BCK-ALGEBRAS OF FRACTIONS

R. A. BORZOOEI AND J. SHOHANI

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Abstract. In this paper, by considering the notion of a $\wedge$-closed set in BCK-algebras, we construct the fractions of BCK-algebras and prove some related results. Moreover, we study the notion of a BCK-module and prove that any BCK-algebra is a BCK-module on itself. Finally, we construct the fractions of BCK-modules.

1. Introduction

A BCK-algebra is an important class of logical algebras introduced by Y. Imai and K. Iséki in 1966 [4, 5]. This notion is originated in two different ways: One is based on set theory; another comes from the classical and non-classical propositional calculi. As is well known there is a close relation between the notion of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship. What is the most essential and fundamental common properties? Can we establish a good theory of general algebras? To give an answer to these problems, Y. Imai and K. Iséki introduced the notion of a new class of general algebras called BCK- algebras. This name is taken from BCK-system of C. A. Meredith. The BCK-action was introduced by H. Abujabal, M. Aslam and A. B. Thaheem in 1994 [1] as an action of a BCK-algebra over a commutative group. This concept is extended by Z. Perveen, M. Aslam and A. B. Thaheem in 2006 [10], as a BCK-module. Now, in this paper we follow [10] and construct the fractions of BCK-algebras. Moreover, we prove that any BCK-algebras is a BCK-module on itself and we construct the fractions of BCK-modules. It should be noted that in this paper the main idea is to observe does the BCK-algebra has the ability to obtain and prove some well-known concepts and theorems in commutative algebra, especially fraction algebra, by use of the structure and characteristics of BCK-algebras.

2. Preliminaries

Definition 2.1. [7] A *BCK-algebra* is a set X with a binary operation "$*$" and a constant "0" satisfying the following axioms:
(BCK1) $((x * y) * (x * z)) * (z * y)) = 0$,
(BCK2) $(x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = 0$,
(BCK5) $x * y = y * x = 1$ imply $x = y$.

A BCK-algebra X is called *implicative* if $x * (y * x) = x$, *commutative* if $x * (x * y) = y * (y * x)$, *bounded* if there exists a unique element $1 \in X$ such that $x * 1 = 0$, for all $x, y, z \in X$.

Definition 2.2. [6] Let $(X, *, 0)$ be a BCK- algebra, I be a nonempty subset of X and $0 \in I$. Then I is called an ideal of X if $x * y \in I$ and $y \in I$ imply $x \in I$, for any $x, y \in X$. An ideal I is called *proper*, if $I \neq X$.

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Theorem 2.3. [9] *If X is a BCK-algebra and $\emptyset \neq A \subseteq X$, then the ideal generated by A (the intersection of all ideals of X containing A) will be denoted by $(A]$ and*

$$(A] = \{x \in X | \exists a_1, a_2, \cdots, a_n \in A \text{ such that } (\cdots((x * a_1) * a_2) * \cdots) * a_n = 0\}$$

Theorem 2.4. [9] *Any implicative BCK-algebra is a commutative BCK-algebra.*

Theorem 2.5. [9] *Let $(X, *, 0)$ be a bounded implicative BCK-algebra. Then $(X, \wedge, \vee, 0, 1)$ is a complimented distributive lattice and so Boolean algebra, where*

$$x \wedge y = y * (y * x) \quad , \quad x \vee y = N(Nx \wedge Ny)$$

and $Nx = 1 * x$, for any $x, y \in X$.

Theorem 2.6. [9] *Let X be a bounded implicative BCK-algebra. Then we have the following properties for all $x, y \in X$:*
(i) $NNx = x$,
(ii) $Nx \vee Ny = N(x \wedge y)$, $Nx \wedge Ny = N(x \vee y)$,
(iii) $Nx * Ny = y * x$,
(iv) $x \wedge Nx = 0$,
(v) $x \vee Nx = 1$,
(vi) $x * (x * Ny) = x * y$, i.e. $Ny \wedge x = x * y$,
(x) $x * (x * y) = x * Ny$, i.e. $y \wedge x = x * Ny$
(xi) $N0 = 1, N1 = 0$.

From now on, in this paper we let X to be a bounded implicative BCK-algebra. Note that these algebra is infact a Boolean algebra.

3. BCK-algebras of Fractions

In this section, by using techniques of BCK-algebras, we introduce and study the notion of fraction for bounded implicative BCK(Boolean)-algebras.

Definition 3.1. [9] Let S be a non empty subset of X. Then S is called $\wedge$-closed if $1 \in S$ and $x \wedge y \in S$, for all $x, y \in S$.

From now on, in this paper we let S to be a $\wedge$-closed subset of X.

Lemma 3.2. *If the relation " $\sim$ " on $X \times S$ is defined by:*

$$(x_1, t_1) \sim (x_2, t_2) \Longleftrightarrow \exists s \in S, s \wedge x_1 \wedge t_2 = s \wedge x_2 \wedge t_1,$$

then " $\sim$ " is an equivalence relation.

Proof. By Theorem 2.5, $(X, \wedge)$ is a $\wedge$-semi lattice. Hence, we prove the equivalence relation properties. Reflexive and symmetric properties are clear.

Transitive property:
Let $(x_1, t_1) \sim (x_2, t_2)$ and $(x_2, t_2) \sim (x_3, t_3)$. Then there exist $s_1, s_2 \in S$ such that $s_1 \wedge x_1 \wedge t_2 = s_1 \wedge x_2 \wedge t_1$ and $s_2 \wedge x_2 \wedge t_3 = s_2 \wedge x_3 \wedge t_2$. Hence,

$$s_1 \wedge s_2 \wedge t_2 \wedge t_3 \wedge x_1 = s_1 \wedge s_2 \wedge t_1 \wedge t_3 \wedge x_2$$

and

$$s_1 \wedge s_2 \wedge t_1 \wedge t_3 \wedge x_2 = s_1 \wedge s_2 \wedge t_1 \wedge t_2 \wedge x_3$$

and so

$$s_1 \wedge s_2 \wedge t_2 \wedge t_3 \wedge x_1 = s_1 \wedge s_2 \wedge t_1 \wedge t_2 \wedge x_3$$

Now, let $s' = s_1 \wedge s_2 \wedge t_2 \in S$. Hence $s' \wedge x_1 \wedge t_3 = s' \wedge x_3 \wedge t_1$, and this implies that $(x_1, t_1) \sim (x_3, t_3)$. □

Notation. From now on, for each element $(x, s) \in X \times S$, the class $[(x, s)]$ will be denoted by x/s and the set $X \times S/\sim$ by $S^{-1}X$. Hence we have $S^{-1}X = \{x/s : x \in X, s \in S\}$.

Corollary 3.3. *(i) $x/t = y/s$, for any $x/t, y/s \in S^{-1}X$, if and only if there exists $s' \in S$ such that $s' \wedge x \wedge s = s' \wedge y \wedge t$,*
(ii) $0/s = 0/t$, for any $t, s \in S$.(So, for any $s \in S$, $0/s$ will be denoted by $0_{S^{-1}X}$),
(iii) $x/t = 0_{S^{-1}X}$ if and only if there exists $l \in S$ such that $x \wedge l = 0$,
(iv) $0 \in S$ if and only if $S^{-1}X = \{0_{S^{-1}X}\}$.

Proof. The proofs of (i), (ii) and (iv) are clear.
(iii) Let $x/t = 0_{S^{-1}X}$, for $x \in X$ and $t \in S$. By (ii), we can let $0_{S^{-1}X} = 0/t$. Then $x/t = 0/t$ and so there exists $s \in S$ such that $x \wedge t \wedge s = 0$. Let $l = t \wedge s$. Hence, there is $l \in S$, such that $x \wedge l = 0$. □

Lemma 3.4. *For any $x, y, z \in X$, we have;*
(i) $x \wedge (y * z) = (x \wedge y) * (x \wedge z)$,
(ii) $x * (x \wedge y) = x * y$.

Proof. (i) Let $x, y, z \in X$. Then by Theorem 2.6,

$$\begin{aligned} (x \wedge y) * (x \wedge z) &= (x \wedge y) \wedge N(x \wedge z), \\ &= (x \wedge y) \wedge (Nx \vee Nz) \\ &= [(x \wedge y) \wedge Nx] \vee [(x \wedge y) \wedge Nz] \quad , \quad \text{(by Theorem 2.5)} \\ &= [y \wedge (x \wedge Nx)] \vee [(x \wedge y) \wedge Nz] \\ &= [y \wedge 0)] \vee [(x \wedge y) \wedge Nz] \\ &= 0 \vee [(x \wedge y) \wedge Nz] \\ &= (x \wedge y) \wedge Nz, \\ &= x \wedge (y \wedge Nz), \\ &= x \wedge (y * z). \end{aligned}$$

(ii) Since X is implicative,

$$x * (x \wedge y) = x * (x * (x * y)) = x * y$$

□

Theorem 3.5. *If the binary relation "$\star$" on $S^{-1}X$ is defined by*

$$x/t \star y/s = [(x \wedge s) * (y \wedge t)]/(t \wedge s)$$

then $(S^{-1}X, \star, 0_{S^{-1}X})$ is a bounded implicative BCK-algebra.

Proof. First we show that "$\star$" is well defined. Let $a/s = a'/s'$ and $b/t = b'/t'$. Then there exist $u, v \in S$ such that

$$u \wedge s' \wedge a = u \wedge s \wedge a' \quad , \quad v \wedge t' \wedge b = v \wedge t \wedge b'$$

and so

$$v \wedge t \wedge t' \wedge u \wedge s' \wedge a = v \wedge t \wedge t' \wedge u \wedge s \wedge a'$$

and

$$s \wedge s' \wedge u \wedge v \wedge t' \wedge b = s \wedge s' \wedge u \wedge v \wedge t \wedge b'$$

So, we conclude that

$$[(u \wedge v) \wedge (s' \wedge t') \wedge (t \wedge a)] = [(u \wedge v) \wedge (s \wedge t) \wedge (a' \wedge t')]$$

and

$$[(u \wedge v) \wedge (s' \wedge t') \wedge (s \wedge b) = [(u \wedge v) \wedge (s \wedge t) \wedge (s' \wedge b')]$$

Hence,

$[(u\wedge v)\wedge(s'\wedge t')\wedge(t\wedge a)]*[(u\wedge v)\wedge(s'\wedge t')\wedge(s\wedge b)] = [(u\wedge v)\wedge(s\wedge t)\wedge(a'\wedge t')]*[(u\wedge v)\wedge(s\wedge t)\wedge(s'\wedge b')]$

Then, by Lemma 3.4, we have,

$$[(u \wedge v) \wedge (s' \wedge t')] \wedge [(t \wedge a) * (s \wedge b)] = [(u \wedge v) \wedge (s \wedge t)] \wedge [(a' \wedge t') * (s' \wedge b')]$$

Now, let $s'' = u \wedge v \in S$. Hence, $s'' \in S$ and

$$s'' \wedge (s' \wedge t') \wedge [(t \wedge a) * (s \wedge b)] = s'' \wedge (s \wedge t) \wedge [(a' \wedge t') * (s' \wedge b')]$$

and this implies that

$$[(t \wedge a) * (s \wedge b)]/(s \wedge t) = [(a' \wedge t') * (s' \wedge b')]/(s' \wedge t')$$

Hence,

$$a/s \star b/t = a'/s' \star b'/t'$$

Therefore, "$\star$" is well-defined. Now, we show that $(S^{-1}X, \star, 0_{S^{-1}X})$ is a BCK-algebra. Let $a/s, b/t, d/f \in S^{-1}X$. Then:

(BCK1):

$$\begin{aligned}
& [(a/s \star b/t) \star (a/s \star d/f)] \star (d/f \star b/t) \\
= \ & [[((a \wedge t) * (b \wedge s))/(s \wedge t)] \star [((a \wedge f) * (s \wedge d))/(s \wedge f)]] \star [((d \wedge t) * (b \wedge f))/(f \wedge t)]] \\
= \ & [[[(s \wedge f) \wedge ((a \wedge t) * (b \wedge s))] * [(s \wedge t) \wedge ((a \wedge f) * (s \wedge d))]]/(s \wedge t \wedge f)] \star [((d \wedge t) * (b \wedge f))/(f \wedge t)] \\
= \ & [[(f \wedge t) \wedge [[(s \wedge f) \wedge ((a \wedge t) * (b \wedge s))] * [(s \wedge t) \wedge ((a \wedge f) * (s \wedge d))]]] \\
& *[[(s \wedge t \wedge f) \wedge [(d \wedge t) * (b \wedge f)]]/(s \wedge t \wedge f)] \\
= \ & [[(f \wedge t) \wedge [[((s \wedge f) \wedge (a \wedge t)) * ((s \wedge f) \wedge (b \wedge s))] * [((s \wedge t) \wedge (a \wedge f)) * ((s \wedge t) \wedge (s \wedge d))]]] \\
& *[((s \wedge t \wedge f) \wedge (d \wedge t)) * ((s \wedge t \wedge f) \wedge (b \wedge f))]]/(s \wedge t \wedge f) \\
= \ & [[[(f \wedge t) \wedge ((s \wedge f) \wedge (a \wedge t))] * [(f \wedge t) \wedge ((s \wedge f) \wedge (b \wedge s))]] \\
& *[[(f \wedge t) \wedge ((s \wedge t) \wedge (a \wedge f))] * [(f \wedge t) \wedge ((s \wedge t) \wedge (s \wedge d))]] \\
& *[((s \wedge t \wedge f) \wedge (d \wedge t)) * ((s \wedge t \wedge f) \wedge (b \wedge f))]]/(s \wedge t \wedge f) \\
= \ & [[[(f \wedge t \wedge s \wedge a) * (f \wedge t \wedge s \wedge b)] * [(f \wedge t \wedge s \wedge a) * (f \wedge t \wedge s \wedge d)]] \\
& *[(f \wedge t \wedge s \wedge d) * (f \wedge t \wedge s \wedge b)]]/(s \wedge t \wedge f) \\
= \ & [[[(l \wedge a) * (l \wedge b)] * [(l \wedge a) * (l \wedge d)]] * [(l \wedge d) * (l \wedge b)]]/(s \wedge t \wedge f) \quad , \quad (\text{ if } l = f \wedge t \wedge s) \\
= \ & 0/(s \wedge t \wedge f) \quad , \quad (\text{by (BCK1)}) \\
= \ & 0_{S^{-1}X}
\end{aligned}$$

(BCK2):

$$\begin{aligned}
[a/s \star (a/s \star b/t)] \star b/t &= [a/s \star ((a \wedge t) * (s \wedge b))/(s \wedge t)] \star (b/t) \\
&= [(a \wedge s \wedge t) * (s \wedge ((a \wedge t) * (s \wedge b)))/(s \wedge t)] \star (b/t) \\
&= [((a \wedge s \wedge t) * ((s \wedge a \wedge t) * (s \wedge b)))/(s \wedge t)] \star (b/t) \quad , \quad (\text{by Lemma 3.4}) \\
&= [(t \wedge [(a \wedge s \wedge t) * ((s \wedge a \wedge t) * (s \wedge b))]) * (b \wedge s \wedge t)]/(s \wedge t) \\
&= [[(a \wedge s \wedge t) * ((s \wedge a \wedge t) * (s \wedge b \wedge t))] * (b \wedge s \wedge t)/(s \wedge t) \quad , \quad (\text{by Lemma 3.4}) \\
&= 0/(s \wedge t) \\
&= 0_{S^{-1}X}
\end{aligned}$$

(BCK3):

$$a/s \star a/s = ((a \wedge s) * (a \wedge s))/s = 0/s = 0_{S^{-1}X}$$

(BCK4):

$$0_{S^{-1}X} \star a/s = 0/s \star a/s = [(0 \wedge s) * (a \wedge s)]/s = 0/s = 0_{S^{-1}X}$$

(BCK5): If $a/s \star b/t = 0_{S^{-1}X}$ and $b/t \star a/s = 0_{S^{-1}X}$, then

$$((a \wedge t) * (s \wedge b))/(s \wedge t) = 0_{S^{-1}X} \quad , \quad ((b \wedge s) * (a \wedge t))/(s \wedge t) = 0_{S^{-1}X}$$

Hence, there exist $l, l' \in S$, such that

$$l \wedge [(a \wedge t) * (s \wedge b)] = 0 \quad , \quad l' \wedge [(b \wedge s) * (a \wedge t)] = 0$$

and so

$$(l \wedge l') \wedge ((a \wedge t) * (s \wedge b)) = 0 \quad (l \wedge l') \wedge ((b \wedge s) * (a \wedge t)) = 0$$

Let $h = l \wedge l'$. Then, by Lemma 3.4,

$$(h \wedge (a \wedge t)) * (h \wedge (s \wedge b)) = 0 \quad , \quad (h \wedge (b \wedge s)) * (h \wedge (a \wedge t)) = 0$$

Hence by (BCK5), $h \wedge (a \wedge t) = h \wedge (s \wedge b)$ and so $(h \wedge t) \wedge a = (h \wedge s) \wedge b$ and this implies that $a/s = b/t$.

Therefore, $(S^{-1}X, \star, 0_{S^{-1}X})$ is a BCK-algebra. Now, we prove that it is implicative. Let $x/t, y/s \in S^{-1}X$. Then,

$$\begin{aligned}
x/t \star ((y/s) \star (x/t)) &= x/t \star [((y \wedge t) * (x \wedge s))/(s \wedge t)], \\
&= [(x \wedge s \wedge t) * (t \wedge [(y \wedge t) * (x \wedge s)])]/(s \wedge t) \\
&= [(x \wedge s \wedge t) * [(t \wedge y) * (x \wedge s \wedge t)]]/(s \wedge t) \quad \text{(by Lemma 3.4)} \\
&= (x \wedge s \wedge t)/(s \wedge t) \quad , \quad \text{(since } X \text{ is implicative)} \\
&= x/t.
\end{aligned}$$

So, $S^{-1}X$ is implicative. Finally, we show that it is bounded. In fact, we claim that $1/1$ is an upper bound of $S^{-1}X$. For this, let $x/s \in S^{-1}X$. Then, by the definition of "$\wedge$" and (BCK3), we have;

$$x/s \star s/s = ((x \wedge s) * (s \wedge s))/(s \wedge s) = ((x \wedge s) * s)/s = ((s \wedge x) * s)/s = ((x * (x * s)) * s)/s = 0/y = 0_{S^{-1}X}$$

Now, it is easy to see that, for any $s \in S$, $s/s = 1/1$. Hence, for any $x/s \in S^{-1}X$, $x/s \star 1/1 = 0_{S^{-1}X}$.

Therefore, $(S^{-1}X, \star, 0_{S^{-1}X})$ is a bounded implicative BCK-algebra. $\square$

Corollary 3.6. *If the relation "$\preceq$" on $S^{-1}X$ is defined by:*

$$x/t \preceq y/s \Longleftrightarrow x/t \star y/s = 0_{S^{-1}X}$$

then $(S^{-1}X, \preceq)$ is a poset.

Proof. By Theorem 3.5, $(S^{-1}X, \star)$ is a BCK-algebra and so $(S^{-1}X, \preceq)$ is a poset(See [5]). $\square$

Lemma 3.7. *For any $x, y \in X$ and $s, t \in S$, we have;*
(i) $(x \wedge s)/s = x/s$,
(ii) $(x \wedge t)/(s \wedge t) = x/s$,
(iii) $x/t \wedge y/s = (x \wedge y)/(t \wedge s)$.

Proof. (i) Since for any $t \in S$, $x \wedge s \wedge s \wedge t = x \wedge s \wedge t$, so $(x \wedge s)/s = x/s$.

(ii) The proof is similar to (i).

(iii) Let $x/t, y/s \in S^{-1}X$. Since $(S^{-1}X, \star)$ is a BCK-algebra, $x/t \wedge y/s = y/s \star (y/s \star x/t)$. Hence,

$$\begin{aligned}
x/t \wedge y/s &= y/s \star (y/s \star x/t) \\
&= y/s \star [((y \wedge t) * (x \wedge s))/(s \wedge t)] \\
&= ((y \wedge s \wedge t) * (((y \wedge t) * (x \wedge s)) \wedge s))/(s \wedge t) \\
&= ((x \wedge s) \wedge (y \wedge s \wedge t))/(s \wedge t) \\
&= ((x \wedge t) \wedge (s \wedge y))/(s \wedge t) \\
&= (x \wedge y)/(s \wedge t)
\end{aligned}$$

□

Lemma 3.8. *If I is an ideal of X, then for any $x \in I$ and $y \in X$, $x \wedge y \in I$.*

Theorem 3.9. *If I be an ideal of X, then $S^{-1}I = \{x/s \in S^{-1}X : x \in I\}$ is an ideal of $S^{-1}X$. Moreover, $S^{-1}I$ is proper if and only if $I \cap S = \emptyset$.*

Proof. Let $x/t, y/s \in S^{-1}X$ such that $x/t \star y/s \in S^{-1}I$ and $y/s \in S^{-1}I$. Then, there exist $a, b \in I$ and $u, v \in S$ such that $x/t \star y/s = a/u$ and $y/s = b/v$ and so there exist $h, h' \in S$ such that $h \wedge u \wedge ((x \wedge s) * (y \wedge t)) = h \wedge t \wedge s \wedge a$ and $h' \wedge y \wedge v = h' \wedge s \wedge b$. But, by Lemma 3.4 and some modifications,

$$(h \wedge u \wedge h' \wedge v \wedge x \wedge s) * (h \wedge u \wedge h' \wedge v \wedge y \wedge t) = h \wedge t \wedge s \wedge h' \wedge v \wedge a \ , \quad (1)$$

and

$$h' \wedge y \wedge v \wedge h \wedge u \wedge t = h \wedge s \wedge b \wedge h' \wedge u \wedge t$$

Let $k = h \wedge h' \wedge v \wedge u$. Since $h \wedge h' \wedge v \wedge t \wedge s \wedge a \leq a \in I$ so $h \wedge h' \wedge v \wedge t \wedge s \wedge a \in I$ and so by (1),

$$(k \wedge s \wedge x) * (k \wedge y \wedge t) \in I$$

Now, since $k \wedge y \wedge t = h \wedge s \wedge b \wedge h' \wedge u \wedge t \leq b \in I$, then $k \wedge y \wedge t \in I$ and so $k \wedge s \wedge x \in I$. Hence, $x/t = (k \wedge s \wedge x)/(k \wedge s \wedge t) \in S^{-1}I$. Therefore, $S^{-1}I$ is an ideal of $S^{-1}X$.

Now, let $S^{-1}I$ be proper, but $t \in I \cap S \neq \emptyset$, one the contrary. Let $x/s \in S^{-1}X$. Then by Lemmas 3.7 and 3.8, $x/s = (x \wedge t)/(s \wedge t) \in S^{-1}X$. Hence $S^{-1}X = S^{-1}I$, which is impossible. Moreover, let $I \cap S = \emptyset$, but $S^{-1}X = S^{-1}I$, one the contrary. Since $1/1 \in S^{-1}X$, then $1/1 \in S^{-1}I$ and so there exists $a \in I$ and $s \in S$ such that $1/1 = a/s$ and so there exists $t \in S$ such that $1 \wedge s \wedge t = 1 \wedge a \wedge t$. Hence, $s \wedge t = a \wedge t$. Since $s \wedge t \in S$ and by Lemma 3.8, $s \wedge t = a \wedge t \in I$. Hence $s \wedge t \in S \cap I = \emptyset$, which is impossible. Therefore, $S^{-1}I$ is proper. □

Definition 3.10. [9] A proper ideal P of X is called *prime* if $a \wedge b \in P$ implies $a \in P$ or $b \in P$, for any $a, b \in P$.

Theorem 3.11. *If J is an ideal of $S^{-1}X$, then there exists an ideal I of X such that $J = S^{-1}I$. Moreover, if J is a prime ideal then I is a prime ideal, too, and $I \cap S = \emptyset$.*

Proof. Let J be an ideal of $S^{-1}X$ and $I = \{x \in X : x/1 \in J\}$. First, we show that I is an ideal of X. Let $x * y \in I$ and $y \in I$. Then $x/1 \star y/1 = (x * y)/1 \in J$ and $y/1 \in J$. Since J is an ideal, then $x/1 \in J$ and so $x \in I$. Hence, I is an ideal of X. Now, let $x/t \in J$. Since $t/1 \in S^{-1}X$ then by Lemma 3.8, $x/t \wedge t/1 \in J$. Since, by Lemma 3.7, $x/t \wedge y/t = (x \wedge t)/t = x/1$, then $x/1 \in J$ and so $x \in I$. Hence $x/t \in S^{-1}I$. Therefore, $J \subseteq S^{-1}I$. Now, let $x/t \in S^{-1}I$. Hence there exist $a \in I$ and $s \in S$ such that $x/t = a/s$. Since $a/1 \in J$ and $1/s \in S^{-1}X$, then by Lemma 3.8, $a/s = a/1 \wedge 1/s \in J$ and so $x/t \in J$. Hence, $S^{-1}I \subseteq J$. Therefore, $J = S^{-1}J$. Now, let J be prime. Then $J = S^{-1}I$ is proper and so by Theorem 3.9, $I \cap S = \emptyset$. Now, let $x \wedge y \in I$, for $x, y \in X$. Then $x/1 \wedge y/1 = (x \wedge y)/1 \in S^{-1}I$. Since $S^{-1}I$ is prime, then $x/1 \in S^{-1}I$ or $y/1 \in S^{-1}I$ and so by definition of I, $x \in I$ or $y \in I$. Hence, I is a prime ideal. □

Theorem 3.12. *If P is a prime ideal of X such that $P \cap S = \emptyset$, then $S^{-1}P$ is a prime ideal of $S^{-1}X$.*

Proof. Since P is an ideal of $S^{-1}X$, then by Theorem 3.9, $S^{-1}P$ is an ideal of $S^{-1}X$. Now, first we show that $S^{-1}P$ is proper. Let $S^{-1}P = S^{-1}X$, on the contrary. Since $1/1 \in S^{-1}X = S^{-1}P$, then there exist $s \in S$ and $p \in P$ such that $1/1 = p/s$ and so there exists $t \in S$ such that $t \wedge s = p \wedge t$. Since $p \wedge t \leq p \in P$ and P is an ideal, $t \wedge s = p \wedge t \in P$. Moreover, since $t \wedge s \in S$, then $t \wedge s \in P \cap S = \emptyset$, which is a contradiction. Hence, $S^{-1}P \neq S^{-1}X$. Now, let $x/t \wedge y/s \in S^{-1}P$, for $x/t, y/s \in S^{-1}X$. By Lemma 3.7, $x/t \wedge y/s = (x \wedge y)/(t \wedge s)$. Hence, $(x \wedge y)/(t \wedge s) \in S^{-1}P$ and so there exist $q \in P$ and $r \in S$ such that $(x \wedge y)/(t \wedge s) = q/r$ and this means that there exists

$h \in S$ such that $h \wedge r \wedge x \wedge y = h \wedge t \wedge s \wedge q$. Since $q \in P$, $h \wedge t \wedge s \wedge q \leq q$ and P is an ideal, then $h \wedge r \wedge x \wedge y = h \wedge t \wedge s \wedge q \in P$. Now, since P is prime and $h \wedge r \notin P$, then $x \wedge y \in P$ and so $x \in P$ or $y \in P$. Hence, $x/t \in S^{-1}P$ or $y/s \in S^{-1}P$. Therefore, $S^{-1}P$ is a prime ideal of $S^{-1}X$. $\square$

Lemma 3.13. *Let $f : (X, *) \longrightarrow (S^{-1}X, \star)$ be defined by $f(x) = x/1$. Then;*
(i) f is a BCK-homomorphism(that is; $f(0) = 0_{S^{-1}(X)}$ and $f(x*y) = f(x)\star f(y)$ for any $x, y \in X$,
(ii) If I is an ideal of X, then $I^e[(f(I)] = S^{-1}I$;
(iii) If J is an ideal of $S^{-1}X$, then there exists an ideal I of X such that $J^c[f^{-1}(J)] = I$ and $J = S^{-1}I$.

Proof. (i) The proof is clear.

(ii) Since $f(I) = \{x/1 : x \in I\} \subseteq S^{-1}I$ and by Theorem 3.9, $S^{-1}I$ is an ideal of $S^{-1}X$, then $I^e = [f(I)] \subseteq S^{-1}I$. Now, let $x/s \in S^{-1}I$. Hence, there exists $a \in I$ and $t \in S$ such that $x/s = a/t$. Since $t \wedge a \leq t$, then by Lemma 3.4(i),

$$(a * t) \wedge t = t \wedge (a * t) = (t \wedge a) * (t \wedge t) = (t \wedge a) \wedge t = 0$$

and so by Corollary 3.3(iii), $(a * t)/t = 0_{S^{-1}X}$. Now, by Lemma 3.4(ii) and since $a/1 \in f(I)$, then

$$x/s \star a/1 = a/t \star a/1 = ((a \wedge 1) * (a \wedge t))/(t \wedge 1) = (a * (a \wedge t))/t = (a * t)/t = 0_{S^{-1}X}$$

Hence, $x/s \in (f(I)] = I^e$ and so $S^{-1}I \subseteq I^e$. Therefore, $S^{-1}I = I^e$.

(iii) Let J be an ideal of $S^{-1}X$. Let $I = \{x \in X : x/1 \in J\}$. Then by Theorem 3.11, I is an ideal of X, $J = S^{-1}I$ and $I = \{x \in X : f(x) \in J\} = f^{-1}(J) = J^c$. $\square$

Theorem 3.14. *Let $Spec(X)$ be the set of all prime ideals of X, $A = \{P \in Spec(X) | P \cap S = \emptyset\}$ and $B = \{J | J \in Spec(S^{-1}X)\}$. Then $A \cong B$.*

Proof. Let $\varphi : A \longrightarrow B$ be defined by $\varphi(P) = P^e$ and $\psi : B \longrightarrow A$ be defined by $\psi(J) = J^c$. By Theorems 3.11 and 3.12 and Lemma 3.13, φ and ψ are well-defined. Now, let $P \in A$. By Lemma 3.13(ii), $P^e = S^{-1}P$ and $(S^{-1}P)^c = P$. Hence, $P^{ec} = P$. Moreover, if $J \in B$, then by Theorem 3.9 and Lemma 3.13, $J = S^{-1}P$ such that $J^c = P$. Hence, $J^{ce} = p^e = S^{-1}P = J$. Thus, $\varphi \circ \psi(J) = \phi(J^c) = J^{ce} = J$ and $\psi \circ \varphi(P) = \psi(P^e) = P^{ec} = P$. Therefore, $A \cong B$. $\square$

4. BCK-modules of fractions

Definition 4.1. [1] Let $(X, *, 0)$ be a BCK-algebra, $(M, +)$ be an Abelian group and $\cdot : X \times M \longrightarrow M$ with $(x, m) \longrightarrow x \cdot m$ be an operation such that:
(i) $(x \wedge y) \cdot m = x \cdot (y \cdot m)$,
(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
(iii) $0 \cdot m = 0$.
for all $x, y \in X$ and $m, m_1, m_2 \in M$, where $x \wedge y = y * (y * x)$. Then M is called a *left X-module*. Similarly, we can define a *right X-module*.

Note. If X is a commutative BCK-algebra, then the notions of a left X-module and a right X-module quinsied and so, for simplicity, we use of X-module instead of the left X-module. It is clear that in any left X-module M, $s \cdot (-m) = -(s \cdot m)$, for any $s \in S$ and $m \in M$.

Proposition 4.2. *Let $(X, *, 0)$ be bounded implicative algebra, if the operation $+ : X \times X \longrightarrow X$ is defined as follows:*

$$x + y = (x * y) \vee (y * x)$$

then $M = (X, +)$ is an Abelian group and M is an X-module. Infact, X is an X-module on itself.

Proof. First, we prove that $(X, +)$ is an Abelian group.

(i) Associative law:

Let $x, y, z \in X$. First we prove the following identity:

$$x \wedge (Ny \vee z) \wedge (y \vee Nz) = (x \wedge Ny \wedge Nz) \vee (x \wedge z \wedge y) \tag{1}$$

For this, by Theorems 2.5 and 2.6, we have:

$$\begin{aligned}
x \wedge (Ny \vee z) \wedge (y \vee Nz) &= [(x \wedge Ny) \vee (x \wedge z)] \wedge (y \vee Nz)\\
&= (y \vee Nz) \wedge [(x \wedge Ny) \vee (x \wedge z)]\\
&= [(y \vee Nz) \wedge (x \wedge Ny)] \vee [(y \vee Nz) \wedge (x \wedge z)]\\
&= [(x \wedge Ny) \wedge (y \vee Nz)] \vee [(x \wedge z) \wedge (y \vee Nz)]\\
&= [(x \wedge Ny \wedge y) \vee (x \wedge Ny \wedge Nz)] \vee [(x \wedge z \wedge y) \vee (x \wedge z \wedge Nz)]\\
&= [(x \wedge 0) \vee (x \wedge Ny \wedge Nz)] \vee [(x \wedge z \wedge y) \vee (x \wedge 0)]\\
&= [0 \vee (x \wedge Ny \wedge Nz)] \vee [(x \wedge z \wedge y) \vee 0]\\
&= (x \wedge Ny \wedge Nz) \vee (x \wedge z \wedge y)
\end{aligned}$$

Hence we have (1). Moreover, we should prove the following identity:

$$(x \wedge y) \vee (Nx \wedge Ny) = (Nx \vee y) \wedge (Ny \vee x) \tag{2}$$

For this, by Theorems 2.5 and 2.6, we have:

$$\begin{aligned}
(x \wedge y) \vee (Nx \wedge Ny) &= [(x \wedge y) \vee Nx] \wedge [(x \wedge y) \vee Ny]\\
&= [Nx \vee (x \wedge y)] \wedge [Ny \vee (x \wedge y)]\\
&= [(Nx \vee x) \wedge (Nx \vee y)] \wedge [(Ny \vee x) \wedge (Ny \vee y)]\\
&= [1 \wedge (Nx \vee y)] \wedge [(Ny \vee x) \wedge 1]\\
&= (Nx \vee y) \wedge (Ny \vee x)
\end{aligned}$$

Hence we have (2).

Now, by Theorem 2.6,(i),(ii) and (vi) we have:

$$\begin{aligned}
x + (y + z) &= x + [(y * z) \vee (z * y)]\\
&= (x * [(y * z) \vee (z * y)]) \vee ([(y * z) \vee (z * y)] * x)\\
&= [x \wedge N[(y \wedge Nz) \vee (z \wedge Ny)]] \vee [[(y \wedge Nz) \vee (z \wedge Ny)] \wedge Nx]\\
&= [x \wedge [N(y \wedge Nz) \wedge N(z \wedge Ny)]] \vee [Nx \wedge [(y \wedge Nz) \vee (z \wedge Ny)]]\\
&= [x \wedge [(Ny \vee z) \wedge (y \vee Nz)]] \vee [[Nx \wedge (y \wedge Nz)] \vee (Nx \wedge (Ny \wedge z))]\\
&= [x \wedge Ny \wedge Nz] \vee [x \wedge y \wedge z] \vee [Nx \wedge y \wedge Nz] \vee [Nx \wedge Ny \wedge z] \quad \text{, (by (1))}\\
&= [x \wedge Ny \wedge Nz] \vee [Nx \wedge y \wedge Nz] \vee [x \wedge y \wedge z] \vee [Nx \wedge Ny \wedge z]\\
&= [[(x \wedge Ny) \vee (Nx \wedge y)] \wedge Nz] \vee [[(x \wedge y) \vee (Nx \wedge Ny)] \wedge z]\\
&= [[(x * y) \vee (y * x)] \wedge Nz] \vee [[(Nx \vee y) \wedge (Ny \vee x)] \wedge z] \quad \text{,(by (2))}\\
&= [[(x * y) \vee (y * x) \wedge Nz] \vee [N[(x \wedge Ny) \vee (y \wedge Nx)] \wedge z]\\
&= [[(x * y) \vee (y * x)] \wedge Nz] \vee [N[(x * y) \vee (y * x)] \wedge z]\\
&= [[(x * y) \vee (y * x)] * z] \vee [z * [(x * y) \vee (y * x)]]\\
&= (x + y) + z.
\end{aligned}$$

Therefore, we have the associative law.

(ii) Identity element: Let $x \in X$. Then, by Theorem 2.6,

$$x + 0 = (x * 0) \vee (0 * x) = x \vee 0 = N(Nx \wedge N0) = N(Nx \wedge 1) = NNx = x$$

(iii) <u>Inverse element</u>: Let $x \in X$. We claim that x is an inverse of x, since

$$x + x = (x * x) \vee (x * x) = 0 \vee 0 = 0$$

(iv) <u>Abelian law</u>: Let $x, y \in X$. Then

$$x + y = (x * y) \vee (y * x) = (y * x) \vee (x * y) = y + x$$

Hence $M = (X, +)$ is an Abelian group.

Now, we show that M is an X- module. For this we define the operation $\cdot : X \times M \longrightarrow M$ by $x \cdot m = x \wedge m$. Hence:

$$(i): \quad (x \wedge y) \cdot m = (x \wedge y) \wedge m = x \wedge (y \wedge m) = x \wedge (y \cdot m) = x \cdot (y \cdot m)$$

$$\begin{aligned}
(ii) : x \cdot (m_1 + m_2) &= x \wedge (m_1 + m_2) \\
&= x \wedge [(m_1 * m_2) \vee (m_2 * m_1)] \\
&= x \wedge [(m_1 \wedge Nm_2) \vee (m_2 \wedge Nm_1)] \\
&= [x \wedge (m_1 \wedge Nm_2)] \vee [x \wedge (m_2 \wedge Nm_1)] \\
&= [(x \wedge m_1) \wedge Nm_2] \vee [(x \wedge m_2) \wedge Nm_1] \\
&= [0 \vee ((x \wedge m_1) \wedge Nm_2)] \vee [0 \vee ((x \wedge m_2) \wedge Nm_1)] \\
&= [(x \wedge Nx \wedge m_1) \vee ((x \wedge m_1) \wedge Nm_2)] \vee [(x \wedge Nx \wedge m_2) \vee ((x \wedge m_2) \wedge Nm_1)] \\
&= [[(x \wedge m_1) \wedge Nx] \vee [(x \wedge m_1) \wedge Nm_2]] \vee [[(x \wedge m_2) \wedge Nx] \vee [(x \wedge m_2) \wedge Nm_1]] \\
&= [(x \wedge m_1) \wedge (Nx \vee Nm_2)] \vee [(x \wedge m_2) \wedge (Nx \vee Nm_1)] \\
&= [(x \wedge m_1) \wedge N(x \wedge m_2)] \vee [(x \wedge m_2) \wedge N(x \wedge m_1)] \\
&= [(x \wedge m_1) * (x \wedge m_2)] \vee [(x \wedge m_2) * (x \wedge m_1)] \\
&= (x \wedge m_1) + (x \wedge m_2) \\
&= x \cdot m_1 + x \cdot m_2.
\end{aligned}$$

(iii)

$$0 \cdot m = 0 \wedge m = 0 \wedge m = m * (m * 0) = m * m = 0$$

for any $m \in M$.

Therefore, M is an X-module. ☐

Lemma 4.3. *Let M be an X-module and the relation "$\sim$" on $M \times S$ be defined by:*

$$(m, s) \sim (m', s') \Longleftrightarrow \exists t \in S \ s.t, \ t \cdot (s' \cdot m - s \cdot m') = 0$$

Then "$\sim$" is an equivalence relation.

Proof. The proof is straightforward. ☐

Notation. From now on, for each element $(m, s) \in M \times S$, the class $[(m, s)]$ will be denoted by m/s and the set $M \times S/\sim$ will be denoted by $S^{-1}M$. Hence we have $S^{-1}M = \{m/s : m \in M, s \in S\}$.

Theorem 4.4. *Let M be a X-module and the operation "$\oplus$" on $S^{-1}M$ be defined by:*

$$m/s \oplus m'/s' = (s' \cdot m + s \cdot m')/(s \wedge s').$$

Then $(S^{-1}M, \oplus)$ is an Abelian group.

Proof. First we prove that the operation "$\oplus$" is well-defined. Let $m_1/s_1, m_2/s_2, m'_1/s'_1, m'_2/s'_2 \in S^{-1}M$ such that $m_1/s_1 = m_2/s_2$ and $m'_1/s'_1 = m'_2/s'_2$. Then, there exist $t, t' \in S$ such that

$$t \cdot (s_2 \cdot m_1 - s_1 \cdot m_2) = 0 \quad , \quad t' \cdot (s'_2 \cdot m'_1 - s'_1 \cdot m'_2) = 0$$

and so

$$(t \wedge s_2) \cdot m_1 = (t \wedge s_1) \cdot m_2 \quad , \quad (t' \wedge s_2') \cdot m_1' = (t' \wedge s_1') \cdot m_2'$$

Hence, we get that;

$$(t' \wedge s_2' \wedge s_1') \cdot ((t \wedge s_2) \cdot m_1) = (t' \wedge s_2' \wedge s_1') \cdot ((t \wedge s_1) \cdot m_2)$$

and

$$(t \wedge s_2 \wedge s_1) \cdot ((t' \wedge s_2') \cdot m_1') = (t \wedge s_2 \wedge s_1) \cdot ((t' \wedge s_1') \cdot m_2')$$

Hence;

$$[(t' \wedge s_2' \wedge s_1') \wedge (t \wedge s_2)] \cdot m_1 = [(t' \wedge s_2' \wedge s_1') \wedge (t \wedge s_1)] \cdot m_2 \ , \quad (1)$$

and

$$[(t \wedge s_2 \wedge s_1) \wedge (t' \wedge s_2')] \cdot m_1' = [(t \wedge s_2 \wedge s_1) \wedge (t' \wedge s_1')] \cdot m_2' \ , \quad (2)$$

Now, by (1) and (2), we have

$$(t\wedge t'\wedge s_1'\wedge s_2'\wedge s_2)\cdot m_1+(t\wedge t'\wedge s_1\wedge s_2'\wedge s_2)\cdot m_1' = (t\wedge t'\wedge s_1\wedge s_1'\wedge s_2')\cdot m_2+(t\wedge t'\wedge s_2\wedge s_1\wedge s_1')\cdot m_2'$$

and so

$$(t \wedge t' \wedge s_2' \wedge s_2) \cdot (s_1' \cdot m_1 + s_1 \cdot m_1') = (t \wedge t' \wedge s_1 \wedge s_1') \cdot (s_2' \cdot m_2 + s_2 \cdot m_2')$$

and this implies that

$$(t \wedge t') \cdot [(s_2' \wedge s_2) \cdot (s_1' \cdot m_1 + s_1 \cdot m_1')] = (t \wedge t') \cdot [(s_1 \wedge s_1') \cdot (s_2' \cdot m_2 + s_2 \cdot m_2')]$$

Hence, the definition of "$\sim$", we have

$$(s_1' \cdot m_1 + s_1 \cdot m_1')/(s_1 \wedge s_1') = (s_2' \cdot m_2 + s_2 \cdot m_2')/(s_2 \wedge s_2')$$

and this means that "$\oplus$" is well-defined. Now, the proof of group properties are easy by some modifications. $\square$

Theorem 4.5. *Let M be a X-module and the operation $\circ : S^{-1}X \times S^{-1}M \longrightarrow S^{-1}M$ be defined by $x/s \circ m/t = (x \cdot m)/(s \wedge t)$. Then $S^{-1}M$ is an $S^{-1}X$-module.*

Proof. First we show that "$\circ$" is well defined. Let $x_1/s_1, x_2/s_2 \in S^{-1}X$ and $m_1/t_1, m_2/t_2 \in S^{-1}M$ such that $x_1/s_1 = x_2/s_2$ and $m_1/t_1 = m_2/t_2$. Then, there exist $s, t \in S$, such that

$$s \wedge s_2 \wedge x_1 = s \wedge s_1 \wedge x_2 \quad , \quad t \cdot (t_2 \cdot m_1 - t_1 \cdot m_2) = 0 \quad , \quad (1)$$

By definition of a BCK-module and (1), $t \cdot (t_2 \cdot m_1) = t \cdot (t_1 \cdot m_2)$ and so $(t \wedge t_2) \cdot m_1 = (t \wedge t_1) \cdot m_2$. Hence, by (1) we have

$$(s \wedge s_2 \wedge x_1) \cdot ((t \wedge t_2) \cdot m_1) = (s \wedge s_1 \wedge x_2) \cdot ((t \wedge t_1) \cdot m_2)$$

and so

$$(s \wedge s_2 \wedge x_1 \wedge t \wedge t_2) \cdot m_1 = (s \wedge s_1 \wedge x_2 \wedge t \wedge t_1) \cdot m_2$$

Hence, by the definition of BCK-module,

$$((s \wedge t) \wedge (s_2 \wedge t_2)) \cdot (x_1 \cdot m_1) = ((s \wedge t) \wedge (s_1 \wedge t_1)) \cdot (x_2 \cdot m_2)$$

and so

$$(s \wedge t) \cdot ((s_2 \wedge t_2) \cdot (x_1 \cdot m_1)) = (s \wedge t) \cdot ((s_1 \wedge t_1) \cdot (x_2 \cdot m_2))$$

and this implies that

$$(x_1 \cdot m_1)/(s_1 \wedge t_1) = (x_2 \cdot m_2)/(s_2 \wedge t_2)$$

Therefore, the operation "$\circ$" is well-defined.

Now, we should prove the axioms of a BCK-module.

(i) $(x/t \dot{\wedge} y/s) \circ m/l = x/t \circ (y/s \circ m/l)$:

For this, first we prove the following identity.

$$\begin{aligned}(x/t\dot{\wedge}y/s) &= y/s\star(y/s\star x/t), \text{ (since } S^{-1}X \text{ is commutative)}\\ &= y/s\star[((y\wedge t)*(x\wedge s))/(s\wedge t)]\\ &= [(s\wedge t\wedge y)*(s\wedge((y\wedge t)*(x\wedge s)))]/(s\wedge t)\\ &= [(s\wedge t\wedge y)*((s\wedge y\wedge t)*(x\wedge s))]/(s\wedge t), \text{ (by Lemma 3.4)}\\ &= [(x\wedge s)\wedge(s\wedge t\wedge y)]/(s\wedge t) \text{ (by definition of } \wedge \text{ in } X)\\ &= [(s\wedge t)\wedge(x\wedge y)]/(s\wedge t)\\ &= (x\wedge y)/(s\wedge t) \text{ (by Lemma 3.7(i))}\end{aligned}$$

Now, by the above identity, we have:

$$\begin{aligned}(x/t\dot{\wedge}y/s)\circ m/l &= [(x\wedge y)/(s\wedge t)]\circ m/l\\ &= ((x\wedge y)\cdot m)/(s\wedge t\wedge l)\\ &= (x\cdot(y\cdot m))/(s\wedge t\wedge l)\\ &= x/s\circ((y\cdot m)/(t\wedge l))\\ &= x/s\circ(y/t\circ m/l)\end{aligned}$$

(ii) $x/s\circ(m_1/t_1\oplus m_2/t_2)=(x/s\circ m_1/t_1)\oplus(x/s\circ m_2/t_2)$:

$$\begin{aligned}x/s\circ(m_1/t_1\oplus m_2/t_2) &= x/s\circ[(t_2\cdot m_1+t_1\cdot m_2)/(t_1\wedge t_2)]\\ &= [x\cdot(t_2\cdot m_1+t_1\cdot m_2)/(s\wedge t_1\wedge t_2)]]\\ &= [x\cdot(t_2\cdot m_1)+x\cdot(t_1\cdot m_2)]/(s\wedge t_1\wedge t_2)\\ &= ((x\wedge t_2)\cdot m_1)/(s\wedge t_1\wedge t_2)\oplus((x\wedge t_1)\cdot m_2)/(s\wedge t_1\wedge t_2)\\ &= (x\cdot m_1)/(s\wedge t_1)\oplus(x\cdot m_2)/(s\wedge t_2))\\ &= (x/s\circ m_1/t_1)\oplus(x/s\circ m_2/t_2)\end{aligned}$$

(iii) $0_{S^{-1}X}\circ m/t=0_{S^{-1}}M$:

$$0_{S^{-1}X}\circ m/t=0/t\circ m/t=(0\cdot m)/(t\wedge t)=0/t=0_{S^{-1}M}$$

Therefore, $S^{-1}M$ is a $S^{-1}X$-module. □

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R. A. Borzooei, Department of Mathematics, Shahid Beheshti University, Tehran, Iran.
E-mail address: borzooei@sbu.ac.ir

J. Shohani, Department of Mathematics, Sistan and Baluchestan University, Zahedan, Iran.