KANTOROVICH TYPE INEQUALITIES FOR THE DIFFERENCE WITH TWO NEGATIVE PARAMETERS

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Abstract. In this paper, we shall show Kantorovich type inequalities for the difference with two negative parameters as follows: Let A and B be positive invertible operators on a Hilbert space H such that $MI \geq A \geq mI$ for some positive numbers $M > m > 0$. If the usual order $A \geq B$ holds, then

$$B^q + C(m, M, p, q)I \geq B^q + C\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right) I \geq A^p \qquad \text{for all } p, q < -1,$$

where $C(m, M, p, q)$, the Kantorovich constant for the difference with two parameters, is defined as

$$(q-1)\left\{\frac{M^p - m^p}{q(M-m)}\right\}^{\frac{q}{q-1}} + \frac{m^p M - mM^p}{M-m} \quad \text{if} \quad m \leq \left\{\frac{M^p - m^p}{q(M-m)}\right\}^{\frac{1}{q-1}} \leq M.$$

As applications, we show some characterizations of the chaotic order. Thereby, we observe the difference between the usual order and the chaotic one by virtue of Kantorovich constant for the difference.

1 Introduction. Throughout this paper, we consider bounded linear operators on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The positivity defines the usual order $A \geq B$ for selfadjoint operators A and B. For the sake of convenience, $T > 0$ means T is positive and invertible. The Löwner-Heinz inequality asserts that $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for all $0 \leq \alpha \leq 1$. However $A \geq B \geq 0$ does not ensure $A^\alpha \geq B^\alpha$ for $\alpha > 1$ in general. As a complementary result to the Löwner-Heinz inequality, Furuta [4] firstly showed Kantorovich type inequalities for the ratio and afterward Yamazaki [7] showed the following Kantorovich type inequality for the difference:

$$(1.1) \qquad A \geq B,\ MI \geq B \geq mI > 0 \quad \text{imply} \quad A^p + C(m, M, p)I \geq B^p \qquad \text{for all } p \geq 1,$$

where the Kantorovich constant for the difference $C(m, M, p)$ is defined by

$$C(m, M, p) = (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m} \qquad \text{for all } p \in \mathbb{R},$$

see also [5]. We note that $\lim_{p\to 0} C(m, M, p) = 0$ and $\lim_{p\to 1} C(m, M, p) = 0$. Kantorovich type inequalities for the difference provide a new view on the usual order and the chaotic one. The following theorem was obtained in [6] as a two positive parameters version of (1.1).

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Theorem A 1. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some positive numbers $M > m > 0$. Then the usual order $A \geq B$ is equivalent to*

$$A^q + C(m, M, p, q)I \geq B^p \quad \text{for all} \;\; p, \; q > 1,$$

where $C(m, M, p, q)$, the Kantorovich constant for the difference with two parameters is defined by

$$C(m, M, p, q) = \begin{cases} \frac{Mm^p - mM^p}{M-m} + & (q-1)\left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{q}{q-1}} \\ & \text{if } m \leq \left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{1}{q-1}} \leq M \\ m^p - m^q & \text{if } \left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{1}{q-1}} < m \\ M^p - M^q & \text{if } M < \left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{1}{q-1}} \end{cases} \tag{1.2}$$

for all $p, q \in \mathbb{R}$ such that $pq > 0$.

In particular, if we put $q = p$ in (1.2), then we see that $C(m, M, p, p) = C(m, M, p)$, because the condition $m \leq \left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{1}{q-1}} \leq M$ automatically holds. We also note that $C(m, M, p, 1) = m^p - m(\text{resp.} M^p - M)$ if $M^p - m^p \leq M - m(\text{resp.} M^p - M \geq M - m)$.

In this note, as a continuation of [1], we discuss Kantorovich type inequalities for the difference with two negative parameters: If $A \geq B > 0$ such that $MI \geq A \geq mI > 0$, then

$$B^q + C(m, M, p, q)I \geq B^q + C\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right) I \geq A^p \qquad \text{for all } p, q < -1.$$

As applications, we show some characterizations of the chaotic order by means of Kantorovich type inequalities for the difference. Thereby, we observe the difference between the usual order and the chaotic one by virtue of the Kantorovich constant for the difference.

2 Kantorovich type inequalities for difference. At the beginning, we show two negative parameters version of Theorem A. For this, we clarify the meaning of the Kantorovich constant for the difference $C(m, M, p, q)$ defined as (1.2).

Lemma 2.1. *For given $M > m > 0$ and $p, q \in \mathbb{R}$ with $p, q > 1$ or $p, q < 0$,*

$$C(m, M, p, q) = \max\left\{\frac{M^p - m^p}{M - m} t + \frac{m^p M - mM^p}{M - m} - t^q : t \in [m, M]\right\}.$$

Proof. Put $h(t) = \frac{M^p - m^p}{M-m} t + \frac{m^p M - mM^p}{M-m} - t^q$. Note that $h'(t) = 0$ has the unique solution

$$t_1 = \left\{\frac{M^p - m^p}{q(M - m)}\right\}^{\frac{1}{q-1}} > 0$$

and $q \notin [0, 1]$ implies

$$h''(t_1) = -q(q-1)\left\{\frac{M^p - m^p}{q(M - m)}\right\}^{\frac{q-2}{q-1}} < 0.$$

Thus, in the case of $m \le t_1 \le M$, we have the upper bound $h(t_1)$ on $[m, M]$, where

$$h(t_1) = (q-1)\left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{q}{q-1}} + \frac{m^p M - m M^p}{M-m}.$$

Next, if $t_1 < m$, then the upper bound $h(m)$ on $[m, M]$ is given by

$$\begin{aligned} h(m) &= -m^q + \frac{M^p - m^p}{M-m} m + \frac{m^p M - m M^p}{M-m} \\ &= m^p - m^q. \end{aligned}$$

Similarly, if $M < t_1$, then the upper bound $h(M)$ on $[m, M]$ attains at $t = M$:

$$h(M) = M^p - M^q.$$

□

By virtue of Lemma 2.1, we have the following Kantorovich type inequality for the difference with two negative parameters:

Corollary 2.2. *Let A and B be positive invertible operators on H such that $MI \ge A \ge mI$ for some positive numbers $M > m > 0$. If $A \ge B$, then*

$$B^q + C(m, M, p, q)I \ge A^p \quad \text{for all } p, q < -1.$$

Proof. First of all, we note that $t = (Ax, x) \in [m, M]$ for every unit vector $x \in H$ by the assumption. For $p, q < -1$, it follows from Lemma 2.1 that

$$(A^p x, x) \le (Ax, x)^q + C(m, M, p, q) \qquad \text{for every unit vector } x \in H$$

and hence

$$\begin{aligned} (A^p x, x) &\le (Ax, x)^q + C(m, M, p, q) \\ &\le (Bx, x)^q + C(m, M, p, q) \quad \text{by } A \ge B > 0 \\ &\le (B^q x, x) + C(m, M, p, q) \quad \text{by the Hölder-McCarthy inequality} \end{aligned}$$

for every unit vector $x \in H$. □

If we let $q = p \to -1$ in Corollary 2.2, then $C(m, M, p, p) \to \frac{(\sqrt{M} - \sqrt{m})^2}{Mm} \ne 0$, so that $B^q + C(m, M, p, q)I \ge A^p$ for all $p, q < -1$ does not imply $A \ge B$.

To discuss more precise estimation than Corollary 2.2, we prepare the following lemma:

Lemma 2.3. *For $M > m > 0$*

$$M(M^{p-1} - m^{p-1}) \frac{M^{\frac{p-1}{q-1}} - m}{M-m} \ge C(m, M, p, q) \tag{2.1}$$

for all $p,\ q > 1$ such that $m \le \left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{1}{p-1}} \le M$.

Proof. First of all, we put $h = \frac{M}{m}$. Then the condition $m \le \left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{1}{p-1}} \le M$ is equivalent to

$$q \le (h^p - 1)/(h-1) \le qh^{p-1}.$$

Therefore we have the following inequality as a tool.

$$h^{p-1} \ge \frac{(q-1)^{q-1}}{q^q} \frac{(h^p-1)^q}{(h-1)(h^p-h)^{q-1}}.$$

As a matter of fact, $q(h-1) \le h^p - 1$ has an equivalent expression

$$(0 <) \frac{q-1}{q} \cdot \frac{h^p-1}{h^p-h} \le 1.$$

Moreover $\frac{h^p-1}{h-1} \le qh^{p-1}$ implies that

$$\begin{aligned} \frac{(q-1)^{q-1}}{q^q} \frac{(h^p-1)^q}{(h-1)(h^p-h)^{q-1}} &\le qh^{p-1} \left(\frac{(q-1)^{q-1}}{q^q} \cdot \frac{(h^p-1)^{q-1}}{(h^p-h)^{q-1}} \right) \\ &= h^{p-1} \left(\frac{q-1}{q} \cdot \frac{h^p-1}{h^p-h} \right)^{q-1} \le h^{p-1}. \end{aligned}$$

Now both sides of (2.1) are written as follows:

$$M(M^{p-1} - m^{p-1}) \frac{M^{\frac{p-1}{q-1}} - m}{M-m} = \frac{Mm^p - mM^p}{M-m} + \frac{M^{\frac{p-1}{q-1}}(M^p - Mm^{p-1})}{M-m}$$

and by Lemma 2.1

$$C(m, M, p, q) \le \frac{Mm^p - mM^p}{M-m} + (q-1) \left(\frac{M^p - m^p}{q(M-m)} \right)^{\frac{q}{q-1}}.$$

So, aiming at the second terms, it suffices to show that

$$\frac{M^{\frac{p-1}{q-1}}(M^p - Mm^{p-1})}{M-m} \ge (q-1) \left(\frac{M^p - m^p}{q(M-m)} \right)^{\frac{q}{q-1}}.$$

Actually the prepared inequality ensures that

$$\begin{aligned} \frac{M^{\frac{p-1}{q-1}}(M^p - Mm^{p-1})}{M-m} &= h^{\frac{p-1}{q-1}} \cdot \frac{m^{\frac{p-1}{q-1}} m^p (h^p - h)}{m(h-1)} \\ &\ge \frac{(q-1)}{q^{\frac{q}{q-1}}} \frac{(h^p-1)^{\frac{q}{q-1}}}{(h-1)^{\frac{1}{q-1}}(h^p-h)} \frac{m^{\frac{p-1}{q-1}} m^p (h^p-h)}{m(h-1)} \\ &= \frac{(q-1)}{q^{\frac{q}{q-1}}} \frac{(h^p-1)^{\frac{q}{q-1}} (m^p)^{\frac{q}{q-1}}}{(h-1)^{\frac{q}{q-1}} m^{\frac{q}{q-1}}} \\ &= (q-1) \left(\frac{h^p m^p - m^p}{q(hm - m)} \right)^{\frac{q}{q-1}} \\ &= (q-1) \left(\frac{M^p - m^p}{q(M-m)} \right)^{\frac{q}{q-1}}, \end{aligned}$$

as desired. If we put $p = q$ in (2.1), then it follows that $M(M^{p-1} - m^{p-1}) \ge C(m, M, p, p) > 0$ for all $p > 1$. Hence we have $\lim_{p \to 1} C(m, M, p, p) = C(m, M, 1, 1) = 0$. □

Now we show the following two Kantorovich type inequalities for the difference with two negative parameters, which characterize the usual order $A \geq B$:

Theorem 2.4. *Let A and B be positive invertible operators on H such that $MI \geq A \geq mI$ for some positive numbers $M > m > 0$. Then the usual order $A \geq B$ is equivalent to*

$$B^q + C\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right) I \geq A^p \qquad \text{for all } p, q < -1. \tag{2.2}$$

Proof. Since $B^{-1} \geq A^{-1}$ and $\frac{1}{m}I \geq A^{-1} \geq \frac{1}{M}I$, we have

$$(B^{-1})^{q_1} + C(\frac{1}{M}, \frac{1}{m}, p_1, q_1)I \geq (A^{-1})^{p_1} \quad \text{for } p_1, q_1 > 1 \quad \text{by Theorem A .}$$

Put $q_1 = -q\ (> 1), p_1 = -p\ (> 1)$ for $p = q \to -1$. Then it follows that

$$B^q + C\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right) I \geq A^p \quad \text{for } p,\ q < -1.$$

Conversely, suppose (2.2). Since $C\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right) \to 0$ as $p = q \to -1$ by Lemma 2.3, the inequality (2.2) implies $B^{-1} \geq A^{-1}$, i.e., $A \geq B$. □

In particular, if we put $q = -2$ in Theorem 2.4, then we have the following corollary which is utilized well later:

Corollary 2.5. *Let A and B be positive invertible operators on H such that $MI \geq A \geq mI$ for some positive numbers $M > m > 0$. If $A \geq B$, then*

$$B^{-2} + \frac{M^2m^2(m^p - M^p)^2 + 4(M-m)(M^{p+1} - m^{p+1})}{4(M-m)^2} I \geq A^p \qquad \text{for all } p < -1. \tag{2.3}$$

Theorem 2.6. *Let A and B be positive invertible operators on H such that $MI \geq A \geq mI$ for some positive numbers $M > m > 0$. Then the usual order $A \geq B$ is equivalent to*

$$B^q + \frac{(m^{p+1} - M^{p+1})(Mm^{-\frac{p+1}{q+1}} - 1)}{M - m} I \geq B^q + C(\frac{1}{M}, \frac{1}{m}, -p, -q)I \geq A^p \tag{2.4}$$

for all $p,\ q < -1$ such that $\frac{1}{M} \leq \left(\frac{mM(m^p - M^p)}{-q(M-m)}\right)^{\frac{-1}{p+1}} \leq \frac{1}{m}$.

Proof. Since $B^{-1} \geq A^{-1}$ and $\frac{1}{m}I \geq A^{-1} \geq \frac{1}{M}I$, we have the following inequalities by Lemma 2.3:

$$\frac{1}{m}\left(\frac{1}{m^{p_1-1}} - \frac{1}{M^{p_1-1}}\right) \frac{\frac{1}{m^{\frac{p_1-1}{q_1-1}}} - \frac{1}{M}}{\frac{1}{m} - \frac{1}{M}} \geq C\left(\frac{1}{M}, \frac{1}{m}, p_1, q_1\right)$$

for all $p_1,\ q_1 > 1$ such that

$$\frac{1}{M} \leq \left(\frac{\frac{1}{m^{p_1}} - \frac{1}{M^{p_1}}}{q_1(\frac{1}{m} - \frac{1}{M})}\right)^{\frac{1}{p_1-1}} \leq \frac{1}{m}.$$

Taking $q_1 = -q$ (> 1) and $p_1 = -p$ (> 1) for given $p,\ q < -1$, it follows that

$$\begin{aligned}(m^{p+1} - M^{p+1})\frac{(Mm^{-\frac{p+1}{q+1}} - 1)}{M-m} &= \frac{1}{m}(m^{p+1} - M^{p+1})\frac{Mm(m^{-\frac{p+1}{q+1}} - M^{-1})}{M-m} \\ &= \frac{1}{m}\left(\frac{1}{m^{-p-1}} - \frac{1}{M^{-p-1}}\right)\frac{\frac{1}{m^{\frac{p+1}{q+1}}} - \frac{1}{M}}{\frac{1}{m} - \frac{1}{M}} \\ &\geq C\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right).\end{aligned}$$

Therefore, we have (2.4) by Theorem 2.4.

Conversely, suppose (2.4). If $p = q$, then we have

$$B^p + \frac{(m^{p+1} - M^{p+1})}{m} I \geq A^p \qquad \text{for all } p < -1.$$

Since $m^{p+1} - M^{p+1} \to 0$ as $p \to -1$, the inequality above implies $B^{-1} \geq A^{-1}$, i.e., $A \geq B$. □

Theorem 2.4 is better than Corollary 2.2 by the following comparison:

Theorem 2.7. *If $M > m > 0$ for positive numbers M and m, then*

$$C(m, M, -p, -q) \geq C\left(\frac{1}{M}, \frac{1}{m}, p, q\right) \quad \text{for } p,\ q > 1.$$

Proof. For each $s \in \mathbb{R}$, $G_s(x)$ is an affine function corresponding to x^s on an interval $[a, b]$. We define $G_{-p}(x), G_p(x)$ on $[m, M]$ and $[\frac{1}{M}, \frac{1}{m}]$, respectively. For any $x \in [m, M]$, there exist positive numbers α and β such that $x = \beta m + \alpha M$ and $\alpha + \beta = 1$. Then $G_p(\frac{\beta}{m} + \frac{\alpha}{M})$ can be defined. Now the (weighted) arithmetic-harmonic mean inequality says that

$$\left(\frac{\beta}{m} + \frac{\alpha}{M}\right)^{-1} \leq \beta m + \alpha M,$$

so that

$$\frac{\beta}{m} + \frac{\alpha}{M} \geq \frac{1}{\beta m + \alpha M}.$$

Since $G_p(x)$ is an increasing function, we have

$$G_p\left(\frac{\beta}{m} + \frac{\alpha}{M}\right) \geq G_p\left(\frac{1}{\beta m + \alpha M}\right).$$

Moreover it follows from the affinity of $G_s(x)$ that

$$\begin{aligned}G_{-p}(x) &= G_{-p}(\beta m + \alpha M) = \beta G_{-p}(m) + \alpha G_{-p}(M) \\ &= \beta\left(\frac{1}{m}\right)^p + \alpha\left(\frac{1}{M}\right)^p = \beta G_p\left(\frac{1}{m}\right) + \alpha G_p\left(\frac{1}{M}\right) = G_p\left(\frac{\beta}{m} + \frac{\alpha}{M}\right).\end{aligned}$$

Thus,

$$G_{-p}(x) \geq G_p\left(\frac{1}{x}\right).$$

Hence we have the desired inequality as follows:

$$\begin{aligned} C\left(\frac{1}{M},\frac{1}{m},p,q\right) &= \max\left\{G_p\left(\frac{1}{x}\right)-\left(\frac{1}{x}\right)^q :\ x\in[m,M]\right\} \quad \text{by Lemma 2.1} \\ &= G_p\left(\frac{1}{x_0}\right)-\left(\frac{1}{x_0}\right)^q \quad \text{for some} \quad x_0\in[m,M] \\ &\le G_{-p}(x_0)-{x_0}^{-q} \\ &\le \max\{G_{-p}(x)-x^{-q} :\ x\in[m,M]\} \\ &= C(m,M,-p,-q). \end{aligned}$$

□

Remark 1. *By Theorem 2.7, it follows that $A\ge B>0$ with $MI\ge A\ge mI>0$ implies*

$$B^q+C(m,M,p,q)I\ge B^q+C(\frac{1}{M},\frac{1}{m},-p,-q)I\ge A^p \qquad \text{for } p,q<-1.$$

As a generalization of Theorem 2.4, we shall show the following theorem:

Theorem 2.8. *Let A and B be positive invertible operators on H such that $MI\ge A\ge mI$ for some positive numbers $M>m>0$. Then the usual order $A\ge B$ is equivalent to*

$$B^q+M^r\cdot C\left(\frac{1}{M^{1+r}},\frac{1}{m^{1+r}},\frac{r-p}{1+r},\frac{r-q}{1+r}\right)I\ge A^p \quad \text{for all } r>0 \text{ and } p,q<-1. \tag{2.5}$$

Proof. By the Furuta inequality [3], [5, Chapter 7], it follows that $A\ge B$ ensures that

$$A^{1+r}\ge\left(A^{\frac{r}{2}}B^qA^{\frac{r}{2}}\right)^{\frac{1+r}{q+r}} \qquad \text{for all } q>1,\ r>0.$$

For $p,q>1$ and $r>0$, put $A_1=A^{1+r}$, $B_1=\left(A^{\frac{r}{2}}B^qA^{\frac{r}{2}}\right)^{\frac{1+r}{q+r}}$, $p_1=-\frac{p+r}{1+r}<-1$, and $q_1=-\frac{q+r}{1+r}<-1$ in Theorem 2.4. Since $A_1\ge B_1$ and $M^{1+r}I\ge A_1\ge m^{1+r}I$, then we have

$$B_1^{q_1}+C\left(\frac{1}{M^{1+r}},\frac{1}{m^{1+r}},\frac{r+p}{1+r},\frac{r+q}{1+r}\right)I\ge A_1^{p_1}$$

and hence

$$\left(A^{\frac{r}{2}}B^qA^{\frac{r}{2}}\right)^{-1}+C\left(\frac{1}{M^{1+r}},\frac{1}{m^{1+r}},\frac{r+p}{1+r},\frac{r+q}{1+r}\right)I\ge A^{-p-r}.$$

Multiply $A^{\frac{r}{2}}$ on both sides and replace p and q by $-p$ and $-q$ respectively, we have (2.5).

Conversely, suppose (2.5). If we put $r\to 0$ and $p=q\to -1$ in (2.5), then we have $B^{-1}\ge A^{-1}$. □

Corollary 2.9. *Let A and B be positive invertible operators on H such that $MI\ge A\ge mI$ for some positive numbers $M>m>0$. If $A\ge B$, then*

$$B^q+\frac{4m^{q+1}M^{q+1}(m^{q+1}-M^{q+1})(M^{p+1}-m^{p+1})+(m^{q+p+2}-M^{q+p+2})^2}{4M^{q+2}(m^{q+1}-M^{q+1})^2}I\ge A^p$$

for all $p<-1, q<-2$.

Proof. If we put $r=-q-2$ in (2.5) of Theorem 2.8, then we have the desired inequality. □

3 Characterizations of chaotic order. In this section, as an application of Kantorovich type inequalities for the difference with two negative parameters, we give characterizations of the chaotic order. For positive invertible operators A and B, the order defined by $\log A \geq \log B$ is called the chaotic order. We shall show a chaotic order version of Theorem 2.4.

Theorem 3.1. *Let A and B be positive invertible operators on H such that $MI \geq A \geq mI$ for some positive numbers $M > m > 0$. If $\log A \geq \log B$, then*

$$B^q + M \cdot C\left(\frac{1}{M}, \frac{1}{m}, 1-p, 1-q\right) I \geq A^p \qquad \textit{for all } p, q < 0. \tag{3.1}$$

Proof. Put $p, q > 0$. By the chaotic Furuta inequality [2], [5, Chapter 7], it follows that $\log A \geq \log B$ ensures that $A \geq (A^{1/2}B^qA^{1/2})^{1/1+q}$. Since $-(q+1), -(p+1) < -1$, we apply Theorem 2.4 to obtain

$$(A^{\frac{1}{2}}B^qA^{\frac{1}{2}})^{-1} + C(\frac{1}{M}, \frac{1}{m}, p+1, q+1)I \geq A^{-p-1}$$

and hence

$$B^{-q} + C(\frac{1}{M}, \frac{1}{m}, p+1, q+1)A \geq A^{-p}.$$

Replacing p and q by $-p$ and $-q$ respectively, we have

$$B^q + M \cdot C\left(\frac{1}{M}, \frac{1}{m}, 1-p, 1-q\right) I \geq A^p \qquad \text{for all } p, q < 0.$$

□

Though (2.2) in Theorem 2.4 characterizes the usual order, it follows that (3.1) in Theorem 3.1 does not characterize the usual order, because $C(1/M, 1/m, 1-p)/p \not\to 0$ as $p \to 0$. By using (2.3) in Corollary 2.5, we show the following characterization of the chaotic order.

Theorem 3.2. *Let A and B be positive invertible operators on H such that $MI \geq A \geq mI$ for some positive numbers $M > m > 0$. Then the chaotic order $\log A \geq \log B$ is equivalent to*

$$B^q + \frac{(m^{p+q} - M^{p+q})^2 + 4M^qm^q(m^q - M^q)(M^p - m^p)}{4M^q(m^q - M^q)^2} I \geq A^p \qquad \textit{for all } p, q < 0.$$

Proof. By the chaotic Furuta inequality [2], [5, Chapter 7], it follows that $\log A \geq \log B$ ensures that $A^q \geq (A^{q/2}B^qA^{q/2})^{1/2}$ for all $q > 0$. Put $A_1 = A^q$, $B_1 = (A^{q/2}B^qA^{q/2})^{1/2}$, $p_1 = -\frac{q+p}{q}(< -1)$, $q_1 = -2$, $M_1 = M^q$ and $m_1 = m^q$ in Corollary 2.5. Then we have

$$(A^{\frac{q}{2}}B^qA^{\frac{q}{2}})^{-1} + \frac{M^{2q}m^{2q}(m^{-p-q} - M^{-p-q})^2 + 4(M^q - m^q)(M^{-p} - m^{-p})}{4(M^q - m^q)^2} I \geq A^{-p-q}$$

and hence

$$B^{-q} + M^q \cdot \frac{M^{2q}m^{2q}(m^{-p-q} - M^{-p-q})^2 + 4(M^q - m^q)(M^{-p} - m^{-p})}{4(M^q - m^q)^2} I \geq A^{-p}$$

for all $p, q > 0$. Replacing p and q by $-p$ and $-q$, we have

$$B^q + M^{-q} \cdot \frac{M^{-2q} m^{-2q} (m^{p+q} - M^{p+q})^2 + 4(M^{-q} - m^{-q})(M^p - m^p)}{4(M^{-q} - m^{-q})^2} I \geq A^p$$

for all $p, q < 0$ and hence we have the desired inequality.

Conversely, if we put $p = q$, then we have

$$B^p + \frac{(m^{2p} - M^{2p})^2 + 4M^p m^p (m^p - M^p)(M^p - m^p)}{4M^p (m^p - M^p)^2} I \geq A^p$$

for all $p < 0$ and hence

$$B^p + \frac{(m^p - M^p)^2}{4M^p} I \geq A^p \qquad \text{for all } p < 0.$$

This implies that

$$\frac{B^p - I}{p} + \frac{(m^p - M^p)^2}{4pM^p} I \leq \frac{A^p - I}{p} \qquad \text{for all } p < 0$$

and as $p \to 0$ we have $\log B \leq \log A$. □

As a generalization of Theorem 3.1, we shall show the following theorem. Comparing Theorem 2.8 with Theorem 3.3 in the below, we observe that variables p, q and r in the Kantorovich constant change to $p-1, q-1$ and $r-1$ respectively. This is explained by the fact that $\lim_{p\to 0} \frac{t^p - 1}{p} = \log t$ for $t > 0$, as used in above, that is, the exponent of $\log t$ can be regarded as 0.

Theorem 3.3. *Let A and B be positive invertible operators on H such that $MI \geq A \geq mI$ for some positive numbers $M > m > 0$. Then the chaotic order $\log A \geq \log B$ is equivalent to*

$$B^q + M^r \cdot C\left(\frac{1}{M^r}, \frac{1}{m^r}, \frac{r-p}{r}, \frac{r-q}{r}\right) I \geq A^p \qquad \textit{for all } r > 0 \textit{ and } p, q < 0. \tag{3.2}$$

Proof. (⇒) By the chaotic Furuta inequality [2], [5, Chapter 7], it follows that $\log A \geq \log B$ ensures that $A^r \geq (A^{\frac{r}{2}} B^q A^{\frac{r}{2}})^{\frac{r}{q+r}}$ for all $q,\ r \geq 0$. Put $A_1 = A^r, B_1 = (A^{\frac{r}{2}} B^q A^{\frac{r}{2}})^{\frac{r}{r+q}}$, $p_1 = -\frac{r+p}{r} (< -1), q_1 = -\frac{r+q}{r} (< -1), M_1 = M^r$ and $m_1 = m^r$ in Theorem 2.4. Then we have

$$(A^{\frac{r}{2}} B^q A^{\frac{r}{2}})^{-1} + C\left(\frac{1}{M^r}, \frac{1}{m^r}, \frac{r+p}{r}, \frac{r+q}{r}\right) I \geq A^{-r-p} \qquad \text{for all} \ \ p,\ q > 0,$$

so that

$$B^{-q} + C\left(\frac{1}{M^r}, \frac{1}{m^r} \frac{r+p}{r}, \frac{r+q}{r}\right) A^r \geq A^{-p} \qquad \text{for all} \ \ p,\ q > 0.$$

Since $M^r I \geq A^r \geq m^r I$, we have

$$B^q + M^r \cdot C\left(\frac{1}{M^r}, \frac{1}{m^r}, \frac{r-p}{r}, \frac{r-q}{r}\right) I \geq A^p \qquad \text{for all} \ \ p,\ q < 0.$$

($\Leftarrow$) Put $p=q=-r(r>0)$ in the assuming inequality. Then we have

$$B^{-r}+M^rC\left(\frac{1}{M^r},\frac{1}{m^r},2\right)I\geq A^{-r}\quad \text{for all } r>0$$

and

$$M^rC\left(\frac{1}{M^r},\frac{1}{m^r},2\right)=\frac{(M^r-m^r)^2}{4m^{2r}M^r}\quad \text{by the definition of}\quad C\left(\frac{1}{M^r},\frac{1}{m^r},2\right).$$

Since

$$\frac{B^{-r}-I}{r}+\frac{1}{r}\frac{(M^r-m^r)^2}{4m^{2r}M^r}I\geq\frac{A^{-r}-I}{r}\quad \text{for all } r>0$$

and

$$\lim_{r\to 0}\frac{(M^r-m^r)^2}{4rm^{2r}M^r}=\lim_{r\to 0}\frac{(M^r-1)-(m^r-1)}{r}\cdot\frac{M^r-m^r}{4m^{2r}M^r}=0,$$

we have $\log B^{-1}\geq\log A^{-1}$, i.e., $\log A\geq\log B$.

□

Corollary 3.4. *Let A and B be positive invertible operators such that $MI\geq A\geq mI\geq I>0$ for some positive numbers $M>m>0$. If $\log A\geq\log B$, then*

$$B^q+M^r(m^p-M^p)\frac{(M^rm^{-\frac{rp}{q}}-1)}{M^r-m^r}I\geq B^q+M^rC\left(\frac{1}{M^r},\frac{1}{m^r},\frac{r-p}{r},\frac{r-q}{r}\right)I\geq A^p$$

for all $p,\ q<0$ such that $\frac{1}{M^r}\leq\left(\frac{(r-q)(M^r-m^r)}{rm^rM^r(m^{p-r}-M^{p-r})}\right)^{\frac{r}{p}}\leq\frac{1}{m^r}$.

Proof. Put $m_1=\frac{1}{M^r},\ M_1=\frac{1}{m^r},\ p_1=\frac{r-p}{r}(>1)$, and $q_1=\frac{r-q}{r}(>1)$ for $p,\ q<0$ in Lemma 2.3. Then we have

$$\frac{1}{m^r}\left\{\left(\frac{1}{m^r}\right)^{-\frac{p}{r}}-\left(\frac{1}{M^r}\right)^{-\frac{p}{r}}\right\}\frac{(\frac{1}{m^r})^{\frac{p}{q}}-(\frac{1}{M^r})}{\frac{1}{m^r}-\frac{1}{M^r}}\geq C\left(\frac{1}{M^r},\frac{1}{m^r},\frac{r-p}{r},\frac{r-q}{r}\right).$$

Moreover, since the left hand side of the above coincides with

$$\frac{1}{m^r}(m^p-M^p)\frac{m^{-\frac{rp}{q}}-M^{-r}}{\frac{M^r-m^r}{m^rM^r}}=(m^p-M^p)\frac{(M^rm^{-\frac{rp}{q}}-1)}{M^r-m^r},$$

it follows that

$$(m^p-M^p)\frac{(M^rm^{-\frac{rp}{q}}-1)}{M^r-m^r}\geq C\left(\frac{1}{M^r},\frac{1}{m^r},\frac{r-p}{r},\frac{r-q}{r}\right).$$

Therefore we have

$$B^q+M^r(m^p-M^p)\frac{(M^rm^{-\frac{rp}{q}}-1)}{M^r-m^r}I\geq B^q+M^rC\left(\frac{1}{M^r},\frac{1}{m^r},\frac{r-p}{r},\frac{r-q}{r}\right)I\geq A^p$$

by Theorem 3.3. □

Remark 2. *Theorem 3.3 will give an alternative proof to the first half of a proof of Theorem 3.2: As a matter of fact, if we put $r=-q$ in (3.2) of Theorem 3.3, then we have*

$$B^q+M^{-q}C(M^q,m^q,\frac{p+q}{q},2)I\geq A^p.$$

This is just the required inequality.

4 Complementary inequalities. In this final section, we consider complementary inequalities to results with respect to the usual order in §2. For this, we prepare some notations. For $0 < m < M$ and $p, q \in \mathbb{R}$ with $pq > 0, p \neq q$,

$$\beta_1(m, M, p, q) = \max\{m^p - m^q, M^p - M^q\}$$

and

$$\beta_2(m, M, p, q) = \begin{cases} m^p - m^q & \text{if } \left(\frac{p}{q}\right)^{\frac{1}{q-p}} < m, \\ \left(\frac{p}{q}\right)^{\frac{p}{q-p}} - \left(\frac{p}{q}\right)^{\frac{q}{q-p}} & \text{if } m \leq \left(\frac{p}{q}\right)^{\frac{1}{q-p}} \leq M, \\ M^p - M^q & \text{if } M < \left(\frac{p}{q}\right)^{\frac{1}{q-p}}. \end{cases}$$

We here recall the following result proved in [6]:

Lemma 4.1. *Let A and B be positive operators on H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$ for some scalars $M_1 > m_1 > 0$ and $M_2 > m_2 > 0$. If $A \geq B$, then the following inequalities hold:*
(1) $0 < q < p < 1 \Rightarrow A^q + \beta_1(m_1, M_1, p, q)I \geq B^p$,
(2) $p > 1$, $0 < q < 1 \Rightarrow A^q + \beta_1(m_2, M_2, p, q)I \geq B^p$,
(3) $0 < p < 1$, $p < q \Rightarrow A^q + \beta_2(m_1, M_1, p, q)I \geq B^p$.

Based on Lemma 4.1, we give complementary inequalities:

Theorem 4.2. *Let A and B be as in above. Then the following inequalities hold:*
(1) $-1 < p < q < 0 \Rightarrow B^q + \beta_1(m_2, M_2, p, q)I \geq A^p$,
(1') $-1 < q < p < 0 \Rightarrow B^p + \beta_1(m_2, M_2, q, p)I \geq A^q$,
(2) $-1 < q < 0, p < -1 \Rightarrow B^q + \beta_1(m_1, M_1, p, q)I \geq A^p$,
(2') $-1 < p < 0, q < -1 \Rightarrow B^p + \beta_1(m_1, M_1, q, p)I \geq A^q$,
(3) $-1 < p < 0, q < p \Rightarrow B^q + \beta_2(m_2, M_2, p, q)I \geq A^p$,
(3') $-1 < q < 0, p < q \Rightarrow B^p + \beta_2(m_2, M_2, q, p)I \geq A^q$.

Proof. (1) Since $B^{-1} \geq A^{-1} > 0$ and $\frac{1}{m_2} I \geq B^{-1} \geq \frac{1}{M_2} I$, it follows from (1) of Lemma 4.1 that

$$(B^{-1})^{q_1} + \beta_1\left(\frac{1}{M_2}, \frac{1}{m_2}, p_1, q_1\right) I \geq (A^{-1})^{p_1} \quad \text{for } 0 < q_1 < p_1 < 1.$$

Putting $q_1 = -q$ and $p_1 = -p$, we have

$$B^q + \beta_1\left(\frac{1}{M_2}, \frac{1}{m_2}, -p, -q\right) I \geq A^p \quad \text{for } -1 < p < q < 0.$$

Since

$$\begin{aligned} \beta_1\left(\frac{1}{M_2}, \frac{1}{m_2}, -p, -q\right) &= \max\left\{\left(\frac{1}{M_2}\right)^{-p} - \left(\frac{1}{M_2}\right)^{-q}, \left(\frac{1}{m_2}\right)^{-p} - \left(\frac{1}{m_2}\right)^{-q}\right\} \\ &= \max\{{M_2}^p - {M_2}^q, {m_2}^p - {m_2}^q\} \\ &= \beta_1(m_2, M_2, p, q), \end{aligned}$$

we have the desired inequality as follows:

$$B^q + \beta_1(m_2, M_2, p, q)I \geq A^p \quad \text{for } -1 < p < q < 0.$$

(2) Since $B^{-1} \geq A^{-1} > 0$ and $\frac{1}{m_1}I \geq A^{-1} \geq \frac{1}{M_1}I$, it follows from (2) of Lemma 4.1 that

$$(B^{-1})^{q_1} + \beta_1\left(\frac{1}{M_1}, \frac{1}{m_1}, p_1, q_1\right)I \geq (A^{-1})^{p_1} \quad \text{for } p_1 > 1,\ 0 < q_1 < 1.$$

Putting $q_1 = -q,\ p_1 = -p$, then

$$B^q + \beta_1\left(\frac{1}{M_1}, \frac{1}{m_1}, -p, -q\right)I \geq A^p \quad \text{for } p < -1,\ -1 < q < 0.$$

Hence we have

$$B^q + \beta_1(m_1, M_1, p, q)I \geq A^p \quad \text{for } p < -1,\ -1 < q < 0.$$

(3) Since $B^{-1} \geq A^{-1} > 0$ and $\frac{1}{m_2}I \geq B^{-1} \geq \frac{1}{M_2}I$, it follows from (3) of Lemma 4.1 that

$$(B^{-1})^{q_1} + \beta_2\left(\frac{1}{M_2}, \frac{1}{m_2}, p_1, q_1\right)I \geq (A^{-1})^{p_1} \quad \text{for} \quad 0 < p_1 < 1,\ p_1 < q_1.$$

Putting $p_1 = -p,\ q_1 = -q$, then

$$B^q + \beta_2\left(\frac{1}{M_2}, \frac{1}{m_2}, -p, -q\right)I \geq A^p \quad \text{for} \quad -1 < p < 0,\ q < p.$$

Since $\beta_2\left(\frac{1}{M_2}, \frac{1}{m_2}, -p, -q\right) = \beta_2(m_2, M_2, p, q)$ by the definition of β_2, we have

$$B^q + \beta_2(m_2, M_2, p, q)I \geq A^p \quad \text{for } -1 < p < 0,\ q < p.$$

Proofs of (1'), (2') and (3') are given by replacing p and q by q and p in (1), (2) and (3), respectively.

□

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