A GENERALIZED PÓLYA-SZEGÖ INEQUALITY FOR THE HADAMARD PRODUCT

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Abstract. In this paper, we show a generalized Pólya-Szegö inequality for the Hadamard product: Let A and B be $k \times k$-positive definite matrices such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then

$$\sqrt{(A^2x,x)(B^2x,x)} \leq \; k \cdot \frac{M^2+m^2}{2Mm}(A \circ B\; x,x)$$

for every vector x, where I is the identity matrix and the symbol $\circ$ is the Hadamard product.

1 Introduction. Let $\mathbb{M}_k = \mathbb{M}_k(\mathbb{C})$ denote the space of $k \times k$ complex matrices. For a pair A, B of Hermitian matrices the order relation $A \geq B$ means as usual that $A - B$ is positive semidefinite. In particular, $A > 0$ means that A is positive definite. For $A = (a_{ij})$ and $B = (b_{ij})$, their Hadamard product is the $k \times k$ matrix of entrywise products

$$A \circ B = (a_{ij}b_{ij})\,.$$

It is commutative unlike the usual matrix product:

$$A \circ B = B \circ A.$$

The diagonal matrix formed a matrix A can be obtained by Hadamard multiplication with the identity matrix $A \circ I$. As Styan pointed out in [7], the most widely used and possibly most important result concerning the Hadamard product is as follows:

Theorem A (Shur). *If A_i is positive defnite ($i = 1, 2, \cdots, n$), then so is $A_1 \circ A_2 \circ \cdots \circ A_n$.*

It is likely that many matrix inequalities for the Hadamard product is based on this fact. For example, Ando [1] showed the following Cauchy-Schwarz inequality for the Hadamard product: If A_i is positive definite ($i = 1, 2, \cdots, n,\ n \geq 2$), then

$$A_1 \circ A_2 \circ \cdots \circ A_n \leq (A_1^n \circ I)^{\frac{1}{n}} (A_2^n \circ I)^{\frac{1}{n}} \cdots (A_n^n \circ I)^{\frac{1}{n}}. \tag{1.1}$$

In fact, in the case of $n = 2$, if A and B are diagonal matrices, then we have the Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}. \tag{1.2}$$

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In [4], Pólya-Szegö showed a reverse of Cauchy-Schwarz inequality (1.2): If the real numbers a_i and b_i $(i = 1, 2, \cdots, n)$ satisfies the condition

$$0 < m \le a_i, b_i \le M \qquad \text{for } i = 1, \cdots, n, \tag{1.3}$$

then

$$\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \le \frac{M^2+m^2}{2Mm}\sum_{i=1}^{n} a_i b_i. \tag{1.4}$$

In [2], Grueb-Rheinboldt pointed out that the Pólya-Szegö inequality is a direct specialization of the following inequality which is equivalent to the Kantorovich inequality: If $\{a_i\}$ and $\{b_i\}$ $(i = 1, 2, \cdots)$ are two sequences of real numbers with the condition (1.3) and $\{\xi_i\}$ denotes another sequence with $\sum_{i=1}^{\infty} \xi_i^2 < \infty$, then

$$\sqrt{\sum_{i=1}^{\infty} a_i^2\xi_i^2}\sqrt{\sum_{i=1}^{\infty} b_i^2\xi_i^2} \le \frac{M^2+m^2}{2Mm}\sum_{i=1}^{\infty} a_i b_i \xi_i^2. \tag{1.5}$$

From this viewpoint, Grueb-Rheinboldt showed a generalized form of the inequality (1.5), which is called a generalized Pólya-Szegö inequality: Let A and B be commuting positive definite matrices such that $mI \le A, B \le MI$ for some scalars $0 < m < M$. Then

$$\sqrt{(A^2x, x)(B^2x, x)} \le \frac{M^2+m^2}{2Mm}(ABx, x) \tag{1.6}$$

for every vector x.

In this paper, we show a generalized Pólya-Szegö inequality for the Hadamard product and a reverse inequality of n-variables of the Cauchy-Schwarz one (1.1) due to Ando.

2 Hadamard product version

The tensor product $\mathbb{M}_k \otimes \cdots \otimes \mathbb{M}_k$ of n copies of $\mathbb{M}_k$ is identified with $\mathbb{M}_{k^n}$ in a natural way. It has been known that the Hadamard product is a principal square submarix of the tensor product. This fact is formulated as follows:

Lemma 1. *For each positive integer n there is a normalized positive linear map Φ_n from the open cone of positive definite matirces in $\mathbb{M}_{k^n}$ to ones in $\mathbb{M}_k$ that satisfies*

$$\Phi_n(A_1 \otimes \cdots \otimes A_n) = A_1 \circ \cdots \circ A_n \qquad \text{for all } A_i \in \mathbb{M}_k \text{ and } i = 1, \cdots, n.$$

To prove our main results, we need the following well-known two lemmas. We give a proof for convenience.

Lemma 2 ([3, 6]). *If A is a positive definite matrix in $\mathbb{M}_k$, then $A \circ I \ge \frac{1}{k}A$.*

Proof. Let P be the matrix with all entries 1. We define a linear map by $\Phi_X(A) = A \circ X$ for a fixed X. Then it follows that Φ_X is positive if and only if X is positive semidefinite. If we put $X = I - \lambda P$, then we have the desired inequality since $\frac{1}{k} = \max\{\lambda : I - \lambda P \ge 0\}$. □

Lemma 3. *Let A and B be commuting positive definite matrices such that $mI \le A, B \le MI$ for some scalars $0 < m < M$. Then*

$$\frac{A^2+B^2}{2} \le \frac{M^2+m^2}{2Mm}AB.$$

Proof. Put $C = A^{-1}B$ and it follows that C is positive definite and $\frac{m}{M}I \le C \le \frac{M}{m}I$. Then $(\frac{M}{m}I - C)(C - \frac{m}{M}I) \ge 0$ implies

$$\frac{I + C^2}{2} \le \frac{M^2 + m^2}{2Mm} C$$

and hence we have $\frac{A^2+B^2}{2} \le \frac{M^2+m^2}{2Mm} AB$. □

We show a generalized Pólya-Szegö inequality for the Hadamard product.

Theorem 4. *Let A and B be $k \times k$-positive definite matrices in $\mathbb{M}_k$ such that $mI \le A, B \le MI$ for some scalars $0 < m < M$. Then*

$$\sqrt{(A^2x,x)(B^2x,x)} \le \; k \cdot \frac{M^2 + m^2}{2Mm}(A \circ Bx, x)$$

for every vector x.

Proof. Since $A \otimes I$ and $I \otimes B$ are commutative and $mI \otimes I \le A \otimes I, I \otimes B \le MI \otimes I$, it follows from Lemma 3 that

$$\frac{(A \otimes I)^2 + (I \otimes B)^2}{2} \le \frac{M^2 + m^2}{2Mm}(A \otimes I)(I \otimes B) = \frac{M^2 + m^2}{2Mm}(A \otimes B).$$

From Lemma 1 we have

$$\frac{A^2 \circ I + I \circ B^2}{2} \le \frac{M^2 + m^2}{2Mm} A \circ B.$$

Therefore, by the arithmetic-geometric mean inequality and Lemma 2 we have

$$\begin{aligned}\sqrt{(A^2x,x)(B^2x,x)} &\le \frac{1}{2}((A^2x,x) + (B^2x,x)) \le \frac{k}{2}(((A^2 \circ I)x,x) + ((B^2 \circ I)x,x)) \\ &= k\left((\frac{A^2 \circ I + B^2 \circ I}{2})x, x \right) \\ &\le k \cdot \frac{M^2 + m^2}{2Mm}(A \circ B\ x, x)\end{aligned}$$

for every vector x. □

Remark 5. *The inequality $\sqrt{(A^2x,x)(B^2x,x)} \le \frac{M^2+m^2}{2Mm}(A \circ B\ x,x)$ does not hold in general. In fact, put*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $I \le A, B \le 3I$. Then we have $\sqrt{(A^2x,x)(B^2x,x)} = 6\sqrt{5} = 13.4164$. On the other hand, we have $\frac{M^2+m^2}{2Mm}(A \circ B\ x,x) = \frac{5}{3} \cdot 8 = 13.33 \cdots$. Therefore,

$$\sqrt{(A^2x,x)(B^2x,x)} \not\le \frac{M^2 + m^2}{2Mm}(A \circ B\ x, x).$$

3 n-variables version We recall the Specht ratio: As a reverse of the arithmetic-geometric mean inequality, Specht [5] estimated the ratio of the arithmetic mean to the geometric one: For $x_1, \cdots, x_n \in [m, M]$ with $0 < m < M$,

$$\frac{x_1 + \cdots + x_n}{n} \leq S(h)\sqrt[n]{x_1 \cdots x_n} \tag{3.1}$$

where $h = \frac{M}{m}$ and S(h) is defined for $h \geq 1$ as

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h > 1) \qquad \text{and} \quad S(1) = 1. \tag{3.2}$$

The following lemma is regarded as a reverse of the arithmetic-geometric mean inequality for the Hadamard product:

Lemma 6. *Let A_i be positive definite matrices in $\mathbb{M}_k$ such that $mI \leq A_i \leq MI$ for some scalars $0 < m < M$ and $i = 1, 2, \cdots, n$, $n \geq 2$. Put $h = \frac{M}{m}$. Then*

$$\frac{1}{n}(A_1 \circ I + \cdots + A_n \circ I) \leq S(h)(A_1^{\frac{1}{n}} \circ \cdots \circ A_n^{\frac{1}{n}}).$$

Proof. Since $A_1 \otimes I \otimes \cdots \otimes I, \cdots, I \otimes \cdots \otimes I \otimes A_n$ are mutually commutative and the spectrum is contained in $[m, M]$, by the Specht theorem (3.1) it follows that

$$\begin{aligned}
&\frac{1}{n}(A_1 \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A_n) \\
&\leq S(h)\sqrt[n]{(A_1 \otimes I \otimes \cdots \otimes I) \cdots (I \otimes \cdots \otimes I \otimes A_n)} \\
&= S(h)(A_1 \otimes \cdots \otimes A_n)^{\frac{1}{n}} = S(h)(A_1^{\frac{1}{n}} \otimes \cdots \otimes A_n^{\frac{1}{n}})
\end{aligned}$$

and hence from Lemma 1

$$\frac{1}{n}(A_1 \circ I \circ \cdots \circ I + \cdots I \circ \cdots \circ I \circ A_n) \leq S(h)(A_1^{\frac{1}{n}} \circ \cdots \circ A_n^{\frac{1}{n}}).$$

Therefore, we have

$$\frac{1}{n}(A_1 \circ I + \cdots + A_n \circ I) \leq S(h)(A_1^{\frac{1}{n}} \circ \cdots \circ A_n^{\frac{1}{n}}).$$

□

Now, we show n-variables version of Theorem 4:

Theorem 7. *Let A_i be positive definite matrices in $\mathbb{M}_k$ such that $mI \leq A_i \leq MI$ for some scalars $0 < m < M$ and for $i = 1, 2, \cdots, n$, $n \geq 2$. Put $h = \frac{M}{m}$. Then*

$$\sqrt[n]{(A_1^n x, x)(A_2^n x, x) \cdots (A_n^n x, x)} \leq \ k \cdot S(h^n)(A_1 \circ A_2 \circ \cdots \circ A_n \ x, x)$$

for every vector x, where the Specht ratio $S(h)$ is defined by (3.2).

Proof. By Lemma 2 and Lemma 6, it follows that

$$\begin{aligned}
\sqrt[n]{(A_1^n x, x) \cdots (A_n^n x, x)} &\leq \frac{1}{n}((A_1^n x, x) + \cdots + (A_n^n x, x)) \\
&\leq \frac{k}{n}(((A_1^n \circ I)x, x) + \cdots + ((A_n^n \circ I)x, x) \\
&= k\left(\frac{1}{n}(A_1^n \circ I + \cdots + A_n^n \circ I)x, x\right) \\
&\leq \ k \cdot S(h^n)(A_1 \circ \cdots \circ A_n x, x)
\end{aligned}$$

for every vector x. □

Remark 8. *In the case of $n = 2$, Theorem 4 is more precise estimates than Theorem 7. In fact, Yamazaki [8] pointed out that*

$$\frac{M^2 + m^2}{2Mm} = \frac{h^2 + 1}{2h} \le S(h^2) \qquad \text{for } h = \tfrac{M}{m}.$$

Finally, we show an n-variables Pólya-Szegö type inequality for the Cauchy-Schwarz one (1.1) for the Hadamard product due to Ando:

Theorem 9. *Let A_i be positive definite matrices in $\mathbb{M}_k$ such that $mI \le A_i \le MI$ for some scalars $0 < m < M$ and for $i = 1, 2, \cdots, n$, $n \ge 2$. Put $h = \frac{M}{m}$. Then*

$$(A_1^n \circ I)^{\frac{1}{n}} \cdots (A_n^n \circ I)^{\frac{1}{n}} \le S(h^n)(A_1 \circ \cdots \circ A_n),$$

where the Specht ratio $S(h)$ is defined by (3.2).

Proof. By the arithmetic-geometric mean inequality and Lemma 6, it follows that

$$\begin{aligned} &\sqrt[n]{(A_1 \circ I) \cdots (A_n \circ I)} \le \frac{1}{n}(A_1 \circ I + \cdots + A_n \circ I) \\ &\le S(h)(A_1^{\frac{1}{n}} \circ \cdots \circ A_n^{\frac{1}{n}}). \end{aligned}$$

Replacing A_i by A_i^n, we have the desired inequality. □

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