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WEAK AND STRONG CONVERGENCE THEOREMS FOR GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first obtain a weak mean convergence theorem of Baillon's type for generalized hybrid mappings in a Hilbert space. Further, using an idea of mean convergence, we prove a strong convergence theorem of Halpern's type for generalized hybrid mappings in a Hilbert space.

1 Introduction Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. Then, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Baillon [4] proved the first nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, for any $x \in C$, $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges weakly to a fixed point of T.*

We also know the following weak convergence theorem of Mann's type; see, for instance, [17], [18] and [23].

Theorem 1.2. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$ for $n = 1, 2, ...,$ where $\{\alpha_n\} \subset [0,1]$ satisfies $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

The following strong convergence theorem of Halpern's type was proved by Wittmann; see [9], [23] and [28].

Theorem 1.3. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by $x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n$ for $n = 1, 2, ...,$ where $\{\alpha_n\} \subset [0,1]$ satisfies $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping $F : C \to C$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [6] and Goebel and Kirk [8]. It is also known that a firmly nonexpansive mapping F is deduced from an equilibrium problem in a Hilbert space;

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see [5] and [7]. Kohsaka and Takahashi [15] introduced the following nonlinear mapping: A mapping $S : C \to C$ is called *nonspreading* if

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2$$

for all $x, y \in C$. A nonspreading mapping was first defined in a Banach space; see also [14]. A nonspreading mapping in a Hilbert space is also deduced from a firmly nonexpansive mapping; see [10], [11], [15] and [25]. Kurokawa and Takahashi [16] proved a nonlinear ergodic theorem of Baillon's type and a strong convergence theorem of Halpern's type for nonspreading mappings in a Hilbert space. Takahashi [25] defined another nonlinear mapping which is deduced from a firmly nonexpansive mapping in a Hilbert space: A mapping $T : C \to C$ is called *hybrid* if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Motivated by the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings, Aoyama, Iemoto, Kohsaka and Takahashi [2] introduced a class of mappings called λ-hybrid and then obtained a generalization of Baillon's nonlinear ergodic theorem; see also Takahashi and Yao [27]. Very recently, Kocourek, Takahashi and Yao [13] introduced a more wide class of nonlinear mappings containing the class of λ-hybrid mappings. Then, they proved fixed point theorems and weak convergence theorems of Baillon's type and Mann's type for the mappings in a Hilbert space.

In this paper, motivated by these results, we first obtain a nonlinear mean ergodic theorem of Baillon's type for generalized hybrid mappings which generalizes Akatsuka, Aoyama and Takahashi [1], Kurokawa and Takahashi [16] and Kocourek, Takahashi and Yao [13] in a Hilbert space. Further, using an idea of mean convergence by Shimizu and Takahashi [19] and [20], we prove a strong convergence theorem of Halpern's type for generalized hybrid mappings in a Hilbert space.

2 Preliminaries

Throughout this paper, we denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We also denote by $\mathbb{N}$ the set of natural numbers. In a Hilbert space, it is known that

$$\|y\|^2 - \|x\|^2 \leq 2\langle y - x, y \rangle \tag{2.1}$$

for all $x, y \in H$; see, for instance, [24]. Let $\{x_n\}$ be a sequence in H and let $x \in H$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \to x$. Let C be a nonempty closed convex subset of H. We can define the metric projection of H onto C: For each $x \in H$, there exists a unique point $z \in C$ such that

$$\|x - z\| = \min\{\|x - y\| : y \in C\}.$$

For each $x \in H$, such a point z is denoted by Px and P is called the *metric projection* of H onto C. It is known that

$$\langle x - Px, Px - y \rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$; see [22] for more details. Let T be a mapping from C into itself. The set of fixed points of T is denoted by $F(T)$. A mapping T is said to be *nonspreading* [15] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2$$

for all $x, y \in C$. Iemoto and Takahashi [10] proved that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{2.3}$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - u\| \le \|x - u\|$ for all $x \in C$ and $u \in F(T)$. If T is a nonspreading mapping from C into itself and $F(T)$ is nonempty, then T is quasi-nonexpansive. Further, we know that the set of fixed points of a quasi-nonexpansive mapping is closed and convex; see [12]. Then we can define the metric projection of H onto $F(T)$. A mapping $T : C \to C$ is called *generalized hybrid* [13] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in C$. We call such a mapping an (α, β)-*generalized hybrid* mapping. Notice that the class of the mappings above covers classes of several well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We can also show that if $x = Tx$, then for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|x - y\|^2 + (1-\beta)\|x - y\|^2$$

and hence $\|x - Ty\| \le \|x - y\|$. This means that an (α, β)-generalized hybrid mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive. To prove our main results, we need the following lemmas:

Lemma 2.1 (Aoyama-Kimura-Takahashi-Toyoda [3]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \le 0$. Suppose that*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2 (Takahashi-Toyoda [26]). *Let D be a nonempty closed convex subset of a real Hilbert space H. Let P be the metric projection of H onto D and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \le \|x_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.*

3 Weak Convergence Theorem In this section, using the technique developed by Takahashi [21], we prove the following weak convergence theorem for generalized hybrid mapping which generalizes Akatsuka, Aoyama and Takahashi [1], Kurokawa, Takahashi [16], and Kocourekand, Takahashi and Yao [13].

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping from C into itself. Define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for all $n \in \mathbb{N}$, where $0 \le \alpha_n < 1$ and $\alpha_n \to 0$. If $F(T) \neq \emptyset$, then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$. In particular, for any $x \in C$, define

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x.$$

Then, $\{S_n x\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} PT^n x$.

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. Since a generalized hybrid mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive, we have that for all $u \in F(T)$,

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - u\| \\ &\le \alpha_n\|x_n - u\| + (1-\alpha_n)\|Tx_n - u\| \\ &\le \|x_n - u\|.\end{aligned} \tag{3.2}$$

So, $\lim_{n\to\infty}\|x_n - u\|$ exists. Then, $\{x_n\}$ and $\{Tx_n\}$ are bounded. So $\{z_n\}$ is bounded. Let $\{z_{n_i}\}$ be a weakly convergent subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup v$. Then, we can show $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N}$, we have that

$$\begin{aligned}0 &\le \beta\|Tx_k - y\|^2 + (1-\beta)\|x_k - y\|^2 \\ &\quad - \alpha\|Tx_k - Ty\|^2 - (1-\alpha)\|x_k - Ty\|^2 \\ &= \beta\big\{\|Tx_k - Ty\|^2 + 2\langle Tx_k - Ty, Ty - y\rangle + \|Ty - y\|^2\big\} \\ &\quad + (1-\beta)\big\{\|x_k - Ty\|^2 + 2\langle x_k - Ty, Ty - y\rangle + \|Ty - y\|^2\big\} \\ &\quad - \alpha\|Tx_k - Ty\|^2 - (1-\alpha)\|x_k - Ty\|^2 \\ &= \|Ty - y\|^2 + 2\langle \beta Tx_k + (1-\beta)x_k - Ty, Ty - y\rangle \\ &\quad + (\beta - \alpha)\big\{\|Tx_k - Ty\|^2 - \|x_k - Ty\|^2\big\} \\ &= \|Ty - y\|^2 + 2\langle x_k - Ty + \beta(Tx_k - x_k), Ty - y\rangle \\ &\quad + (\beta - \alpha)\big\{\|Tx_k - Ty\|^2 - \|x_k - Ty\|^2\big\}.\end{aligned}$$

Since $Tx_k = x_{k+1} + \alpha_k(Tx_k - x_k)$ and $(1-\alpha_k)(Tx_k - x_k) = x_{k+1} - x_k$, we have that

$$\begin{aligned}0 &\le \|Ty - y\|^2 + 2\langle x_k - Ty, Ty - y\rangle + 2\beta(1-\alpha_k)^{-1}\langle x_{k+1} - x_k, Ty - y\rangle \\ &\quad + (\beta - \alpha)\big\{\|x_{k+1} - Ty + \alpha_k(Tx_k - x_k)\|^2 - \|x_k - Ty\|^2\big\} \\ &= \|Ty - y\|^2 + 2\langle x_k - Ty, Ty - y\rangle + 2\beta(1-\alpha_k)^{-1}\langle x_{k+1} - x_k, Ty - y\rangle \\ &\quad + (\beta - \alpha)\big\{\|x_{k+1} - Ty\|^2 + 2\alpha_k\langle x_{k+1} - Ty, Tx_k - x_k\rangle \\ &\quad + \|\alpha_k(Tx_k - x_k)\|^2 - \|x_k - Ty\|^2\big\} \\ &= \|Ty - y\|^2 + 2\langle x_k - Ty, Ty - y\rangle + 2\beta(1-\alpha_k)^{-1}\langle x_{k+1} - x_k, Ty - y\rangle \\ &\quad + (\beta - \alpha)\big\{\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2 + 2\alpha_k\langle x_{k+1} - Ty, Tx_k - x_k\rangle \\ &\quad + \|\alpha_k(Tx_k - x_k)\|^2\big\}.\end{aligned}$$

From $1 - \alpha_k > 0$, we also have

$$0 \le (1-\alpha_k)\|Ty - y\|^2 + 2(1-\alpha_k)\langle x_k - Ty, Ty - y\rangle + 2\beta\langle x_{k+1} - x_k, Ty - y\rangle$$

$$
\begin{aligned}
&+ (\beta - \alpha)(1 - \alpha_k)\big\{\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2 + 2\alpha_k \langle x_{k+1} - Ty, Tx_k - x_k\rangle \\
&+ \|\alpha_k(Tx_k - x_k)\|^2\big\} \\
= &\|Ty - y\|^2 + 2\langle x_k - Ty, Ty - y\rangle + 2\beta\langle x_{k+1} - x_k, Ty - y\rangle \\
&+ (\beta - \alpha)\big\{\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2\big\} - \alpha_k\|Ty - y\|^2 \\
&- 2\alpha_k\langle x_k - Ty, Ty - y\rangle - (\beta - \alpha)\alpha_k\big\{\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2\big\} \\
&+ (\beta - \alpha)(1 - \alpha_k)\big\{2\alpha_k\langle x_{k+1} - Ty, Tx_k - x_k\rangle + \alpha_k^2\|Tx_k - x_k\|^2\big\} \\
\leq &\|Ty - y\|^2 + 2\langle x_k - Ty, Ty - y\rangle \\
&+ 2\beta\langle x_{k+1} - x_k, Ty - y\rangle + (\beta - \alpha)\big\{\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2\big\} \\
&+ \alpha_k\big\{-2\langle x_k - Ty, Ty - y\rangle - (\beta - \alpha)(\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2)\big\} \\
&+ \alpha_k|\beta - \alpha||2\langle x_{k+1} - Ty, Tx_k - x_k\rangle + \alpha_k\|Tx_k - x_k\|^2| \\
\leq &\|Ty - y\|^2 + 2\langle x_k - Ty, Ty - y\rangle + 2\beta\langle x_{k+1} - x_k, Ty - y\rangle \\
&+ (\beta - \alpha)\big\{\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2\big\} + \alpha_k M + \alpha_k|\beta - \alpha|K,
\end{aligned}
$$

where

$$M = \sup_{k\in\mathbb{N}}\{-2\langle x_k - Ty, Ty - y\rangle - (\beta - \alpha)(\|x_{k+1} - Ty\|^2 - \|x_k - Ty\|^2)\}$$

and

$$K = \sup_{k\in\mathbb{N}}\{|2\langle x_{k+1} - Ty, Tx_k - x_k\rangle + \alpha_k\|Tx_k - x_k\|^2|\}.$$

Summing up these inequalities with respect to $k = 1, 2, \ldots, n$,

$$
\begin{aligned}
0 \leq n\|Ty - y\|^2 &+ 2\left\langle \sum_{k=1}^{n} x_k - nTy, Ty - y\right\rangle + 2\beta\langle x_{n+1} - x_1, Ty - y\rangle \\
&+ (\beta - \alpha)\big\{\|x_{n+1} - Ty\|^2 - \|x_1 - Ty\|^2\big\} + \sum_{k=1}^{n}\alpha_k M + \sum_{k=1}^{n}\alpha_k|\beta - \alpha|K.
\end{aligned}
$$

Deviding this inequality by n, we have

$$
\begin{aligned}
0 \leq \|Ty - y\|^2 &+ 2\langle z_n - Ty, Ty - y\rangle + \frac{1}{n}2\beta\langle x_{n+1} - x_1, Ty - y\rangle \\
&+ \frac{1}{n}(\beta - \alpha)\big\{\|x_{n+1} - Ty\|^2 - \|x_1 - Ty\|^2\big\} + \frac{1}{n}\sum_{k=1}^{n}\alpha_k M + \frac{1}{n}\sum_{k=1}^{n}\alpha_k|\beta - \alpha|K,
\end{aligned}
$$

where $z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$. Replacing n by n_i and letting $n_i \to \infty$, we obtain from $z_{n_i} \rightharpoonup v$ and $\alpha_n \to 0$ that

$$0 \leq \|Ty - y\|^2 + 2\langle v - Ty, Ty - y\rangle.$$

Putting $y = v$, we have $0 \leq -\|Tv - v\|^2$ and hence $Tv = v$. To show that $\{z_n\}$ converges weakly to a fixed point of T, we first show that $\lim_{n\to\infty} Px_n$ exists. Since $F(T) \neq \emptyset$, from (3.2) we have that for all $u \in F(T)$,

$$\|x_{n+1} - u\| \leq \|x_n - u\|.$$

On the other hand, since T is quasi-nonexpansive, $F(T)$ is closed and convex. So, we can define the metric projection P of H onto $F(T)$. Putting $D = F(T)$ in Lemma 2.2, we have that $\lim_{n\to\infty} Px_n$ converges strongly. Put $z = \lim_{n\to\infty} Px_n$. Then we can prove $z_n \rightharpoonup z$. In fact, let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup w$. From the above argument, we have $w \in F(T)$. To complete the proof of the first part, it is sufficient to prove $z = w$. From $w \in F(T)$ and (2.2), we have

$$\begin{aligned}\langle w - z, x_k - Px_k\rangle &= \langle w - Px_k, x_k - Px_k\rangle + \langle Px_k - z, x_k - Px_k\rangle \\ &\le \langle Px_k - z, x_k - Px_k\rangle \\ &\le \|Px_k - z\|\|x_k - Px_k\| \\ &\le \|Px_k - z\|L\end{aligned}$$

for all $k \in \mathbb{N}$, where $L = \sup\{\|x_k - Px_k\| : k \in \mathbb{N}\}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i, we have

$$\left\langle w - z, z_{n_i} - \frac{1}{n_i}\sum_{k=1}^{n_i} Px_k \right\rangle \le \frac{1}{n_i}\sum_{k=1}^{n_i} \|Px_k - z\|L.$$

Since $z_{n_i} \rightharpoonup w$ as $i \to \infty$ and $Px_n \to z$ as $n \to \infty$, we have $\langle w - z, w - z\rangle \le 0$. This implies $z = w$. This completes the proof of the first part. In particular, putting $\alpha_n = 0$ for all $n \in \mathbb{N}$, we see that $x_{n+1} = T^n x$ and $z_n = 1/n\sum_{k=0}^{n-1} T^k x$ for all $n \in \mathbb{N}$, where $T^0 = I$. So, we obtain $S_n x = z_n$. Therefore, $\{S_n x\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} PT^n x$. So, we get the desired result. ☐

Using Theorem 3.1, we obtain the following results proved by Akatsuka, Aoyama and Takahashi [1] and Kurokawa and Takahashi [16].

Theorem 3.2 (Akatsuka, Aoyama and Takahashi [1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping from C into itself. Define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for all $n \in \mathbb{N}$, where $0 \le \alpha_n < 1$ and $\alpha_n \to 0$. If $F(T) \neq \emptyset$, then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$. In particular, for any $x \in C$, define

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x.$$

Then $\{S_n x\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} PT^n x$.

Proof. Since an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, we obtain the desired result from Theorem 3.1. ☐

Theorem 3.3 (Kurokawa and Takahashi [16]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonspreading mapping from C into itself. Define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for all $n \in \mathbb{N}$, *where* $0 \leq \alpha_n < 1$ *and* $\alpha_n \to 0$. *If* $F(T) \neq \emptyset$, *then* $\{z_n\}$ *converges weakly to* $z \in F(T)$, *where* $z = \lim_{n\to\infty} Px_n$ *and* P *is the metric projection of* H *onto* $F(T)$. *In particular, for any* $x \in C$, *define*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x.$$

Then $\{S_n x\}$ *converges weakly to* $z \in F(T)$, *where* $z = \lim_{n\to\infty} PT^n x$.

Proof. Since an (α, β)-generalized hybrid mapping is nonspreading for $\alpha = 2$ and $\beta = 1$, we obtain the desired result from Theorem 3.1. □

4 Strong Convergence Theorem In this section, using an idea of mean convergence by Shimizu and Takahashi [19] and [20], we prove the following strong convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* T *be a generalized hybrid mapping of* C *into itself. Let* $u \in C$ *and define two sequences* $\{x_n\}$ *and* $\{z_n\}$ *in* C *as follows:* $x_1 = x \in C$ *and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \ldots$, *where* $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $F(T)$ *is nonempty, then* $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* Pu, *where* P *is the metric projection of* H *onto* $F(T)$.

Proof. We follow Kurokawa and Takahashi [16] for the proof. Since $T : C \to C$ be a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{4.1}$$

for all $x, y \in C$. Since $F(T) \neq \emptyset$, T is quasi-nonexpansive. So, we have that for all $q \in F(T)$ and $n = 1, 2, 3, \ldots$,

$$\begin{aligned} \|z_n - q\| &= \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n - q \right\| \leq \frac{1}{n}\sum_{k=0}^{n-1} \|T^k x_n - q\| \\ &\leq \frac{1}{n}\sum_{k=0}^{n-1} \|x_n - q\| = \|x_n - q\|. \end{aligned} \tag{4.2}$$

Then we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n u + (1-\alpha_n) z_n - q\| \\ &\leq \alpha_n\|u - q\| + (1-\alpha_n)\|z_n - q\| \\ &\leq \alpha_n\|u - q\| + (1-\alpha_n)\|x_n - q\|. \end{aligned}$$

Hence, by induction, we obtain

$$\|x_n - q\| \leq \max\{\|u - q\|, \|x - q\|\}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{z_n\}$ are bounded. Since $\|T^n x_n - q\| \le \|x_n - q\|$, we have also that $\{T^n x_n\}$ is bounded. Let $n \in \mathbb{N}$. Since T is generalized hybrid, we have that for all $y \in C$ and $k = 0, 1, 2, ..., n-1$,

$$\begin{aligned} 0 &\le \beta\|T^{k+1}x_n - y\|^2 + (1-\beta)\|T^k x_n - y\|^2 \\ &\quad - \alpha\|T^{k+1}x_n - Ty\|^2 - (1-\alpha)\|T^k x_n - Ty\|^2 \\ &= \beta\big\{\|T^{k+1}x_n - Ty\|^2 + 2\left\langle T^{k+1}x_n - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\ &\quad + (1-\beta)\big\{\|T^k x_n - Ty\|^2 + 2\left\langle T^k x_n - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\ &\quad - \alpha\|T^{k+1}x_n - Ty\|^2 - (1-\alpha)\|T^k x_n - Ty\|^2 \\ &= \|Ty - y\|^2 + 2\left\langle \beta T^{k+1}x_n + (1-\beta)T^k x_n - Ty, Ty - y\right\rangle \\ &\quad + (\beta - \alpha)\big\{\|T^{k+1}x_n - Ty\|^2 - \|T^k x_n - Ty\|^2\big\} \\ &= \|Ty - y\|^2 + 2\left\langle T^k x_n - Ty + \beta(T^{k+1}x_n - T^k x_n), Ty - y\right\rangle \\ &\quad + (\beta - \alpha)\big\{\|T^{k+1}x_n - Ty\|^2 - \|T^k x_n - Ty\|^2\big\}. \end{aligned}$$

Summing these inequalities from $k = 0$ to $n-1$ and dividing by n, we have

$$\begin{aligned} 0 \le &\|Ty - y\|^2 + 2\langle z_n - Ty, Ty - y\rangle + 2\beta\frac{1}{n}\left\langle T^n x_n - x_n, Ty - y\right\rangle \\ &+ (\beta - \alpha)\frac{1}{n}\big\{\|T^n x_n - Ty\|^2 - \|x_n - Ty\|^2\big\}. \end{aligned}$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup w \in C$. Replacing n by n_i, we have

$$\begin{aligned} 0 \le &\|Ty - y\|^2 + 2\langle z_{n_i} - Ty, Ty - y\rangle + 2\beta\frac{1}{n_i}\left\langle T^{n_i} x_{n_i} - x_{n_i}, Ty - y\right\rangle \\ &+ (\beta - \alpha)\frac{1}{n_i}\big\{\|T^{n_i} x_{n_i} - Ty\|^2 - \|x_{n_i} - Ty\|^2\big\}. \end{aligned}$$

Since $\{x_n\}$ and $\{T^n x_n\}$ are bounded, we have that

$$0 \le \|Ty - y\|^2 + 2\langle w - Ty, Ty - y\rangle$$

as $i \to \infty$. Putting $y = w$, we have

$$0 \le \|Tw - w\|^2 + 2\langle w - Tw, Tw - w\rangle = -\|Tw - w\|^2.$$

Hence, $w \in F(T)$. On the other hand, since $x_{n+1} - z_n = \alpha_n(u - z_n)$, $\{z_n\}$ is bounded and $\alpha_n \to 0$, we have $\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0$. Let us show $\limsup_{n\to\infty}\langle u - Pu, x_{n+1} - Pu\rangle \le 0$. We may assume without loss of generality that there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\langle u - Pu, x_{n+1} - Pu\rangle = \lim_{i\to\infty}\langle u - Pu, x_{n_i+1} - Pu\rangle$$

and $x_{n_i+1} \rightharpoonup v$. From $\|x_{n+1} - z_n\| \to 0$, we have $z_{n_i} \rightharpoonup v$. From the above argument, we have $v \in F(T)$. Since P is the metric projection of H onto $F(T)$, we have

$$\lim_{i\to\infty}\langle u - Pu, x_{n_i+1} - Pu\rangle = \langle u - Pu, v - Pu\rangle \le 0.$$

This implies

$$\limsup_{n\to\infty}\langle u - Pu, x_{n+1} - Pu\rangle \le 0. \tag{4.3}$$

Since $x_{n+1} - Pu = (1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)$, from (2.1) and (4.2) we have

$$\begin{aligned}\|x_{n+1} - Pu\|^2 &= \|(1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2 \\ &\le (1-\alpha_n)^2\|z_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &\le (1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle.\end{aligned}$$

Putting $s_n = \|x_n - Pu\|^2$, $\beta_n = 0$ and $\gamma_n = 2\langle u - Pu, x_{n+1} - Pu\rangle$ in Lemma 2.1, from $\sum_{n=1}^{\infty}\alpha_n = \infty$ and (4.3) we have

$$\lim_{n\to\infty}\|x_n - Pu\| = 0.$$

By $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we also obtain $z_n \to Pu$ as $n \to \infty$. □

Using Theorem 4.1, we can show the following result obtained by Kurokawa and Takahashi [16].

Theorem 4.2 (Kurokawa and Takahashi [16]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonspreading mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \ldots$, where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

Proof. Since an (α, β)-generalized hybrid mapping is nonspreading for $\alpha = 2$ and $\beta = 1$, we obtain the desired result from Theorem 4.1. □

Remark. We do not know whether a strong convergence theorem of Halpern's type for generalized hybrid mappings holds or not.

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