdots)|n \in N; I_i \in M_2(F);$

$$I_i = 0, i \in B\},$$

where A is a finite subset of natural numbers and B is an arbitrary subset of natural numbers.

If $B = \{k\}$ for some natural number k, then $(Q'_B)^2 \subset Q''_B$, and so Q''_B is not prime. If B has more than two elements, then, for any $s, t \in B$ with $s \neq t, Q''_{N-\{s\}}Q''_{N-\{t\}} = 0 \subset Q''_B$ and hence Q''_B is not prime. Therefore ideals of type III are not prime.

An ideal of type I is prime if and only if its index set A is a singleton, and ideals of type II are prime if and only if its index set A is empty. Therefore, $P(R) = (\cap Q_{\{n\}}) \cap Q'_\emptyset = 0$ and R is semiprime. Also, every prime factor of R is simple Artinian and R satisfies conditions (1) through (4), but R is not von Neumann regular, since the element

$$x = (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \cdots)$$

is not regular in R. Clearly, R is a P. I. ring of bounded index, and primitive factors of R are Artinian. This example shows that possible extensions of our Theorem 1 are limited.

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