Scientiae Mathematicae **Vol. 1,** No. 3(1998), 347–354

ON BH–ALGEBRAS

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Received April 14, 1998

Abstract. In this paper, we introduce a new notion, called an BH-algebra, which is a generalization of $BCH/BCI/BCK$-algebras. We define the notions of ideals and boundedness in BH-algebras, and show that there is a maximal ideal in bounded BH-algebras. Furthermore, we establish construct the quotient BH-algebras via translation ideals and obtain the fundamental theorem of homomorphisms for BH-algebras as a consequence.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2, 3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. BCK-algebras have some connections with other areas: D. Mundici [8] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [7] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. In this paper, we introduce a new notion, called an BH-algebra, which is a generalization of $BCH/BCI/BCK$-algebras. We define the notions of ideals and boundedness in BH-algebras, and show that there is a maximal ideal in bounded BH-algebras. Furthermore, we establish construct the quotient BH-algebras via translation ideals and obtain the fundamental theorem of homomorphisms for BH-algebras as a consequence.

2. BH-algebras

In 1983, Q. P. Hu and X. Li [1] introduced a very interesting class of algebras, called a BCH-algebra. An algebra $(X;*,0)$ of type (2,0) with the following axioms: for all $x, y, z \in X$,

(1) $x * x = 0$,

(2) $(x * y) * z = (x * z) * y$,

(3) $x * y = 0$ and $y * x = 0$ imply $x = y$.

is called a *BCH-algebra*. It is well known that for any BCH-algebra X

(4) $x * 0 = x$ for all $x \in X$.

It is natural to pose a question: Can we construct more generalized algebraic class of the BCH-algebras ? In this section, we will discuss this question and introduce the new notion of BH-algebras which is a generalization of BCH-algebras.

1980 *Mathematics Subject Classification* (1985 *Revision*). 06F35, 03G25.

Key words and phrases. BH-algebra, (maximal) ideal, translation ideal, quotient BH-algebra, homomorphism.

†Supported by the BSRI Program, Ministry of Education, 1997, Project No. BSRI-97-1406

‡ Supported by the Korea Research Foundation, 1997 .

Definition 2.1. By a *BH-algebra*, we mean an algebra $(X; *, 0)$ of type (2,0) having the conditions (1), (3) and (4).

Example 2.2. (a) Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

It is easy to verify that $(X; *, 0)$ is a BH-algebra, but not a BCH-algebra, since $(2*3)*2 = 1 \neq 2 = (2*2)*3$.

(b) Let $\mathbb{R}$ be the set of all real numbers and define

$$x * y := \begin{cases} 0 & \text{if } x = 0, \\ \frac{(x-y)^2}{x} & \text{otherwise}, \end{cases}$$

for all $x, y \in \mathbb{R}$, where "$-$" is the usual substraction of real numbers. Then it is easy to check that $(\mathbb{R}; *, 0)$ is a BH-algebra, but not a BCH-algebra.

We now investigate relations between BH-algebras and BCH-algebras (also, BCK/BCI-algebras). The following theorems are easily proved, and omit the proof.

Theorem 2.3. *Every BCH-algebra is a BH-algebra. Every BH-algebra satisfying the condition* (2) *is a BCH-algebra.*

Theorem 2.4. *Every BH-algebra satisfying the condition*

(5) $((x * y) * (x * z)) * (z * y) = 0, \quad \forall x, y, z \in X,$

is a BCI-algebra.

Theorem 2.5. *Every BH-algebra satisfying the conditions* (5) *and*

(6) $(x * y) * x = 0, \quad \forall x, y \in X,$

is a BCK-algebra.

By [3], [1], Theorem 2.3 and Example 2.2, we know the following relations:

The class of BCK-algebra	$\subset$	The class of BCI-algebras	$\subset$	The class of BCH-algebras	$\subset$	The class of BH-algebras

Theorem 2.6. *Every BH-algebra X satisfying $x * (x * y) = x * y$ for all $x, y, z \in X$ is a trivial algebra.*

Proof. Putting $x = y$ in the equation $x * (x * y) = x * y$, we have $x * 0 = 0$. It follows from (4) that $x = 0$. Hence X is a trivial algebra. $\square$

The following example shows that a BH-algebra may not have the associative law.

Example 2.7. Let $X = \{0, 1, 2\}$ with the Cayley table as follows:

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Then X is an BH-algebra, but associativity does not hold, since $(2 * 1) * 1 \neq 2 * (1 * 1)$.

Theorem 2.8. *Every BH-algebra $(X; *, 0)$ satisfying the associative law is a group under the operation "$*$".*

Proof. Putting $x = y = z$ in the associative law $(x * y) * z = x * (y * z)$ and using (1) and (4), we obtain $0 * x = x * 0 = x$. This means that 0 is the identity of X. By (1), every element x of X has its inverse element x itself. Therefore $(X, *)$ is a group. □

3. Ideals in BH-algebras

In this section, we introduce the notions of ideals, translation ideals in BH-algebras, and we show that there is a maximal ideal in bounded BH-algebras. Finally, we construct the quotient BH-algebra via translation ideals.

Definition 3.1. Let X be a BH-algebra and $I(\neq \emptyset) \subseteq X$. I is called an *ideal* of X if it satisfies: for all $x, y, z \in X$,

(7) $0 \in I$,

(8) $x * y \in I$ and $y \in I$ imply $x \in I$,

Obviously, $\{0\}$ and X are ideals of X. We will call $\{0\}$ and X a *zero ideal* and a *trivial ideal*, respectively. An ideal I said to be *proper* if $I \neq X$.

Example 3.2. Let $X = \{0, 1, 2\}$ be a BH-algebra with the following table:

$*$	0	1	2
0	0	0	1
1	1	0	2
2	2	2	0

We can easily show that $\{0, 1\}$ is a proper ideal of X.

Definition 3.3. Let I be an ideal of a BH-algebra X. Then I is called a *maximal ideal* if I is a proper ideal of X, and not a proper subset of any proper ideal of X.

In Example 3.2, $\{0, 1\}$ is a maximal ideal of X.

Definition 3.4. Let X be a BH-algebra. X is said to be *bounded* if there is an element $m \in X$ satisfying $x * m = 0$ for all $x \in X$,

Theorem 3.5. *Let $(X; *, 0)$ be a bounded BH-algebra with $|X| \geq 2$. Then X has at least one maximal ideal.*

Proof. Let $m \in X$ with $x * m = 0$ for all $x \in X$. First, we prove that an ideal I of X is proper if and only if $m \notin I$. In fact, if $m \notin I$, then $I \neq X$, and so I is a proper ideal. Conversely, assume that I is a proper ideal of X and let $x \in X$. If $m \in I$, then since $x * m = 0$, we have $x \in I$. This means that $I = X$, which contradicts to the assumption. Therefore $m \notin I$.

We now prove that every ideal A of X is contained in a maximal ideal. The set of all proper ideals containing A is denoted by $\mathcal{S}$. Obviously, $(\mathcal{S}, \subseteq)$ is a partially ordered set and $\mathcal{S} \neq \emptyset$. Let $\mathcal{S}_0$ be a chain of $\mathcal{S}$ and let $B := \cup\{I | I \in \mathcal{S}_0\}$. Noticing that A is the least element of $(\mathcal{S}, \subseteq)$, we have $A \subseteq B$. Hence $0 \in B$. Let $x, y \in X$ be such that $x * y \in B$ and $y \in B$. Then there are $I_1, I_2 \in \mathcal{S}_0$ such that $x * y \in I_1$ and $y \in I_2$. We may assume $I_2 \subseteq I_1$, without loss of generality. Thus $x * y \in I_1, y \in I_1$ and so $x \in I_1$. It follows that $x \in B$. Hence the condition (8) holds. This means that B is an ideal of X. Since every ideal of $\mathcal{S}_0$

does not contain the element m, we have $m \notin B$, and so B is a proper ideal. Hence $B \in \mathcal{S}$. This proves that every chain of $\mathcal{S}$ has an upper bound in $\mathcal{S}$. By Zorn's Lemma, $\mathcal{S}$ have a maximal element M. This proves the theorem. □

Definition 3.6. An ideal A of a BH-algebra X is said to be a *translation ideal* of X if it satisfies: for all $x, y, z \in X$,

(9) $x * y \in A$ and $y * x \in A$ imply $(x * z) * (y * z) \in A$ and $(z * x) * (z * y) \in A$.

Obviously, $\{0\}$ and X are translation ideals of X. The next examples show that (8) and (9) are independent.

Example 3.7. Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the Cayley table as follows:

*	0	1	2	3
0	0	1	3	0
1	1	0	2	0
2	2	2	0	3
3	3	3	3	0

Then $A = \{0, 1, 2\}$ satisfies (8), but not (9), since $1*2 = 2$, $2*1 = 2 \in A$ but $(1*1)*(2*1) = 0 * 2 = 3 \notin A$.

Example 3.8. Let $X = \{0, 1, 2\}$ be a BH-algebra with the Cayley table as follows:

*	0	1	2
0	0	0	1
1	1	0	0
2	2	1	0

Then $A = \{0, 2\}$ satisfies (9), but not (8), since $1 * 2 = 0 \in A$ and $2 \in A$ but $1 \notin A$.

Example 3.9. Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the Cayley table as follows:

*	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

Then we can easily show that $A = \{0, 1\}$ is a translation ideal of X.

Now we construct the quotient BH-algebras via translation ideals. Let A be a translation ideal of a BH-algebra $(X; *, 0)$. For any $x, y \in X$, we define

$$x \sim_A y \text{ if and only if } x * y \in A \text{ and } y * x \in A.$$

Since $0 \in A$, we have $x * x = 0 \in A$, i.e., $x \sim_A x$ for any $x \in X$. This means that $\sim_A$ is reflexive.

Let $x \sim_A y$ and $y \sim_A z$. Then $x * y, y * x \in A$ and $y * z, z * y \in A$. Thus by (9) we have $(x * z) * (y * z) \in A$ and $(z * x) * (z * y) \in A$. Hence by (8), we get $x * z \in A$ and $z * x \in A$, and so $x \sim_A z$. This proves that $\sim_A$ is transitive.

The symmetry of $\sim_A$ is trivial from the definition. Therefore $\sim_A$ is an equivalence relation on X.

If $x \sim_A y$ and $u \sim_A v$, then $x * y, y * x \in A$ and $u * v, v * u \in A$. Thus by (9) we have $(x * u) * (y * u) \in A$ and $(y * u) * (x * u) \in A$, and so $x * u \sim_A y * u$. Similarly, we have $y * u \sim_A y * v$. By transitivity of $\sim_A$, we obtain $x * u \sim_A y * v$. Consequently $\sim_A$ is a congruence relation on X.

Denote the equivalence class containing x by $[x]_A$, i.e.,

$$[x]_A = \{y \in X | x \sim_A y\}.$$

We note that $x \sim_A y$ if and only if $[x]_A = [y]_A$.

Denote $X/A = \{[x]_A | x \in X\}$ and define

$$[x]_A *' [y]_A = [x * y]_A.$$

The operation "$*'$" is well-defined, since $\sim_A$ is a congruence relation on X. We claim that $(X/A, *', [0]_A)$ is a BH-algebra. Clearly $[x]_A *' [x]_A = [0]_A$ and $[x]_A *' [0]_A = [x]_A$ for all $[x]_A \in X/A$. Let $[x]_A, [y]_A \in X/A$ be such that $[x]_A *' [y]_A = [0]_A = [y]_A *' [x]_A$. Then $[x * y]_A = [0]_A = [y * x]_A$ and so $x * y \sim_A 0$ and $y * x \sim_A 0$. If $u \in [x]_A$ then $x \sim_A u$. It follows that $u * y \sim_A x * y \sim_A 0$, which shows $u \in [y]_A$. Hence $[x]_A \subseteq [y]_A$. Similarly, we have $[y]_A \subseteq [x]_A$. Thus $[x]_A = [y]_A$. We summarize:

Theorem 3.10. *Let A be a translation ideal of a BH-algebra $(X; *, 0)$. If we define*

$$[x]_A *' [y]_A = [x * y]_A \ \ \forall x, y \in X,$$

then $(X/A; *', [0]_A)$ is a BH-algebra, which is called the *quotient BH-algebra* via A.

The notion of translation ideal is necessary for constructing the quotient BH-algebras.

Example 3.11. Let $X = \{0, 1, 2, 3, 4\}$ be a BH-algebra with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then the set $A = \{0, 1\}$ is an ideal of X, but not a translation ideal of X, since $1*0, 0*1 \in A$, but $(1 * 4) * (0 * 4) = 4 \notin A$.

4. Homomorphisms in BH-algebras

In this section, we state a fundamental theorem of a homomorphism.

Definition 4.1. Let X and Y be BH-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if

$$f(x * y) = f(x) * f(y), \ \ \forall x, y \in X.$$

A homomorphism f is called a *monomorphism* (resp., *epimorphism*) if it is injective (resp., surjective). A bijective homomorphism is called an *isomorphism*. Two BH-algebras X and Y are said to be *isomorphic*, written $X \cong Y$, if there exists an isomorphism $f : X \to Y$. For any homomorphism $f : X \to Y$, the set $\{x \in X | f(x) = 0\}$ is called the *kernel* of f, denoted by $Ker(f)$ and the set $\{f(x) | x \in X\}$ is called the *image* of f, denoted by $Im(f)$. Notice that $f(0) = 0$ for any homomorphism f.

Theorem 4.2. *Let $f : X \to Y$ be a homomorphism of BH-algebras. Then $Ker(f)$ is a translation ideal of X.*

Proof. Obviously, $0 \in Ker(f)$. Let $x * y \in ker(f)$ and $y \in ker(f)$. Then $0 = f(x * y) = f(x) * f(y) = f(x) * 0 = f(x)$. Hence $x \in ker(f)$. Let $x * y \in ker(f)$ and $y * x \in ker(f)$. Then we get $f(x) = f(y)$. Thus for any $z \in X$, we have

$$f((x * z) * (y * z)) = 0 \text{ and } f((z * x) * (z * y)) = 0.$$

Hence we obtain $(x * z) * (y * z) \in Ker(f)$ and $(z * x) * (z * y) \in Ker(f)$. Therefore $ker(f)$ is a translation ideal of X. □

Theorem 4.3 (Homomorphism Theorem). *If $f : X \to Y$ is a homomorphism from a BH-algebra X onto a BH-algebra Y, then $X/Ker(f) \cong Y$.*

Proof. Define a mapping $\mu : X/Ker(f) \to Y$ by $\mu([x]_{Ker(f)}) = f(x)$. If $[x]_{Ker(f)} = [y]_{Ker(f)}$ then $x * y, y * x \in Ker(f)$, and so $f(x) * f(y) = 0 = f(y) * f(x)$. By (3) we have $f(x) = f(y)$, i.e., $\mu([x]_{Ker(f)}) = \mu([y]_{Ker(f)})$. This means that μ is well-defined. For any $y \in Y$, there is an $x \in X$ such that $y = f(x)$, since f is onto. Hence $\mu([x]_{Ker(f)}) = f(x) = y$, which means that μ is onto. Let $[x]_{Ker(f)}, [y]_{Ker(f)} \in X/Ker(f)$ with $[x]_{Ker(f)} \neq [y]_{Ker(f)}$. Then $x \sim_{Kerf} y$ does not hold, and hence either $x * y \notin Ker(f)$ or $y * x \notin Ker(f)$. Without loss of generality, we may assume $x * y \notin Ker(f)$. It follows that $f(x) * f(y) = f(x * y) \neq 0$ and hence $f(x) \neq f(y)$. This means that μ is one-one. Since $\mu([x]_{Ker(f)} * [y]_{Ker(f)}) = \mu([x * y]_{Ker(f)}) = f(x * y) = f(x) * f(y) = \mu([x]_{Ker(f)}) * \mu([y]_{Ker(f)})$, μ is a homomorphism. Thus we obtain $X/Ker(f) \cong Y$, completing the proof. □

Theorem 4.4. *Let X, Y and Z be BH-algebras, and let $h : X \to Y$ be an epimorphism and $g : X \to Z$ be a homomorphism. If $Ker(h) \subseteq Ker(g)$, then there is a unique homomorphism $f : Y \to Z$ satisfying $f \circ h = g$.*

Proof. For any $y \in Y$, there exists an $x \in X$ such that $y = h(x)$. Given an element x, we put $z := g(x)$. Define a mapping $f : Y \to Z$ by $f(y) = z$.

To prove that f is well-defined and $f \circ h = g$. If $y = h(x_1) = h(x_2), x_1, x_2 \in X$, then $0 = h(x_1) * h(x_2) = h(x_1 * x_2)$. Hence $x_1 * x_2 \in Ker(h)$. Since $Ker(h) \subseteq Ker(g)$, we have $0 = g(x_1 * x_2) = g(x_1) * g(x_2)$. Similarly, we get $g(x_2) * g(x_1) = 0$. Thus $g(x_2) = g(x_1)$. This means that f is well-defined. Clearly $g(x) = f(h(x))$ for any $x \in X$.

To show that f is a homomorphism. Let $y_1, y_2 \in Y$. For any $x_1, x_2 \in X$ such that $y_1 = h(x_1), y_2 = h(x_2)$, we have $f(y_1 * y_2) = f(h(x_1) * h(x_2)) = f(h(x_1 * x_2)) = g(x_1 * x_2) = g(x_1) * g(x_2) = f(h(x_1)) * f(h(x_2)) = f(y_1) * f(y_2)$. Hence f is a homomorphism. The uniqueness of f follows directly from the fact that h is an epimorphism. □

Theorem 4.5. *Let X, Y and Z be BH-algebras, and let $g : X \to Z$ be a homomorphism and $h : Y \to Z$ be a monomorphism with $Im(g) \subseteq Im(h)$, then there is a unique homomorphism $f : X \to Y$ satisfying $h \circ f = g$.*

Proof. For each $x \in X$, $g(x) \subseteq Im(g) \subseteq Im(h)$. Since h is a monomorphism, there exists a unique $y \in Y$ such that $h(y) = g(x)$. Define a map $f : X \to Y$ by $f(x) = y$. Then $h \circ f = g$. We show that f is a homomorphism. If $x_1, x_2 \in X$, then $g(x_1 * x_2) = h(f(x_1 * x_2))$. On the other hand, since g is a homomorphism, $g(x_1 * x_2) = g(x_1) * g(x_2) = h(f(x_1)) * h(f(x_2)) = h(f(x_1) * f(x_2))$. Hence $h(f(x_1 * x_2)) = h(f(x_1) * f(x_2))$. Since h is a monomorphism, we have $f(x_1 * x_2) = f(x_1) * f(x_2)$. The uniqueness of f follows from the fact that h is a monomorphism. □

If A is a translation ideal of a BH-algebra X. Then a map $p : X \to X/A$ defined by $p(x) = [x]_A$ is a homomorphism, which is called the *canonical mapping*. Note that $Ker(p) = A$

Lemma 4.6. *Let X and Y be BH-algebras and let $f : X \to Y$ be a homomorphism. If A is a translation ideal of X such that $A \subseteq Ker(f)$, then a map $\bar{f} : X/A \to Y$ defined by $\bar{f}([x]_A) = f(x)$ for all $x \in X$, is a homomorphism.*

Proof. We show that $\bar{f}$ is well-defined. If $[x]_A = [y]_A$, then $[x * y]_A = [x]_A * [y]_A$. This means $x * y \in A \subseteq Ker(f)$, and so $f(x) * f(y) = f(x * y) = 0$. Similarly, we have $f(y) * f(x) = f(y * x) = 0$. Hence $f(x) = f(y)$. Clearly, $\bar{f}$ is a homomorphism. □

Theorem 4.7. *Let $f : X \to Y$ be a homomorphism of BH-algebras, and let A be a translation ideal of X and $f : X \to Y$ be a homomorphism. Then the following are equivalent:*

(i) *there is a unique homomorphism $\bar{f} : X/A \to Y$ such that $\bar{f} \circ p = f$, where $p : X \to X/A$ is the canonical mapping.*

(ii) $A \subseteq Ker(f)$.

Furthemore, $\bar{f}$ is a monomorphism if and only if $A = Ker(f)$.

Proof. (i) $\Rightarrow$ (ii). If $a \in A$, then $f(a) = \bar{f}(p(a)) = \bar{f}([0]_A) = f(0) = 0$ for all $a \in A$, since $\bar{f} \circ p = f$ and $Ker(f) = A$. Hence $a \in Ker(f)$.

(ii) $\Rightarrow$ (i). By Lemma 4.6, we have a homomorphism $\bar{f} : X/A \to Y$ defined by $\bar{f}([x]_A) = f(x)$ for all $x \in X$. Since $(\bar{f} \circ p)(x) = \bar{f}([x]_A) = f(x)$ for all $x \in X$, we have $\bar{f} \circ p = f$. The uniqueness of $\bar{f}$ follows from the fact that p is surjective.

Furthemore, $\bar{f}$ is a monomorphism if and only if $f(x) = 0$ implies $[x]_A = [0]_A = A$, i.e., if and only if $Ker(f) \subseteq A$. This proves the theorem. □

Theorem 4.8. *Let X and Y be BH-algebras. If a homomorphism $f : X \to Y$ can be expressed as a composite homomorphisms of BH-algebras*

$$X \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} Y,$$

where α is an epimorphism, β is an isomorphism, and γ is a monomorphism, then $A \cong X/Ker(f)$ and $B \cong Im(f)$.

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & X/Ker(f) & \xrightarrow{\bar{f}} & Im(f) & & \\ & \overset{p}{\nearrow} & & & & \overset{i}{\searrow} & \\ X & & \uparrow h & & \downarrow k & & Y \\ & \underset{\alpha}{\searrow} & & & & \underset{\gamma}{\nearrow} & \\ & & A & \underset{\beta}{\longrightarrow} & B & & \end{array}$$

where $i \circ \bar{f} \circ p$ is the canonical decomposition of f and α, β, γ are respectively an epimorphism, an isomorphism and a monomorphism, respectively. Since $f = \gamma \circ \beta \circ \alpha$ and γ, β are each monomorphisms, we have $f(x) = 0$ if and only if $\alpha(x) = 0$. Hence $Ker(\alpha) = Ker(f) = Ker(p)$. By Theorem 4.4, there is a unique homomorphism $h : A \to X/Ker(f)$ such that $h \circ \alpha = p$. Clearly the mapping h is a monomorphism, since $Ker(\alpha) = Ker(p)$. Moreover, h is surjective, since p is surjective. Thus h is an isomorphism.

Since $Im(\gamma) = Im(f)$, by applying Theorem 4.5, we have a unique homomorphism $k : Im(f) \to B$ such that $\gamma \circ k = i$. The mapping k is clearly an epimorphism. The injectivity of k follows from that i is injective. Thus k is an isomorphism. □

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