BALANCED FRACTIONAL 3^m FACTORIAL DESIGNS OF RESOLUTIONS $\mathbf{R}(\{00, 10, 01\} \cup S_1|\Omega)$

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Abstract. This paper presents three kinds of balanced fractional 3^m factorial designs such that the general mean and all the main effects are estimable, and furthermore (A) the linear by linear components of the two-factor interaction are estimable, and the factorial effects of the quadratic by quadratic and linear by quadratic ones of the two-factor interaction are confounded with each other, (B) the quadratic by quadratic ones of the two-factor interaction are estimable, and the effects of the linear by linear and linear by quadratic ones of the two-factor interaction are confounded with each other, and (C) the linear by quadratic ones of the two-factor interaction are estimable, and the effects of the linear by linear and quadratic by quadratic ones of the two-factor interaction are confounded with each other, where the three-factor and higher-order interactions are assumed to be negligible and the number of assemblies is less than the number of non-negligible factorial effects. These designs are concretely given by the indices of a balanced array of full strength, which is called a simple array.

1 Introduction As a generalization of an orthogonal array, the concept of a balanced array (BA) was first introduced by Chakravarti [1] as a partially BA. However it is a generalization of a BIB design and not of a PBIB design, and hence Srivastava and Chopra [9] called it a BA. A design is said to be balanced if the variance-covariance matrix of the estimators of the factorial effects to be of interest is invariant under any permutation on the factors. The relation between a BA of strength four, size N, m constraints, three symbols and index set $\{\mu_{j_0j_1j_2}|j_0 + j_1 + j_2 = 4\}$, which is denoted by $\mathrm{BA}(N, m, 3, 4; \{\mu_{j_0j_1j_2}\})$ for brevity, and a balanced fractional 3^m factorial (3^m-BFF) design of resolution V was presented by Kuwada [5]. Furthermore the same author [6] obtained the explicit expression for the characteristic polynomial of the information matrix of a 3^m-BFF design of resolution V derived from a $\mathrm{BA}(N, m, 3, 4; \{\mu_{j_0j_1j_2}\})$ using the algebraic structure of the multidimensional relationship (MDR). In the design theory, the concept of a relationship was first introduced by James [3]. By use of a different approach, the inversion of the information matrix of a 3^m-BFF design of resolution V was presented by Srivastava and Ariyaratna [8]. As a special case of a 3^m-BFF design of resolution V, the expression for the trace of the variance-covariance matrix of the estimators of non-negligible factorial effects based on a balanced (2,0)-symmetric design was presented Srivastava and Chopra [10]. Some 3^m-BFF designs of resolution IV were obtained by Kuwada and Ikeda [7] using the properties of the MDR algebra and a generalized inverse of a matrix. However their results are given by the matrix formulas and they are very complex.

A BA of strength m and indices $\lambda_{i_0i_1i_2}$ $(i_0 + i_1 + i_2 = m)$ is called a simple array (SA) and it is briefly denoted by $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$. Let S_2 be one of the sets $\{20, 02\}$, $\{02, 11\}$ and $\{02, 11\}$. Then under the assumption that the three-factor and higher-order interactions are negligible and the number of assemblies (or treatment combinations), N, say, is less

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than the number of non-negligible factorial effects ($=\nu(m)$, say), Taniguchi *et al.* [11] has given 3^m-BFF designs derived from SA$(m;\{\lambda_{i_0i_1i_2}\})$'s such that $\boldsymbol{\theta}_{00}$, $\boldsymbol{\theta}_{10}$, $\boldsymbol{\theta}_{01}$ and $\boldsymbol{\theta}_{a_1a_2}$ are estimable for $a_1a_2 \in S_2$ and the factorial effects of $\boldsymbol{\theta}_{b_1b_2}$ are confounded with themselves for $b_1b_2 \in \{20, 02, 11\} \setminus S_2$, whose designs are said to be of resolutions R$(\{00, 10, 01\} \cup S_2|\Omega)$, where $\boldsymbol{\theta}_{00}$ is the general mean, $\boldsymbol{\theta}_{10}$ and $\boldsymbol{\theta}_{01}$ are the vectors of the linear and quadratic components of the main effect, respectively, $\boldsymbol{\theta}_{20}$, $\boldsymbol{\theta}_{02}$ and $\boldsymbol{\theta}_{11}$ are the vectors of the linear by linear, quadratic by quadratic and linear by quadratic ones of the two-factor interaction, respectively, $\Omega = \{00, 10, 01, 20, 02, 11\}$, and $\nu(m) = 1 + 2m^2$.

In this paper, we present 3^m-BFF designs derived from SA$(m;\{\lambda_{i_0i_1i_2}\})$'s such that $\boldsymbol{\theta}_{00}$, $\boldsymbol{\theta}_{10}$, $\boldsymbol{\theta}_{01}$ and $\boldsymbol{\theta}_{c_1c_2}$ are estimable for $c_1c_2 \in S_1$ and the factorial effects of $\boldsymbol{\theta}_{d_1d_2}$ are confounded with each other for $d_1d_2 \in \{20, 02, 11\} \setminus S_1$, whose designs are said to be of resolutions R$(\{00, 10, 01\} \cup S_1|\Omega)$, where $S_1 = \{20\}$, $\{02\}$ and $\{11\}$, the three-factor and higher-order interactions are assumed to be negligible and $N < \nu(m)$. These designs are concretely given by the indices $\lambda_{i_0i_1i_2}$ of an SA. Resolutions R$(\{00, 10, 01\} \cup S_2|\Omega)$ designs given above and resolutions R$(\{00, 10, 01\} \cup S_1|\Omega)$ designs considered here are a part of resolution IV designs. In an even resolution design, $\boldsymbol{\theta}_{00}$ may or may not be estimable. Thus in separate papers, we shall present another resolution IV designs derived from SA$(m;\{\lambda_{i_0i_1i_2}\})$'s such that (I) $\boldsymbol{\theta}_{00}$, $\boldsymbol{\theta}_{10}$ and $\boldsymbol{\theta}_{01}$ are estimable, and the factorial effects of $\boldsymbol{\theta}_{20}$, $\boldsymbol{\theta}_{02}$ and $\boldsymbol{\theta}_{11}$ are confounded with each other, whose designs are said to be of resolution R$(\{00, 10, 01\}|\Omega)$, and (II) $\boldsymbol{\theta}_{10}$ and $\boldsymbol{\theta}_{01}$ are estimable, and $\boldsymbol{\theta}_{00}$ is confounded with some two-factor interactions, whose designs are said to be of resolutions R$(\{10, 01\} \cup S|\Omega)$, where $S = S_2$, S_1 and $\{\phi\}$. In all our evaluations, we code the three levels of a factor as 0, 1 or 2, and employ the standard orthogonal contrasts used in the 3^m case: viz., $-1, 0, 1$ and $1, -2, 1$ for the linear and quadratic contrasts, respectively.

2 Preliminaries Consider a fractional 3^m factorial design T with N assemblies, where $m \geq 4$, and the three-factor and higher-order interactions are assumed to be negligible. Then the vector of non-negligible factorial effects is given by $\boldsymbol{\Theta} = (\boldsymbol{\theta}_{00}'; \boldsymbol{\theta}_{10}'; \boldsymbol{\theta}_{01}'; \boldsymbol{\theta}_{20}'; \boldsymbol{\theta}_{02}'; \boldsymbol{\theta}_{11}')'$, where A' denotes the transpose of a matrix A. Hence the linear model is given by $\boldsymbol{y}(T) = E_T\boldsymbol{\Theta} + \boldsymbol{e}_T$, where $\boldsymbol{y}(T)$, E_T and $\boldsymbol{e}_T$ are, respectively, an $N \times 1$ observation vector based on T, the $N \times \nu(m)$ design matrix and an $N \times 1$ error vector with mean $\boldsymbol{0}_N$ and variance-covariance matrix $\sigma^2 I_N$. The normal equations for estimating $\boldsymbol{\Theta}$ are given by

$$M_T\hat{\boldsymbol{\Theta}} = E_T'\boldsymbol{y}(T), \tag{2.1}$$

where M_T $(= E_T'E_T)$ is the information matrix of order $\nu(m)$.

Let T be a design derived from a BA$(N, m, 3, 4; \{\mu_{j_0j_1j_2}\})$. Then from the properties of the MDR algebra (see [6]), the M_T is given by

$$\begin{aligned} M_T = & \textstyle\sum_{a_1a_2} \sum_{b_1b_2} \sum_{\gamma} \kappa_{\gamma}^{a_1a_2,b_1b_2} D_{\gamma}^{\#(a_1a_2,b_1b_2)} \\ & + \textstyle\sum_{u_1u_2;i} \sum_{v_1v_2;j} \kappa_{f_{ij}}^{u_1u_2,v_1v_2} D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}, \end{aligned} \tag{2.2}$$

where the relations between $\kappa_{\gamma}^{a_1a_2,b_1b_2}$ $(\gamma = 0, 1, 2)$ (or $\kappa_{f_{ij}}^{u_1u_2,v_1v_2}$) and $\mu_{j_0j_1j_2}$ are given in the Appendix of Yamamoto *et al.* [12]. Here the matrices $D_{\gamma}^{\#(a_1a_2,b_1b_2)}$ and $D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ of order $\nu(m)$ are given by some linear combinations of the relationship matrices $D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ and $D_{\boldsymbol{\alpha}}^{(u_1u_2,v_1v_2)}$ (see [6]), respectively. Thus the M_T is isomorphic to the symmetric matrices $||\kappa_{\gamma}^{a_1a_2,b_1b_2}||$ $(= K_\gamma$, say) for $\gamma = 0, 1, 2$ and $||\kappa_{f_{ij}}^{u_1u_2,v_1v_2}||$ $(= K_f$, say) (see [6]), i.e., there exists an orthogonal matrix Q of order $\nu(m)$ such that $Q'M_TQ = \text{diag}[K_0; K_1, \ldots, K_1; K_2, \ldots, K_2; K_f, \ldots, K_f]$, where the multiplicities of K_β are ϕ_β for $\beta = 0, 1, 2, f$. Here $\phi_0 = 1$,

$\phi_1 = m(m-3)/2$, $\phi_2 = \binom{m-1}{2}$ and $\phi_f = m-1$, where $\binom{p}{q}$ is the binomial coefficient, and $\binom{p}{q} = 0$ if and only if $q < 0$ or $p < q$. Note that the K_β are called the irreducible representations of M_T and the order of K_0, K_1, K_2 and K_f are 6, 3, 1 and 6, respectively.

The a_1a_2-th row block and b_1b_2-th column one of $D_\gamma^{\#(a_1a_2,b_1b_2)}$ are concerned with $A_\gamma^{\#(a_1a_2,a_1a_2)}\boldsymbol{\theta}_{a_1a_2}$ and $A_\gamma^{\#(b_1b_2,b_1b_2)}\boldsymbol{\theta}_{b_1b_2}$, respectively, where (i) if $\gamma = 0$, then $a_1a_2,\ b_1b_2 = 00, 10, 01, 20, 02, 11$, (ii) if $\gamma = 1$, then $a_1a_2, b_1b_2 = 20, 02, 11$, and (iii) if $\gamma = 2$, then $a_1a_2, b_1b_2 = 11$, and the u_1u_2-th row block and v_1v_2-th column one of $D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ are also concerned with $A_{f_{ii}}^{\#(u_1u_2,u_1u_2)}\boldsymbol{\theta}_{u_1u_2}$ and $A_{f_{jj}}^{\#(v_1v_2,v_1v_2)}\boldsymbol{\theta}_{v_1v_2}$, respectively, where $(u_1u_2; i)$, $(v_1v_2; j) = (10; 1), (01; 1), (20; 2), (02; 2), (11; 3), (11; 4)$. Here the matrices $A_\gamma^{\#(a_1a_2,b_1b_2)}$ $(= A_\gamma^{\#(b_1b_2,a_1a_2)'})$ and $A_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ $(= A_{f_{ji}}^{\#(v_1v_2,u_1u_2)'})$ of size $n_{a_1a_2} \times n_{b_1b_2}$ and $n_{u_1u_2} \times n_{v_1v_2}$ are given by some linear combinations of the local relationship matrices $A_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ and $A_{\boldsymbol{\alpha}}^{(u_1u_2,v_1v_2)}$ (see [6]), respectively, where $n_{a_1a_2} = \binom{m}{a_1}\binom{m-a_1}{a_2}$.

3 Decomposition of $\boldsymbol{K_\beta}$ An SA$(m; \{\lambda_{i_0i_1i_2}\})$ always exists for any indices $\lambda_{i_0i_1i_2}$ and any m, but a BA$(N, m, 3, 4; \{\mu_{j_0j_1j_2}\})$ does not always exist for given $\mu_{j_0j_1j_2}$ and $m \geq 5$. Furthermore if $N \geq \nu(m)$, then there exists a 3^m-BFF design of resolution R$(\Omega|\Omega)$, i.e., of resolution V, (e.g., [4]). Thus throughout this paper, we only consider a design derived from an SA$(m; \{\lambda_{i_0i_1i_2}\})$ with $N < \nu(m)$. Here the relations between the indices $\mu_{j_0j_1j_2}$ of a BA of strength four and $\lambda_{i_0i_1i_2}$ of an SA are given by

$$(3.1)\qquad \mu_{j_0j_1j_2} = \textstyle\sum_{p_0+p_1+p_2=m-4}\{(m-4)!/(p_0!p_1!p_2!)\}\lambda_{j_0+p_0\,j_1+p_1\,j_2+p_2},$$

and $N = \sum_{i_0+i_1+i_2=m}\{m!/(i_0!i_1!i_2!)\}\lambda_{i_0i_1i_2}$. Note that if T is an SA$(m; \{\lambda_{i_0i_1i_2}\})$, where $m \geq 4$, then it is the BA$(N, m, 3, 4; \{\mu_{j_0j_1j_2}\})$, but the converse is not always true for $m \geq 5$. Since $N < \nu(m)$, the information matrix M_T is singular, and hence at least one of K_β $(\beta = 0, 1, 2, f)$ is singular. Thus it holds that $\sum_\beta[\text{rank}\{K_\beta\}]\phi_\beta \leq N < \nu(m)$.

A necessary and sufficient condition for a parametric function $C\boldsymbol{\Theta}$ of $\boldsymbol{\Theta}$ to be estimable for some matrix C of order $\nu(m)$ is that there exists a matrix X of order $\nu(m)$ such that $XM_T = C$ (e.g., [13]). If $C\boldsymbol{\Theta}$ is estimable, then its BLUE is given by $C\hat{\boldsymbol{\Theta}}$, where $\hat{\boldsymbol{\Theta}}$ is a solution of the Eqs. (2.1), and its variance-covariance matrix is given by $\sigma^2 XM_TX'$.

Let T be an SA$(m; \{\lambda_{i_0i_1i_2}\})$. Then the M_T is given by some linear combinations of the matrices $D_\gamma^{\#(a_1a_2,b_1b_2)}$ and $D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ as in (2.2). Thus we impose some restrictions on C such that it is given by some linear combinations of these matrices, and hence we define C as follows:

$$\begin{aligned} C = D_0^{\#(00,00)} &+ \{D_0^{\#(10,10)} + D_{f_{11}}^{\#(10,10)}\} + \{D_0^{\#(01,01)} + D_{f_{11}}^{\#(01,01)}\} \\ &+ \textstyle\sum^*_{a_1a_2}\sum^*_{b_1b_2}\sum_\gamma g_\gamma^{a_1a_2,b_1b_2}D_\gamma^{\#(a_1a_2,b_1b_2)} + \sum^{**}_{u_1u_2;i}\sum^{**}_{v_1v_2;j} g_{f_{ij}}^{u_1u_2,v_1v_2}D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}, \end{aligned}$$

where $\sum^*_{a_1a_2}$ and $\sum^{**}_{u_1u_2;i}$ are the summations over all the values of a_1a_2 and $(u_1u_2; i)$ such that (i) if $\gamma = 0, 1$, then $a_1a_2 = 20, 02, 11$ and (ii) if $\gamma = 2$, then $a_1a_2 = 11$, and $(u_1u_2; i) =$ (20; 2),(02; 2),(11; 3),(11; 4), respectively, and $g_\gamma^{a_1a_2,b_1b_2}$ $(\gamma = 0, 1, 2)$ and $g_{f_{ij}}^{u_1u_2,v_1v_2}$ are some constants. Similarly we define X as follows:

$$X = \textstyle\sum_{a_1a_2}\sum_{b_1b_2}\sum_\gamma \chi_\gamma^{a_1a_2,b_1b_2}D_\gamma^{\#(a_1a_2,b_1b_2)} + \sum_{u_1u_2;i}\sum_{v_1v_2;j}\chi_{f_{ij}}^{u_1u_2,v_1v_2}D_{f_{ij}}^{\#(u_1u_2,v_1v_2)},$$

where $\chi_\gamma^{a_1a_2,b_1b_2}$ and $\chi_{f_{ij}}^{u_1u_2,v_1v_2}$ are also some constants which depend on $\kappa_\gamma^{a_1a_2,b_1b_2}$ and $g_\gamma^{a_1a_2,b_1b_2}$, and $\kappa_{f_{ij}}^{u_1u_2,v_1v_2}$ and $g_{f_{ij}}^{u_1u_2,v_1v_2}$, respectively. Then C and X are isomorphic to $\varGamma_\beta$

and χ_β $(\beta = 0, 1, 2, f)$, respectively, where

$$(3.2)\quad \begin{aligned} &\mathit{\Gamma}_0 = \mathrm{diag}[I_3; \begin{pmatrix} g_0^{20,20} & g_0^{20,02} & g_0^{20,11} \\ g_0^{02,20} & g_0^{02,02} & g_0^{02,11} \\ g_0^{11,20} & g_0^{11,02} & g_0^{11,11} \end{pmatrix}],\ \mathit{\Gamma}_1 = \begin{pmatrix} g_1^{20,20} & g_1^{20,02} & g_1^{20,11} \\ g_1^{02,20} & g_1^{02,02} & g_1^{02,11} \\ g_1^{11,20} & g_1^{11,02} & g_1^{11,11} \end{pmatrix}, \\ &\mathit{\Gamma}_2 = g_2^{11,11},\ \mathit{\Gamma}_f = \mathrm{diag}[I_2; \begin{pmatrix} g_{f_{22}}^{20,20} & g_{f_{22}}^{20,02} & g_{f_{23}}^{20,11} & g_{f_{24}}^{20,11} \\ g_{f_{22}}^{02,20} & g_{f_{22}}^{02,02} & g_{f_{23}}^{02,11} & g_{f_{24}}^{02,11} \\ g_{f_{32}}^{11,20} & g_{f_{32}}^{11,02} & g_{f_{33}}^{11,11} & g_{f_{34}}^{11,11} \\ g_{f_{42}}^{11,20} & g_{f_{42}}^{11,02} & g_{f_{43}}^{11,11} & g_{f_{44}}^{11,11} \end{pmatrix}], \\ &\chi_\gamma = ||\chi_\gamma^{a_1a_2,b_1b_2}|| \text{ and } \chi_f = ||\chi_{f_{ij}}^{u_1u_2,v_1v_2}||. \end{aligned}$$

Thus $XM_T = C$ is also isomorphic to $\chi_\beta K_\beta = \mathit{\Gamma}_\beta$.

By use of the methods similar to the proof of Theorem 3.4 due to Taniguchi *et al.* [11], the following can be easily proved:

Theorem 3.1. *If there exists a* 3^m-BFF *design of resolutions* $\mathrm{R}(\{00, 10, 01\} \cup S_1|\Omega)$ *derived from an* $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ *with* $N < \nu(m)$*, where* $m \geq 4$*, and* $S_1 = \{20\}, \{02\}$ *and* $\{11\}$*, then it holds that* $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (pm-p0)$ $(1 \leq p \leq m), (0qm-q)$ $(1 \leq q \leq m)$, $(m-r0r)$ $(1 \leq r \leq m), (11m-2), (m-211), (1m-21)$.

It follows from Theorem 3.1 that in the rest of this paper, we consider a design derived from an $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ with $N < \nu(m)$, where the indices $\lambda_{i_0i_1i_2}$ satisfy the conditions of Theorem 3.1. Let F_γ $(\gamma = 0, 1, 2)$ and F_f be some matrices whose rows and columns are concerned with $A_\gamma^{\#(a_1a_2,a_1a_2)}\boldsymbol{\theta}_{a_1a_2}$ and $\lambda_{i_0i_1i_2}$, and $A_{f_{ii}}^{\#(u_1u_2,u_1u_2)}\boldsymbol{\theta}_{u_1u_2}$ and $\lambda_{i_0i_1i_2}$, respectively, and further let $\mathit{\Lambda}_\beta$ $(\beta = 0, 1, 2, f)$ be some diagonal matrices. Here the 6×1 column vectors of F_0 of size $6\times3(m+1)$ concerned with the indices λ_{pm-p0}, λ_{0qm-q}, λ_{m-r0r}, λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$(3.3)\quad \begin{aligned} &\sqrt{\lambda_{pm-p0}}(1\ \ -p\ \ -(2m-3p)\ \ p(p-1)\ \ (2m-3p)^2-(4m-3p)\ \ p(2m-3p+1))', \\ &\sqrt{\lambda_{0qm-q}}(1\ \ m-q\ \ m-3q\ \ (m-q)(m-q-1)\ \ (m-3q)^2-(m+3q)\ \ (m-q)(m-3q-1))', \\ &\sqrt{\lambda_{m-r0r}}(1\ \ -(m-2r)\ \ m\ \ (m-2r)^2-m\ \ m(m-1)\ \ -(m-1)(m-2r))', \\ &\sqrt{\lambda_{11m-2}}(1\ \ m-3\ \ m-3\ \ (m-2)(m-5)\ \ (m-1)(m-6)\ \ (m-3)(m-4))', \\ &\sqrt{\lambda_{m-211}}(1\ \ -(m-3)\ \ m-3\ \ (m-2)(m-5)\ \ (m-1)(m-6)\ \ -(m-3)(m-4))' \text{ and} \\ &\sqrt{\lambda_{1m-21}}(1\ \ 0\ \ -2(m-3)\ \ -2\ \ 2(2m^2-14m+21)\ \ 0)', \end{aligned}$$

where $1 \leq p, q, r \leq m$, the 3×1 column vectors of F_1 of size $3 \times 3(m-2)$ concerned with λ_{pm-p0}, λ_{0qm-q}, λ_{m-r0r}, λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$(3.4)\quad \begin{aligned} &\sqrt{\lambda_{pm-p0}}(1\ \ 1\ \ -1)',\ \sqrt{\lambda_{0qm-q}}(1\ \ 1\ \ 1)',\ \sqrt{\lambda_{m-r0r}}(1\ \ 0\ \ 0)', \\ &\sqrt{\lambda_{11m-2}}(2\ \ 0\ \ 1)',\quad \sqrt{\lambda_{m-211}}(2\ \ 0\ \ -1)' \text{ and } \sqrt{\lambda_{1m-21}}(1\ \ -1\ \ 0)', \end{aligned}$$

where $2 \leq p, q, r \leq m-2$, the elements of F_2 of size 1×3 concerned with λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$(3.5)\quad \sqrt{\lambda_{11m-2}}(1),\ \ \sqrt{\lambda_{m-211}}(1)\ \ \text{and}\ \ \sqrt{\lambda_{1m-21}}(1),$$

and the 6×1 column vectors or the 6×2 submatrices of F_f of size $6 \times 3(m+1)$ concerned with λ_{pm-p0}, λ_{0qm-q}, λ_{m-r0r}, λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$
(3.6) \qquad
\begin{aligned}
&\sqrt{\lambda_{pm-p0}}(1 \quad 1 \quad p-1 \quad 2m-3p-1 \quad m \quad m-3p+1)', \\
&\sqrt{\lambda_{0qm-q}}(-1 \quad 1 \quad m-q-1 \quad -(m-3q+1) \quad -m \quad 2m-3q-1)', \\
&\sqrt{\lambda_{m-r0r}}(2 \quad 0 \quad 2(m-2r) \quad 0 \quad -m \quad -(m-2))', \\
&\sqrt{\lambda_{11m-2}}\begin{pmatrix} 1 & 1 & -(m-2) & -(m-2) & -2(m-3) & m-2 \\ -3 & 1 & 3m-10 & -(m-2) & 0 & 3(m-4) \end{pmatrix}', \\
&\sqrt{\lambda_{m-211}}\begin{pmatrix} -1 & 1 & -(m-2) & -(m-2) & 2(m-3) & -(m-2) \\ 3 & 1 & 3m-10 & -(m-2) & 0 & -3(m-4) \end{pmatrix}' \text{ and} \\
&\sqrt{\lambda_{1m-21}}\begin{pmatrix} 1 & 0 & 0 & 0 & m-3 & m-2 \\ 0 & -1 & 1 & -(2m-7) & 0 & 0 \end{pmatrix}',
\end{aligned}
$$

where $1 \le p, q, r \le m-1$ (see [11]). Furthermore the diagonal elements of Λ_0 of order $3(m+1)$ concerned with the indices λ_{pm-p0}, λ_{0qm-q}, λ_{m-r0r}, λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$
(3.7) \qquad \sqrt{\tbinom{m}{p}},\ \sqrt{\tbinom{m}{q}},\ \sqrt{\tbinom{m}{r}},\ \sqrt{2\tbinom{m}{2}},\ \sqrt{2\tbinom{m}{2}} \text{ and } \sqrt{2\tbinom{m}{2}},
$$

where $1 \le p, q, r \le m$, the diagonal elements of Λ_1 of order $3(m-2)$ concerned with λ_{pm-p0}, λ_{0qm-q}, λ_{m-r0r}, λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$
(3.8) \qquad \sqrt{\tbinom{m-4}{p-2}},\ \sqrt{\tbinom{m-4}{q-2}},\ 4\sqrt{\tbinom{m-4}{r-2}},\ \sqrt{2},\ \sqrt{2} \text{ and } \sqrt{2},
$$

where $2 \le p, q, r \le m-2$, the diagonal elements of Λ_2 of order 3 concerned with λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$
(3.9) \qquad 6,\ 6 \text{ and } 6,
$$

and the diagonal elements or the 2×2 block diagonal ones of Λ_f of order $3(m+1)$ concerned with λ_{pm-p0}, λ_{0qm-q}, λ_{m-r0r}, λ_{11m-2}, λ_{m-211} and λ_{1m-21} are, respectively, given by

$$
(3.10) \qquad
\begin{aligned}
&\sqrt{\tbinom{m-2}{p-1}},\ \sqrt{\tbinom{m-2}{q-1}},\ \sqrt{\tbinom{m-2}{r-1}}, \\
&\mathrm{diag}[\sqrt{m/2}; \sqrt{(m-2)/2}],\ \mathrm{diag}[\sqrt{m/2}; \sqrt{(m-2)/2}] \text{ and } \mathrm{diag}[\sqrt{2m}; \sqrt{2(m-2)}],
\end{aligned}
$$

where $1 \le p, q, r \le m-1$. Then from Theorem 3.1, Lemma A.1 and (3.1), the following yields (see [11]):

Theorem 3.2. *Let T be an* $\mathrm{SA}(m; \{\lambda_{i_0 i_1 i_2}\})$ *satisfying the conditions of* Theorem 3.1, *then*

$$
(3.11) \qquad K_\beta = (D_\beta F_\beta \Lambda_\beta)(D_\beta F_\beta \Lambda_\beta)' \quad \textit{for } \beta = 0, 1, 2, f,
$$

where $m \ge 4$, F_β and Λ_β are given by (3.3) *through* (3.6) *and* (3.7) *through* (3.10), *respectively, and*

$$
\begin{aligned}
&D_0 = \mathrm{diag}[1; 1/\sqrt{m}; 1/\sqrt{m}; 1/\{2\sqrt{\tbinom{m}{2}}\}; 1/\{2\sqrt{\tbinom{m}{2}}\}; 1/\{\sqrt{2\tbinom{m}{2}}\}],\ D_1 = \mathrm{diag}[1; 9; 3\sqrt{2}], \\
&D_2 = 1 \textit{ and } D_f = \mathrm{diag}[-1; 3; 1/\sqrt{m-2}; -3/\sqrt{m-2}; \sqrt{2/m}; \sqrt{2/(m-2)}].
\end{aligned}
$$

By (3.11), it holds that $\mathrm{rank}\{K_\beta\} = \text{r-rank}\{F_\beta\}$ for $\beta = 0, 1, 2, f$, where r-rank$\{A\}$ denotes the row rank of a matrix A.

Note from Theorem 5.1 of Kuwada [6] that

(i) if $A_0^{\#(00,00)}\boldsymbol{\theta}_{00}$ is estimable, then $\boldsymbol{\theta}_{00}$ is estimable,
(ii) if $A_0^{\#(a_1a_2,a_1a_2)}\boldsymbol{\theta}_{a_1a_2}$ and $A_{f_{11}}^{\#(a_1a_2,a_1a_2)}\boldsymbol{\theta}_{a_1a_2}$ $(a_1a_2 = 10, 01)$ are estimable, then $\boldsymbol{\theta}_{a_1a_2}$ is estimable,
(iii) if $A_0^{\#(b_1b_2,b_1b_2)}\boldsymbol{\theta}_{b_1b_2}$, $A_1^{\#(b_1b_2,b_1b_2)}\boldsymbol{\theta}_{b_1b_2}$ and $A_{f_{22}}^{\#(b_1b_2,b_1b_2)}\boldsymbol{\theta}_{b_1b_2}$ $(b_1b_2 = 20, 02)$ are estimable, then $\boldsymbol{\theta}_{b_1b_2}$ is estimable, and
(iv) if $A_\gamma^{\#(11,11)}\boldsymbol{\theta}_{11}$ and $A_{f_{ii}}^{\#(11,11)}\boldsymbol{\theta}_{11}$ are estimable for all $\gamma = 0, 1, 2$ and $i = 3, 4$, then $\boldsymbol{\theta}_{11}$ is estimable.

4 Resolutions $\mathbf{R}(\{00, 10, 01\} \cup S_1|\Omega)$ designs with $N < \nu(m)$ In this section, the focus is on obtaining a 3^m-BFF design of resolutions $\mathrm{R}(\{00, 10, 01\} \cup S_1|\Omega)$ derived from an $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ with $N < \nu(m)$, where $m \geq 4$, $S_1 = \{20\}, \{02\}$ and $\{11\}$, and the indices $\lambda_{i_0i_1i_2}$ satisfy the conditions of Theorem 3.1. The resulting array given by interchanging all of the symbols 0 and 2 of an $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ is also the $\mathrm{SA}(m; \{\lambda^*_{k_0k_1k_2}\})$, where $\lambda^*_{k_0k_1k_2} = \lambda_{k_2k_1k_0}$, and it is briefly denoted by (0, 2)-ISA.

(A) Resolution $\mathrm{R}(\{00, 10, 01, 20\}|\Omega)$ designs

We firstly consider a 3^m-BFF design of resolution $\mathrm{R}(\{00, 10, 01, 20\}|\Omega)$ derived from an $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ with $N < \nu(m)$. Then $\boldsymbol{\theta}_{00}$, $\boldsymbol{\theta}_{10}$, $\boldsymbol{\theta}_{01}$ and $\boldsymbol{\theta}_{20}$ are estimable and the factorial effects of $\boldsymbol{\theta}_{02}$ and $\boldsymbol{\theta}_{11}$ are confounded with each other. Using the row relations of F_β $(\beta = 0, 1, 2, f)$ given by (3.3) through (3.6) and Lemma A.1, we have the following:

Theorem 4.A. *Let T be an* $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ *with $N < \nu(m)$, where $m \geq 4$ and the indices $\lambda_{i_0i_1i_2}$ satisfy the conditions of Theorem* 3.1. *Then a necessary and sufficient condition for T to be a 3^m*-BFF *design of resolution* $\mathrm{R}(\{00, 10, 01, 20\}|\Omega)$ *is that one of the following holds*:

(I) *When* $m = 6$, $\lambda_{051} = \lambda_{015} = \lambda_{330} = \lambda_{303} = 1$, *exactly three out of* $\{\lambda_{150}, \lambda_{510}, \lambda_{105}, \lambda_{501}\}$ *are* 1, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (x6-x0)$ $(x = 1, 3, 5, 6)$, $(0y6-y)$ $(y = 1, 5, 6)$, $(6-z0z)$ $(z = 1, 3, 5, 6)$ *and* $\lambda_{600} + \lambda_{006} + \lambda_{060} < 3$, *or its* (0, 2)-ISA,

(II) *when* $m = 8$, $\lambda_{170}, \lambda_{071}, \lambda_{017}, \lambda_{710} \geq 1$, $\lambda_{107} + \lambda_{701} \geq 1$, $\lambda_{404} = 1$, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (x8-x0)$ $(x = 1, 7, 8)$, $(0y8-y)$ $(y = 1, 7, 8)$, $(8-z0z)$ $(z = 1, 4, 7, 8)$ *and* $\lambda_{800} + \lambda_{008} + \lambda_{080} + 8(\lambda_{170} + \lambda_{071} + \lambda_{017} + \lambda_{710} + \lambda_{107} + \lambda_{701}) < 59$,

(III) *when* $m = 6$ *and* 7,

(i) $\lambda_{0m-11} = \lambda_{01m-1} = \lambda_{2m-20} + \lambda_{m-220} = \lambda_{20m-2} + \lambda_{m-202} = 1$, *and*

(1) $\lambda_{1m-10} + \lambda_{m-110} = \lambda_{30m-3} + \lambda_{m-303}$ (*if* $m = 7$) $= 1$, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (xm-x0)$ $(x = 1, 2, m-2, m-1, m)$, $(0ym-y)$ $(y = 1, m-1, m)$, $(m-z0z)$ $(z = 2, 3$ (*if* $m = 7$), $m-3, m-2, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} < 1 + m(m-2)(7-m)/6$, *or its* (0, 2)-ISA, *or*

(2) $\lambda_{10m-1} + \lambda_{m-101} = \lambda_{3m-30} + \lambda_{m-330}$ (*if* $m = 7$) $= 1$, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (xm-x0)$ $(x = 2, 3, m-3$ (*if* $m = 7$), $m-2, m)$, $(0ym-y)$ $(y = 1, m-1, m)$, $(m-z0z)$ $(z = 1, 2, m-2, m-1, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} < 1 + m(m-2)(7-m)/6$, *or its* (0, 2)-ISA,

(ii) $\lambda_{ab0}, \lambda_{0m-11}, \lambda_{01m-1} \geq 1$ $((ab) = (1m-1), (m-11))$, $\lambda_{cd0} = 1$ $((cd) = (3m-3), (m-33)$ (*if* $m = 7$)), *and*

(1) $\lambda_{b0a} \geq 1$, *where* (ab) *is the same as in* (ii), $\lambda_{30m-3} + \lambda_{m-303}$ (*if* $m = 7$) $= 1$, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (m00), (ab0), (b0a), (0ym-y)$ $(y = 1, m-1, m)$, $(cd0)$, $(m-z0z)$ $(z = 3$ (*if* $m = 7$), $m-3, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} + m(\lambda_{ab0} + \lambda_{0m-11} + \lambda_{01m-1} + \lambda_{b0a}) < 1 + m\{(m-2)(7-m) + 12\}/3$, *or its* (0, 2)-ISA, *or*

(2) $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} \geq 1$ (*if* $m = 6$), $\lambda_{a0b} \geq 1$, $\lambda_{d0c} = 1$, *where* (ab) *and* (cd) *are the same as in* (ii), $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (m00), (00m), (ab0)$,

$(a0b), (0ym-y)$ $(y=1, m-1, m), (cd0), (d0c)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{ab0}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{a0b})<1+m\{(m-2)(7-m)+12\}/3$, *or its* $(0,2)$-ISA,

(iii) $\lambda_{1m-10}+\lambda_{m-110}=\lambda_{0m-11}=\lambda_{01m-1}=\lambda_{20m-2}=\lambda_{m-202}=\lambda_{3m-30}+\lambda_{m-330}$ (*if* $m=7$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1,3, m-3$ (*if* $m=7$)$, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=2, m-2, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}<1+m(m-2)(7-m)/6$, *or its* $(0,2)$-ISA,

(iv) $\lambda_{0m-11}=\lambda_{01m-1}=\lambda_{10m-1}+\lambda_{m-101}=\lambda_{2m-20}=\lambda_{m-220}=\lambda_{30m-3}+\lambda_{m-303}$ (*if* $m=7$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=2, m-2, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, 3$ (*if* $m=7$)$, m-3, m-1, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}<1+m(m-2)(7-m)/6$, *or its* $(0,2)$-ISA, *or*

(v) $\lambda_{1m-10}=\lambda_{0m-11}=\lambda_{01m-1}=\lambda_{m-110}=\lambda_{10m-1}=\lambda_{m-101}=1$, *and*

(1) $\lambda_{2m-20}+\lambda_{m-220}=\lambda_{30m-3}+\lambda_{m-303}$ (*if* $m=7$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, 2, m-2, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, 3$ (*if* $m=7$)$, m-3, m-1, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}<1+m(m-5)(7-m)/6$, *or its* $(0,2)$-ISA,

(2) $\lambda_{20m-2}+\lambda_{m-202}=\lambda_{3m-30}+\lambda_{m-330}$ (*if* $m=7$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, 3, m-3$ (*if* $m=7$)$, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, 2, m-2, m-1, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}<1+m(m-5)(7-m)/6$, *or its* $(0,2)$-ISA, *or*

(3) $\lambda_{20m-2}+\lambda_{m-202}=\lambda_{30m-3}+\lambda_{m-303}$ (*if* $m=7$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, 2, 3$ (*if* $m=7$)$, m-3, m-2, m-1, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}<1+m(m-5)(7-m)/6$,

(IV) *when* $6\leq m\leq 8$, $\lambda_{0m-11}, \lambda_{01m-1}\geq 1$, *and furthermore*

(i) *exactly three out of* $\{\lambda_{1m-10}, \lambda_{m-110}, \lambda_{10m-1}, \lambda_{m-101}\}$ *are non-zero, and*

(1) $\lambda_{2m-20}+\lambda_{m-220}=\lambda_{30m-3}+\lambda_{m-303}$ (*if* $m\neq 6$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, 2, m-2, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, 3$ (*if* $m\neq 6$)$, m-3, m-1, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110}+\lambda_{10m-1}+\lambda_{m-101})<1+m\{(m-4)(8-m)+33\}/6$, *or its* $(0,2)$-ISA, *or*

(2) $\lambda_{20m-2}+\lambda_{m-202}=\lambda_{3m-30}+\lambda_{m-330}$ (*if* $m\neq 6$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, 3, m-3$ (*if* $m\neq 6$)$, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, 2, m-2, m-1, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110}+\lambda_{10m-1}+\lambda_{m-101})<1+m\{(m-4)\times(8-m)+33\}/6$, *or its* $(0,2)$-ISA, *or*

(ii) $\lambda_{1m-10}, \lambda_{m-110}, \lambda_{a0b}\geq 1$ $((ab)=(1m-1), (m-11))$, $\lambda_{20m-2}+\lambda_{m-202}=\lambda_{30m-3}+\lambda_{m-303}$ (*if* $m\neq 6$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(a0b), (xm-x0)$ $(x=1, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=2, 3$ (*if* $m\neq 6$)$, m-3, m-2, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110}+\lambda_{a0b})<1+m\{(m-4)(8-m)+33\}/6$,

(V) *when* $6\leq m\leq 9$, $\lambda_{0m-11}, \lambda_{01m-1}\geq 1$, *and furthermore*

(i) $\lambda_{ab0}, \lambda_{c0d}\geq 1$ $((ab), (cd)=(1m-1), (m-11))$, *and*

(1) $\lambda_{2m-20}+\lambda_{m-220}=\lambda_{30m-3}+\lambda_{m-303}$ (*if* $m\neq 6$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(ab0), (c0d), (xm-x0)$ $(x=2, m-2, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=3$ (*if* $m\neq 6$)$, m-3, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{ab0}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{c0d})<1+m\{(m-3)(9-m)+28\}/6$, *or its* $(0,2)$-ISA, *or*

(2) $\lambda_{20m-2}+\lambda_{m-202}=\lambda_{3m-30}+\lambda_{m-330}$ (*if* $m\neq 6$) $=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(ab0), (c0d), (xm-x0)$ $(x=3, m-3$ (*if* $m\neq 6$)$, m), (0ym-y)$ $(y=$

$1, m-1, m), (m-z0z)$ $(z=2, m-2, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{ab0}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{c0d})<1+m\{(m-3)(9-m)+28\}/6$, *or its* $(0,2)$-ISA, *or*

(ii) $\lambda_{1m-10}, \lambda_{m-110} \geq 1$, $\lambda_{20m-2}+\lambda_{m-202}=\lambda_{30m-3}+\lambda_{m-303}$ $(\text{if } m\neq 6)=1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=2,3$ $(\text{if } m\neq 6), m-3, m-2, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110})<1+m\{(m-3)(9-m)+28\}/6$,

(VI) *when* $6\leq m\leq 12$, $\lambda_{1m-10}, \lambda_{0m-11}, \lambda_{01m-1}, \lambda_{m-110}\geq 1$, $\lambda_{10m-1}+\lambda_{m-101}\geq 1$, $\lambda_{a0b}\geq 1$ $((ab)=(3m-3), (m-33)$ $(\text{if } m\neq 6))$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, m-1, m), (a0b)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110}+\lambda_{10m-1}+\lambda_{m-101})+\binom{m}{3}\lambda_{a0b}<1+2m^2$,

(VII) $\lambda_{0m-11}, \lambda_{01m-1}\geq 1$, *and furthermore*

(i) $\lambda_{ab0}, \lambda_{cd0}\geq 1$ $((ab)=(1m-1), (m-11);$ $(cd)=(2m-2), (m-22)$ $(\text{if } m\neq 4))$, *and*

(1) $\lambda_{e0f}\geq 1$ $((ef)=(2m-2), (m-22)$ $(\text{if } m\neq 4))$, *and*

(a) $\lambda_{b0a}\geq 1$, *where* (ab) *is the same as in* (i), $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(ab0)$, $(b0a), (0ym-y)$ $(y=1, m-1), (cd0), (e0f)$ *and* $m(\lambda_{ab0}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{b0a})+\binom{m}{2}(\lambda_{cd0}+\lambda_{e0f})<1+2m^2$, *or its* $(0,2)$-ISA, *or*

(b) $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}\geq 1$, $\lambda_{g0h}\geq 1$ $((gh)=(1m-1), (m-11))$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(m00), (00m), (ab0), (g0h), (0ym-y)$ $(y=1, m-1, m), (cd0), (e0f)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{ab0}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{g0h})+\binom{m}{2}(\lambda_{cd0}+\lambda_{e0f})<1+2m^2$, *or its* $(0,2)$-ISA, *or*

(2) *when* $m\geq 5$, $\lambda_{a0b}, \lambda_{d0c}\geq 1$, *where* (ab) *and* (cd) *are the same as in* (i), $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(ab0), (a0b), (0ym-y)$ $(y=1, m-1), (cd0), (d0c)$ *and* $m(\lambda_{ab0}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{a0b})+\binom{m}{2}(\lambda_{cd0}+\lambda_{d0c})<1+2m^2$, *or its* $(0,2)$-ISA,

(ii) $\lambda_{1m-10}, \lambda_{m-110}\geq 1$, *and*

(1) $\lambda_{10m-1}+\lambda_{m-101}\geq 1$, $\lambda_{20m-2}+\lambda_{m-202}$ $(\text{if } m\neq 4)\geq 1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, 2$ $(\text{if } m\neq 4), m-2, m-1, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110}+\lambda_{10m-1}+\lambda_{m-101})+\binom{m}{2}(\lambda_{20m-2}+\lambda_{m-202}$ $(\text{if } m\neq 4))<1+2m^2$, *or*

(2) *when* $m\geq 5$, $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}\geq 1$, $\lambda_{20m-2}, \lambda_{m-202}\geq 1$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=2, m-2, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110})+\binom{m}{2}(\lambda_{20m-2}+\lambda_{m-202})<1+2m^2$, *or*

(iii) *at least three out of* $\{\lambda_{1m-10}, \lambda_{m-110}, \lambda_{10m-1}, \lambda_{m-101}\}$ *are non-zero*, $\lambda_{ab0}, \lambda_{c0d}\geq 1$ $((ab), (cd)=(2m-2), (m-22)$ $(\text{if } m\neq 4))$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(xm-x0)$ $(x=1, m-1, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=1, m-1, m), (ab0)$, $(c0d)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{1m-10}+\lambda_{0m-11}+\lambda_{01m-1}+\lambda_{m-110}+\lambda_{10m-1}+\lambda_{m-101})+\binom{m}{2}(\lambda_{ab0}+\lambda_{c0d})<1+2m^2$, *or its* $(0,2)$-ISA, *or*

(VIII) *when* $m\geq 5$, $\lambda_{0m-11}, \lambda_{01m-1}\geq 1$, *and furthermore*

(i) $\lambda_{20m-2}=\lambda_{m-202}=1$, *and*

(1) $\lambda_{2m-20}+\lambda_{m-220}=1$, *and*

(a) $\lambda_{ab0}\geq 1$ $((ab)=(1m-1), (m-11))$, $\lambda_{i_0i_1i_2}=0$ *for* $(i_0i_1i_2)\neq(ab0), (xm-x0)$ $(x=2, m-2, m), (0ym-y)$ $(y=1, m-1, m), (m-z0z)$ $(z=2, m-2, m)$ *and* $\lambda_{m00}+\lambda_{00m}+\lambda_{0m0}+m(\lambda_{ab0}+\lambda_{0m-11}+\lambda_{01m-1})<\binom{m+2}{2}$, *or its* $(0,2)$-ISA,

(b) *exactly two out of* $\{\lambda_{1m-10}, \lambda_{m-110}, \lambda_{10m-1}, \lambda_{m-101}\}$ *except for* $\{\lambda_{10m-1},$

$\lambda_{m-101}\}$ *are non-zero*, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (xm{-}x0)$ $(x = 1, 2, m-2, m-1, m), (0ym-y)$ $(y = 1, m-1, m), (m-z0z)$ $(z = 1, 2, m-2, m-1, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} + m(\lambda_{1m-10} + \lambda_{0m-11} + \lambda_{01m-1} + \lambda_{m-110} + \lambda_{10m-1} + \lambda_{m-101}) < \binom{m+2}{2}$, *or its* $(0, 2)$-ISA, *or*

(c) *when* $m \geq 7$, *exactly three out of* $\{\lambda_{1m-10}, \lambda_{m-110}, \lambda_{10m-1}, \lambda_{m-101}\}$ *are non-zero*, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (xm-x0)$ $(x = 1, 2, m-2, m-1, m), (0ym-y)$ $(y = 1, m-1, m), (m-z0z)$ $(z = 1, 2, m-2, m-1, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} + m(\lambda_{1m-10} + \lambda_{0m-11} + \lambda_{01m-1} + \lambda_{m-110} + \lambda_{10m-1} + \lambda_{m-101}) < \binom{m+2}{2}$, *or its* $(0, 2)$-ISA, *or*

(2) *when* $m \geq 9$, $\lambda_{1m-10}, \lambda_{m-110}, \lambda_{10m-1}, \lambda_{m-101} \geq 1$, $\lambda_{2m-20} + \lambda_{0m-22} + \lambda_{02m-2} + \lambda_{m-220} = 1$, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (xm-x0)$ $(x = 1, 2, m-2, m-1, m), (0ym-y)$ $(y = 1, 2, m-2, m-1, m), (m-z0z)$ $(z = 1, 2, m-2, m-1, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} + m(\lambda_{1m-10} + \lambda_{0m-11} + \lambda_{01m-1} + \lambda_{m-110} + \lambda_{10m-1} + \lambda_{m-101}) < \binom{m+2}{2}$, *or*

(ii) $\lambda_{2m-20} = \lambda_{m-220} = \lambda_{20m-2} + \lambda_{m-202} = 1$, *and*

(1) $\lambda_{a0b} \geq 1$ $((ab) = (1m-1), (m-11))$, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (a0b)$, $(xm-x0)$ $(x = 2, m-2, m), (0ym-y)$ $(y = 1, m-1, m), (m-z0z)$ $(z = 2, m-2, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} + m(\lambda_{0m-11} + \lambda_{01m-1} + \lambda_{a0b}) < \binom{m+2}{2}$, *or its* $(0, 2)$-ISA,

(2) *exactly two out of* $\{\lambda_{1m-10}, \lambda_{m-110}, \lambda_{10m-1}, \lambda_{m-101}\}$ *except for* $\{\lambda_{1m-10}, \lambda_{m-110}\}$ *are non-zero*, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (xm-x0)$ $(x = 1, 2, m-2, m-1, m), (0ym-y)$ $(y = 1, m-1, m), (m-z0z)$ $(z = 1, 2, m-2, m-1, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} + m(\lambda_{1m-10} + \lambda_{0m-11} + \lambda_{01m-1} + \lambda_{m-110} + \lambda_{10m-1} + \lambda_{m-101}) < \binom{m+2}{2}$, *or its* $(0, 2)$-ISA, *or*

(3) *when* $m \geq 7$, *at least three out of* $\{\lambda_{1m-10}, \lambda_{m-110}, \lambda_{10m-1}, \lambda_{m-101}\}$ *are non-zero*, $\lambda_{i_0i_1i_2} = 0$ *for* $(i_0i_1i_2) \neq (xm-x0)$ $(x = 1, 2, m-2, m-1, m), (0ym-y)$ $(y = 1, m-1, m), (m-z0z)$ $(z = 1, 2, m-2, m-1, m)$ *and* $\lambda_{m00} + \lambda_{00m} + \lambda_{0m0} + m(\lambda_{1m-10} + \lambda_{0m-11} + \lambda_{01m-1} + \lambda_{m-110} + \lambda_{10m-1} + \lambda_{m-101}) < \binom{m+2}{2}$, *or its* $(0, 2)$-ISA.

Remark 4.A. In Theorem 4.A, we have the following:
$70 \leq N < 73$ for (I), $110 \leq N < 129$ for (II), $3m + 2\binom{m}{2} + \binom{m}{3} \leq N < \nu(m)$ for (III)(i), (iii) and (iv), $65 \leq N < 73$ (if $m = 6$) and $N = 98$ (if $m = 7$) for (III)(ii), $6m + \binom{m}{2} + \binom{m}{3} \leq N < \nu(m)$ for (III)(v), $5m + \binom{m}{2} + \binom{m}{3} \leq N < \nu(m)$ for (IV), $4m + \binom{m}{2} + \binom{m}{3} \leq N < \nu(m)$ for (V), $5m + \binom{m}{3} \leq N < \nu(m)$ for (VI), $N = km + h\binom{m}{2}$ ($h = 2$ and $4 \leq k \leq m+1$ for $m \geq 4$; $h = 3$ and $4 \leq k \leq (m+3)/2$ for $m \geq 5$) for (VII)(i)(1)(a), $1 + 4m + 2\binom{m}{2} \leq N < \nu(m)$ for (VII)(i)(1)(b) and (ii)(2), $N = km + h\binom{m}{2}$ ($h = 2$ and $4 \leq k \leq m+1$; $h = 3$ and $4 \leq k \leq (m+3)/2$) for (VII)(i)(2), $5m + \binom{m}{2} \leq N < \nu(m)$ for (VII)(ii)(1), $5m + 2\binom{m}{2} \leq N < \nu(m)$ for (VII)(iii), $3m + 3\binom{m}{2} \leq N < \nu(m)$ for (VIII)(i)(1)(a) and (ii)(1), $4m + 3\binom{m}{2} \leq N < \nu(m)$ for (VIII)(i)(1)(b) and (ii)(2), $5m + 3\binom{m}{2} \leq N < \nu(m)$ for (VIII)(i)(1)(c) and (ii)(3), and $6m + 3\binom{m}{2} \leq N < \nu(m)$ for (VIII)(i)(2), and furthermore r-rank$\{F_0\} = 6$, r-rank$\{F_1\} = 2$ and the last row of F_1 equals $w_1(= -1)$ times the second, r-rank$\{F_2\} = 0$ and r-rank$\{F_f\} = 6$ for (I), (III)(i), (ii), (iii), (iv) and (v)(1) and (2), (IV)(i), (V)(i), (VII)(i) and (iii), and (VIII), and r-rank$\{F_0\} = 6$, r-rank$\{F_1\} = 1$ and the last two rows of F_1 are zero, r-rank$\{F_2\} = 0$ and r-rank$\{F_f\} = 6$ for (II), (III)(v)(3), (IV)(ii), (V)(ii), (VI), and (VII)(ii).

(B) Resolution $\mathrm{R}(\{00, 10, 01, 02\}|\Omega)$ designs

Let T be a 3^m-BFF design of resolution $\mathrm{R}(\{00, 10, 01, 02\}|\Omega)$ derived from an SA$(m;$

$\{\lambda_{i_0i_1i_2}\})$ with $N < \nu(m)$. Then $\boldsymbol{\theta}_{00}$, $\boldsymbol{\theta}_{10}$, $\boldsymbol{\theta}_{01}$ and $\boldsymbol{\theta}_{02}$ are estimable and the effects of $\boldsymbol{\theta}_{20}$ and $\boldsymbol{\theta}_{11}$ are confounded with each other. Using the row relations of F_β ($\beta = 0, 1, 2, f$) given by (3.3) through (3.6), and Lemmas A.1 and A.2, we obtain the following:

Theorem 4.B. *There does not exist a* 3^m-BFF *design of resolution* $\mathrm{R}(\{00, 10, 01, 02\}|\Omega)$ *derived from an* $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ *with* $N < \nu(m)$, *where* $m \geq 4$ *and the indices* $\lambda_{i_0i_1i_2}$ *satisfy the conditions of Theorem* 3.1.

(C) Resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ designs

We finally consider a 3^m-BFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from an $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ with $N < \nu(m)$, and hence $\boldsymbol{\theta}_{00}$, $\boldsymbol{\theta}_{10}$, $\boldsymbol{\theta}_{01}$ and $\boldsymbol{\theta}_{11}$ are estimable and the effects of $\boldsymbol{\theta}_{20}$ and $\boldsymbol{\theta}_{02}$ are confounded with each other. By use of the methods similar to Theorem 4.B, the following yields:

Theorem 4.C. *There does not exist a* 3^m-BFF *design of resolution* $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ *derived from an* $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ *with* $N < \nu(m)$, *where* $m \geq 4$ *and the indices* $\lambda_{i_0i_1i_2}$ *satisfy the conditions of Theorem* 3.1.

It follows from Remark 4.A that we have the following theorem:

Theorem 4.1. *Let* T *be a* 3^m-BFF *design of resolution* $\mathrm{R}(\{00, 10, 01, 20\}|\Omega)$ *derived from an* $\mathrm{SA}(m; \{\lambda_{i_0i_1i_2}\})$ *with* $N < \nu(m)$, *where* $m \geq 4$ *and the indices* $\lambda_{i_0i_1i_2}$ *satisfy the conditions of Theorem* 3.1, *then*

(I) r-rank$\{F_0\} = 6$, *and hence* $A_0^{\#(a_1a_2, a_1a_2)}\boldsymbol{\theta}_{a_1a_2}$ ($a_1a_2 = 00, 10, 01, 20, 02, 11$) *are estimable*,

(II) (i) *if* r-rank$\{F_1\} = 1$ *and the last two rows of* F_1 *are zero, then* $A_1^{\#(20,20)}\boldsymbol{\theta}_{20}$ *is estimable*, *and*

(ii) *if* r-rank$\{F_1\} = 2$ *and the last row of* F_1 *equals* w_1 ($\neq 0$) *times the second*, *then* $A_1^{\#(20,20)}\boldsymbol{\theta}_{20}$ *and* $A_1^{\#(02,02)}\boldsymbol{\theta}_{02} + w_1^* A_1^{\#(02,11)}\boldsymbol{\theta}_{11}$ *are estimable*, *where* $w_1^* = (\sqrt{2}/3)w_1$, *and*

(III) r-rank$\{F_f\} = 6$, *and hence* $A_{f_{ii}}^{\#(u_1u_2, u_1u_2)}\boldsymbol{\theta}_{u_1u_2}$ ($(u_1u_2; i) = (10; 1), (01; 1), (20; 2), (02; 2), (11; 3), (11; 4)$) *are estimable*.

Appendix

Let $ZL = H$ be a matrix equation, where Z is a variable matrix of order n, $L = ||L_{ij}||$ ($i, j = 1, 2, 3$) is the positive semidefinite matrix of order n with $\mathrm{rank}\{L\} = \mathrm{rank}\{\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\}$ $= n_1 + n_2$ (≥ 1), and $H = ||H_{ij}||$ ($i, j = 1, 2, 3$) is some matrix of order n with $H_{11} = I_{n_1}$, $H_{12} = H_{21}' = O_{n_1 \times n_2}$ and $H_{13} = H_{31}' = O_{n_1 \times n_3}$. Here L_{ij} and H_{ij} are of size $n_i \times n_j$, and $n_1 + n_2 + n_3 = n$. Then $ZL = H$ has a solution if and only if $\mathrm{rank}\{L'\} = \mathrm{rank}\{(L'; H')\}$. Thus we get the following:

Lemma A.1. (see [2]) *A matrix equation* $ZL = H$ *has a solution if and only if*

(I) $n_3 = 0$, *where* H_{22} (*if* $n_2 \geq 1$) *is arbitrary, or*

(II) $n_3 \geq 1$, *and in addition*

(i) *when* $n_2 = 0$, $L_{33} = O_{n_3 \times n_3}$, *and furthermore* $H_{33} = O_{n_3 \times n_3}$, *or*

(ii) *when* $n_2 \geq 1$, *there exists a matrix* W *of size* $n_3 \times n_2$ *such that* $[L_{31}; L_{32}; L_{33}]$ $= W[L_{21}; L_{22}; L_{23}]$, *and furthermore* $H_{23}' = WH_{22}'$ *and* $H_{33}' = WH_{32}'$, *where* H_{22} *and*

H_{32} are arbitrary.

Lemma A.2. *The existence of a solution Z to the matrix equation $ZL = H$ is equivalent to that of Z^* to $Z^*L^* = H^*$, where $Z^* = P^{'}ZP$, $L^* = P^{'}LP$ and $H^* = P^{'}HP$, and P is a permutation matrix of order n.*

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