EMBEDDING AN ORDERED SEMIGROUP INTO A TRANSLATIONAL HULL

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Abstract. The paper deals with the embedding of an ordered semigroup into the translational hull of its ideals which are both dense and weakly reductive. For a semigroup or an ordered semigroup S, $\Omega(S)$ denotes the set of (all) bitranslations of S. It is well known that if K is a dense ideal of a semigroup S such that K is weakly reductive, then S is isomorphic to a subsemigroup of $\Omega(K)$. In the present paper we generalize this result for ordered semigroups using the concept of pseudoorder –a concept which extends the concept of congruences of semigroups and plays an important role in studying the structure of ordered semigroups. We prove that if S is an ordered semigroup and K a weakly reductive dense ideal of S, then S is embedded into the ordered semigroup $\Omega(K)$ of (all) bitranslations of K (and so S is isomorphic to an (ordered) subsemigroup of the translational hull of K).

1. Introduction and prerequisites

For a semigroup or an ordered semigroup S, $\Omega(S)$ denotes the translational hull of S i.e. the set of (all) bitranslations of S. It is well known that if K is a dense ideal of a semigroup S such that K is weakly reductive, then S is isomorphic to a subsemigroup of $\Omega(K)$ (see [8]). The following question is natural: Is there an analogous result for ordered semigroups? Under what conditions an ordered semigroup S can be embedded into the translational hull of some weakly reductive ideals of S? The concept of weakly reductive ordered semigroups has been introduced by Kehayopulu and Tsingelis in [7]. In this paper we first introduce the concept of dense ideals of ordered semigroups which extends the definition of dense ideals of semigroups (without order). We say that an ideal K of an ordered semigroup $(S,.,\leq)$ is a dense ideal of S if for each pseudoorder σ on S the relation $\sigma\cap(K\times K)\subseteq\leq\cap(K\times K)$ implies $\sigma\subseteq\leq$. Using the concept of pseudoorder, we prove that if S is an ordered semigroup and K a weakly reductive dense ideal of S, then S is embedded into the ordered semigroup $\Omega(K)$ of (all) bitranslations of K (and so S is isomorphic to an (ordered) subsemigroup of the translational hull of K). Our result generalizes the corresponding result of semigroups (without order).

If S is a semigroup or an ordered semigroup, by a *congruence* on S we mean an equivalence relation on S such that $(a,b)\in\sigma$ implies $(ac,bc)\in\sigma$ and $(ca,cb)\in\sigma$ for all $c\in S$ (see [1,9] and [3]). A semigroup S (without order) is called *weakly reductive* if the following assertion is satisfied: If $a,b\in S$ such that $ax=bx$ and $xa=xb$ for all $x\in S$, then $a=b$ [1]. An ideal K of a semigroup (without order) S is called *dense* if $\sigma|K=id_K$ implies $\sigma=id_S$ for every congruence σ of S, where $\sigma|K$ denotes the restriction of σ to K [8], that is if $\sigma\cap(K\times K)=\{(a,a)\mid a\in K\}(:=i_K)$, then $\sigma=\{(a,a)\mid a\in S\}(:=i_S)$. For the necessary definitions and notations for semigroups (without order) we refer to [1,9,8].

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Let $(S, ., \leq)$ be an ordered semigroup. A nonempty subset A of S is called an *ideal* of S if (1) $AS \subseteq A$, $SA \subseteq A$ and (2) $a \in A$, $S \ni b \leq a$ implies $b \in A$ [2]. A nonempty subset T of S is called a *subsemigroup* of S if $a, b \in T$ implies $ab \in T$. Every subsemigroup T of S with the relation " $\leq_T$ " on T defined by $\leq_T := \{(a, b) \in T \times T \mid a \leq b\}$ is an ordered semigroup (called an ordered subsemigroup of S). Clearly, $\leq_T = \leq \cap (T \times T)$. In other words, if T is a subsemigroup of $(S, ., \leq)$, that is $T^2 = T$, then the set T with the multiplication "." and the order " $\leq$ " of S, that is $(T, ., \leq)$, is an ordered semigroup. Every ideal of $(S, ., \leq)$ is a subsemigroup of S, thus ideals are ordered semigroups. An ordered semigroup $(S, ., \leq)$ is said to be *weakly reductive* if $a, b \in S$ such that $ax \leq bx$ and $xa \leq xb$ for every $x \in S$ implies $a \leq b$. A relation σ on S is called *pseudoorder* if the following assertions are satisfied:

(1) $\leq \subseteq \sigma$

(2) $(a, b) \in \sigma$ and $(b, c) \in \sigma \Longrightarrow (a, c) \in \sigma$.

(3) $(a, b) \in \sigma \Longrightarrow (ac, bc) \in \sigma$ and $(ca, cb) \in \sigma \;\; \forall\, c \in S$.

The concept of pseudoorder, first introduced by Kehayopulu and Tsingelis in [4], plays an essential role in studying the structure of ordered semigroups. While for a semigroup S (without order) and a congruence σ of S, the quotient set S/σ is a semigroup, this is not the case for ordered semigroups, in general. We can overcome that difficulty using the concept of pseudoorder. Pseudoorder being a relation on an ordered semigroup S for which there exists a congruence σ on S such that S/σ is an ordered semigroup (see [4]) plays a basic role in studying the structure of ordered semigroups. The concept of pseudoorder for ordered semigroups is an extended form of the concept of congruence defined for semigroups. We recall that if σ is a congruence on a semigroup $(S, .)$ and endow S with the order $\leq := \{(x, y) \mid x = y\}$ (the equality relation) then σ is a pseudoorder on the ordered semigroup $(S, ., \leq)$. In a similar way, the concept of weakly reductive ordered semigroups extends the corresponding concept of weakly reductive semigroups, as each semigroup endowed with the equality relation is a weakly reductive ordered semigroup. Similarly, the notion of dense ideals of ordered semigroups extends the corresponding one for semigroups.

Let $(S, ., \leq)$ be an ordered semigroup. A mapping $\lambda : S \to S$ is called a *left translation* of S if

(1) $\lambda(xy) = \lambda(x)y \;\; \forall\, x, y \in S$.

(2) $x \leq y \;\Rightarrow\; \lambda(x) \leq \lambda(y) \;\; \forall\, x, y \in S$.

A mapping $\rho : S \to S$ is called a *right translation* of S if

(1) $\rho(xy) = x\rho(y) \;\; \forall\, x, y \in S$.

(2) $x \leq y \;\Rightarrow\; \rho(x) \leq \rho(y) \;\; \forall\, x, y \in S$ [5].

A left translation λ and a right translation ρ of S are said to be *linked* if $x\lambda(y) = \rho(x)y$ for each $x, y \in S$. In that case, the pair (λ, ρ) is called a *bitranslation* of S, and the set of (all) bitranslations of S, denoted by $\Omega(S)$, is called the *translational hull* of S [6].

Let $(S, ., \leq_S)$, $(T, \circ, \leq_T)$ be ordered semigroups and $f : S \to T$ a mapping of S into T. The mapping f is called *isotone* if $x \leq_S y$ implies $f(x) \leq_T f(y)$. f is called *reverse isotone* if $x, y \in S$, $f(x) \leq_T f(y)$ implies $x \leq_S y$. Clearly, each reverse isotone mapping is (1–1). The mapping f is called a *homomorphism* if it is isotone and satisfies $f(xy) = f(x) \circ f(y)$ for all $x, y \in S$. f is called an *isomorphism* if it is onto, homomorphism and reverse isotone. The ordered semigroups S and T are called *isomorphic*, in symbol $S \approx T$, if there exists an isomorphism between them. *S is embedded into T* if, by definition, S is isomorphic to a subset of T, i.e. if there exists a mapping $f : S \to T$ which is homomorphism and reverse isotone [5].

Lemma 1. [6] *Let $(S, ., \leq)$ be an ordered semigroup. The set $\Lambda(S)$ of left translations of S with the operation "." and the order " $\leq_\Lambda$ " on $\Lambda(S)$ defined by:*

$. \;:\; \Lambda(S) \times \Lambda(S) \to \Lambda(S) \mid (\lambda_1, \lambda_2) \to \lambda_1.\lambda_2$, *where*

$(\lambda_1.\lambda_2)(x) := \lambda_1(\lambda_2(x)) \;\; \forall\, x \in S$
$\lambda_1 \leq_\Lambda \lambda_2 \;\Leftrightarrow \lambda_1(x) \leq \lambda_2(x) \;\; \forall\, x \in S$
is an ordered semigroup.
The set $P(S)$ of right translations on S with the operation "." and the order " $\leq_P$ " on $P(S)$ defined by:
$. \,:\, P(S) \times P(S) \to P(S) \mid (\rho_1, \rho_2) \to \rho_1.\rho_2$, *where*
$(\rho_1.\rho_2)(x) := \rho_2(\rho_1(x)) \;\; \forall\, x \in S$
$\rho_1 \leq_P \rho_2 \;\Leftrightarrow \rho_1(x) \leq \rho_2(x) \;\; \forall\, x \in S$
is an ordered semigroup.
The set $\Omega(S)$ of bitranslations on S with the operation "." and the order " $\leq_\Omega$ " on $\Omega(S)$ defined by:
$. \,:\, \Omega(S) \times \Omega(S) \to \Omega(S) \mid ((\lambda_1, \rho_1), (\lambda_2, \rho_2)) \to (\lambda_1, \rho_1).(\lambda_2, \rho_2)$, *where*
$(\lambda_1, \rho_1).(\lambda_2, \rho_2) := (\lambda_1.\lambda_2, \rho_1.\rho_2)$
$(\lambda_1, \rho_1) \leq_\Omega (\lambda_2, \rho_2) \Leftrightarrow \lambda_1 \leq_\Lambda \lambda_2, \; \rho_1 \leq_P \rho_2$
is an ordered semigroup.

2. Main result

We use the concept of weakly reductive ordered semigroups introduced in [7] and the concept of dense ideals for ordered semigroups given in this paper. We prove that if S is an ordered semigroup and K a weakly reductive dense ideal of S, then S is embedded into the ordered semigroup of the translational hull of K. As a consequence one gets the corresponding result of semigroup (without order).

Definition 2. [7] An ordered semigroup $(S, ., \leq)$ is called *weakly reductive* if the following assertion is satisfied:
If $a, b \in S$ such that $ax \leq bx$ and $xa \leq xb$ for all $x \in S$, then $a \leq b$.

Remark 3. If S is an weakly reductive ordered semigroup and $a, b \in S$ such that $ax = bx$ and $xa = xb$ for all $x \in S$, then $a = b$.

Definition 4. Let $(S, ., \leq)$ be an ordered semigroup. An ideal K of S is said to be *dense* if for each pseudoorder σ on S, the relation $\sigma \cap (K \times K) \subseteq \leq \cap (K \times K)$ implies $\sigma \subseteq \leq$.
Equivalent Definitions:
(1) $\sigma \cap (K \times K) \subseteq \leq \cap (K \times K) \implies \sigma = \leq$.
(2) $\sigma \cap (K \times K) = \leq \cap (K \times K) \implies \sigma = \leq$.

Notation 5. Let S be an ordered semigroup, K an ideal of S and $s \in S$. Denote by λ^s (resp. ρ^s) the left (resp. right) translation of K defined by:

$$\lambda^s : K \to K \mid x \to \lambda^s(x) := sx$$

$$\rho^s : K \to K \mid x \to \rho^s(x) := xs.$$

Theorem 6. *Let $(S, ., \leq)$ be an ordered semigroup and K a dense ideal of S such that K is weakly reductive. Then $(S, ., \leq)$ is embedded into the ordered semigroup $(\Omega(K), ., \leq_\Omega)$.*

Proof. We consider the mapping:

$$f : (S, ., \leq) \to (\Omega(K), ., \leq_\Omega) \mid s \to (\lambda^s, \rho^s).$$

As the pair (λ^s, ρ^s) is a bitranslation on K, the mapping f is well defined.
1. The mapping f is a homomorphism. In fact: Let $s, t \in S$. Then

$$f(s).f(t) = (\lambda^s, \rho^s).(\lambda^t, \rho^t) = (\lambda^s.\lambda^t, \rho^s.\rho^t).$$

On the other hand, $\lambda^s.\lambda^t = \lambda^{st}$ and $\rho^s.\rho^t = \rho^{st}$. Indeed: If $a \in K$, then

$$\begin{aligned}(\lambda^s.\lambda^t)(a) &= \lambda^s(\lambda^t(a)) = \lambda^s(ta)\\ &= s(ta) \text{ (since } ta \in K)\\ &= (st)a = \lambda^{st}(a).\end{aligned}$$

If $a \in K$, then $as \in K$ and

$$(\rho^s.\rho^t)(a) = \rho^t(\rho^s(a)) = \rho^t(as) = (as)t = a(st) = \rho^{st}(a).$$

Thus we have $f(s).f(t) = (\lambda^s.\lambda^t, \rho^s.\rho^t) = (\lambda^{st}, \rho^{st}) = f(st)$. (Cf. also [8]).

Let now $s \leq t$. Then $f(s) \leq_\Omega f(t)$. In fact: First of all, $f(s) := (\lambda^s, \rho^s)$, $f(t) := (\lambda^t, \rho^t)$. On the other hand, $\lambda^s \leq_\Lambda \lambda^t$ and $\rho^s \leq_P \rho^t$. Indeed, if $a \in K$, then $\lambda^s(a) = sa \leq ta = \lambda^t(a)$ and $\rho^s(a) = as \leq at = \rho^t(a)$, thus we have $f(s) \leq_\Omega f(t)$.

2. The mapping f is reverse isotone: Let $s, t \in S$, $f(s) \leq_\Omega f(t)$. Then $s \leq t$. In fact: We consider the relation σ on S defined as follows:

$$\sigma := \{(a, b) \in S \times S \mid ax \leq bx \text{ and } xa \leq xb \;\; \forall\, x \in K\}.$$

2.1. σ is a pseudoorder on S. Indeed: If $(a, b) \in\leq$ and $x \in K$, then $ax \leq bx$ and $xa \leq xb$, thus $(a, b) \in \sigma$, so $\leq\subseteq \sigma$. Let $(a, b) \in \sigma$, $(b, c) \in \sigma$ and $x \in K$. Since $(a, b) \in \sigma$ and $x \in K$, $ax \leq bx$; since $(b, c) \in \sigma$ and $x \in K$, $bx \leq cx$, thus $ax \leq cx$. Similarly $xa \leq xc$, hence $(a, c) \in \sigma$. Let $(a, b) \in \sigma$, $c \in S$ and $x \in K$. Since $(a, b) \in \sigma$ and $cx \in SK \subseteq K$, we have $(ac)x \leq (bc)x$; since $(a, b) \in \sigma$ and $x \in K$, we have $xa \leq xb$, then $x(ac) \leq x(bc)$, hence $(ac, bc) \in \sigma$. Similarly, if $(a, b) \in \sigma$ and $c \in S$ then $(ca, cb) \in \sigma$.

2.2. $\sigma \cap (K \times K) \subseteq\leq \cap(K \times K)$. Indeed:
Let $(a, b) \in \sigma \cap (K \times K)$. Since $(a, b) \in \sigma$, we have $ax \leq bx$ and $xa \leq xb$ for all $x \in K$. Since K is weakly reductive, we get $a \leq b$. Hence $(a, b) \in\leq \cap(K \times K)$.

We are ready now to prove that $s \leq t$. By hypothesis, $f(s) \leq_\Omega f(t)$, thus $\lambda^s \leq_\Lambda \lambda^t$ and $\rho^s \leq_P \rho^t$. Let $x \in K$. Since $\lambda^s \leq_\Lambda \lambda^t$, we have $\lambda^s(x) \leq \lambda^t(x)$, so $sx \leq tx$. Since $\rho^s \leq_P \rho^t$, we have $\rho^s(x) \leq \rho^t(x)$, so $xs \leq xt$. As $sx \leq tx$ and $xs \leq xt$ for all $x \in K$, we have $(s, t) \in \sigma$. Since σ is a pseudoorder on S, $\sigma \cap (K \times K) \subseteq\leq \cap(K \times K)$ and K is a dense ideal of S, we have $\sigma \subseteq\leq$. Hence we have $(s, t) \in\leq$ i.e. $s \leq t$. □

Corollary 7. *Each ordered semigroup having a weakly reductive dense ideal K is isomorphic to a subsemigroup of $\Omega(K)$.*

Corollary 8. (cf. [8, Theorem 1.37]) *Each semigroup S having a weakly reductive dense ideal K is isomorphic to a subsemigroup of $\Omega(K)$.*

Proof. Let $(S, .)$ be a semigroup, K a weakly reductive ideal of S and σ a congruence on S such that $\sigma \cap (K \times K) = \{(a, a) \mid a \in K\}$ implies $\sigma = \{(a, a) \mid a \in S\}$. Then the semigroup S endowed with the order $\leq := \{(x, y) \in S \times S \mid x = y\}$ is an ordered semigroup, σ is a psudoorder on $(S, ., \leq)$ and K is a weakly reductive dense ideal of $(S, ., \leq)$. By Corollary 7, the ordered semigroup $(S, ., \leq)$ is isomorphic to a subsemigroup of $\Omega(K)$ (the isomorphism between ordered semigroups). As the reverse isotone mappings are (1–1), the semigroup $(S, .)$ is isomorphic to a subsemigroup of $\Omega(K)$ (the isomorphism between semigroups).

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