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FREE ORDERED SEMIGROUPS

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Abstract. In this paper we introduce the concept of free ordered semigroups as follows: If $(X, \leq_X)$ is an ordered set, an ordered semigroup $(F, ., \leq_F)$ is said to be a free ordered semigroup over $(X, \leq_X)$, if there is an isotone mapping $\varepsilon : (X, \leq_X) \to (F, \leq_F)$ satisfying the following "universal" condition: for any ordered semigroup $(S, *, \leq_S)$ and any isotone mapping $f : (X, \leq_X) \to (S, \leq_S)$, there exists a unique homomorphism $\varphi : (F, ., \leq_F) \to (S, *, \leq_S)$ such that $\varphi \circ \varepsilon = f$. Basing on the fact that the mapping ε is reverse isotone, we find relationships between the mappings ε_1 and ε_2 which correspond to free ordered semigroups $((F_1, ., \leq_1), \varepsilon_1)$ and $((F_2, ., \leq_2), \varepsilon_2)$.

1. Introduction and prerequisites

In the present paper we introduce the concept of free ordered semigroup over an ordered set as follows: An ordered semigroup $(F, ., \leq_F)$ is called a free ordered semigroup over an ordered set $(X, \leq_X)$ if there exists an isotone mapping ε of $(X, \leq_X)$ into $(F, \leq_F)$ satisfying the "universal" condition: for any ordered semigroup $(S, *, \leq_S)$ and any isotone mapping $f : (X, \leq_X) \to (S, \leq_S)$, there exists a unique homomorphism $\varphi : (F, ., \leq_F) \to (S, *, \leq_S)$ such that $\varphi \circ \varepsilon = f$. Since the free ordered semigroup $(F, ., \leq_F)$ depends on the mapping ε, it is convenient to use the notation $((F, ., \leq_F), \varepsilon)$. We first prove that for every ordered set X we can construct a free ordered semigroup over X. In fact, the set $F_X := \{(x_1, x_2, ..., x_n) \mid n \in N \text{ and } x_i \in X, i = 1, 2, ..., n\}$ is a free ordered semigroup over $(X, \leq_X)$ unique up to isomorphism (cf. [5, section 2.1]), the natural number n for the element $u = (x_1, x_2, ..., x_n)$, denoted by $l(u)$, is the length of u. In the following, we always use the notation F_X for the free ordered semigroup over the ordered set X. We denote by " $*$ ", " $\preceq$ ", the multiplication and the order on F_X, respectively, the mapping ε in the above construction being the mapping $\varepsilon : (X, \leq_X) \to (F_X, \preceq) \mid x \to (x)$. We remark that the concept of free ordered semigroups defined in this paper generalizes the concept of free semigroup [1] as each free semigroup endowed with the equality relation is a free ordered semigroup. One of the basic results of this paper is that if $((F, ., \leq_F), \varepsilon)$ is a free ordered semigroup over $(X, \leq_X)$, then the mapping $\varepsilon : (X, \leq_X) \to (F, \leq_F)$ is reverse isotone. Moreover, we prove the following: If $((F_1, ., \leq_1), \varepsilon_1)$ and $((F_2, *, \leq_2), \varepsilon_2)$ are free ordered semigroups over the ordered set $(X, \leq_X)$, then there exists an isomorphism $g : (F_1, ., \leq_1) \to (F_2, *, \leq_2)$ such that $g \circ \varepsilon_1 = \varepsilon_2$. If $((F_1, ., \leq_1), \varepsilon_1)$ and $((F_2, *, \leq_2), \varepsilon_2)$ are free ordered semigroups over the ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ respectively, and $\pi : (X, \leq_X) \to (Y, \leq_Y)$ an isotone, reverse isotone and onto mapping, then there exists an isomorphism $g : (F_1, ., \leq_1) \to (F_2, *, \leq_2)$ such that $g \circ \varepsilon_1 = \varepsilon_2 \circ \pi$. If $((F_1, ., \leq_1), \varepsilon_1)$ and $((F_2, *, \leq_2), \varepsilon_2)$ are free ordered semigroups over the ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ respectively, and $\varphi : (F_1, ., \leq_1) \to (F_2, *, \leq_2)$ an isomorphism, then the mapping $\pi : (X, \leq_X) \to (Y, \leq_Y) \mid x \to y$, where $y \in Y$ such that $(\varphi \circ \varepsilon_1)(x) = \varepsilon_2(y)$ is an isomorphism as well. As a consequence, two free

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ordered semigroups over the ordered sets X and Y, respectively, are isomorphic if and only if X and Y are so. So, exactly as in free semigroups, the free ordered semigroup over an ordered set is unique, up to isomorphism. However, if we have two different orders on X, then the free ordered semigroups over X are not isomorphic, in general.

If $(S, ., \leq_S)$ and $(T, *, \leq_T)$ are two ordered semigroups, a mapping $f : (S, ., \leq_S) \to (T, *, \leq_T)$ is called *isotone* if $a, b \in S$ such that $a \leq_S b$ implies $f(a) \leq_T f(b)$. It is called *reverse isotone* if $a, b \in S$ such that $f(a) \leq_T f(b)$ implies $a \leq_S b$. The mapping f is called a *homomorphism* if is isotone and satisfies the condition $f(ab) = f(a) * f(b)$ for each $a, b \in S$. An onto reverse isotone homomorphism is called *isomorphism.* Recall that any reverse isotone mapping is (1–1) [4]. We will also use the following well known

Lemma 1.1. *Let $(A, \leq_A)$, $(B, \leq_B)$ be ordered sets, f an isotone mapping of $(A, \leq_A)$ into $(B, \leq_B)$ and g a mapping of $(B, \leq_B)$ into $(A, \leq_A)$ such that $f \circ g = 1_B$. Then*

(1) *The mapping g is reverse isotone and*

(2) *The mapping f is onto.*

2. Main results

Definition 2.1. Let $(X, \leq_X)$ be an ordered set and $(F, ., \leq_F)$ an ordered semigroup. Suppose there exists an isotone mapping $\varepsilon : (X, \leq_X) \to (F, \leq_F)$ such that the following "universal" condition is satisfied: for any ordered semigroup $(S, *, \leq_S)$ and any isotone mapping $f : (X, \leq_X) \to (S, \leq_S)$ there exists a unique homomorphism $\varphi : (F, ., \leq_F) \to (S, *, \leq_S)$ such that $\varphi \circ \varepsilon = f$. Then we say that the ordered semigroup $(F, ., \leq_F)$ is a *free ordered semigroup over* $(X, \leq_X)$.

Construction 2.2. (cf. also [5; section 2.1]) For each ordered set $(X, \leq_X)$, we can construct a free ordered semigroup over $(X, \leq_X)$. In fact, let $(X, \leq_X)$ be an ordered set. We consider the set

$$F_X := \{(x_1, x_2, ..., x_n) \mid n \in N \text{ and } x_i \in X, i = 1, 2, ..., n\}$$

with the operation and the order on F_X defined by:

$$(x_1, x_2, ..., x_n) * (y_1, y_2, ..., y_m) := (x_1, x_2, ..., x_n, y_1, y_2, ..., y_m)$$

$$(x_1, x_2, ..., x_n) \preceq (y_1, y_2, ..., y_m) \Longleftrightarrow n = m \text{ and } x_i \leq_X y_i \ \forall\, i = 1, 2, ..., n.$$

Let now ε be the isotone mapping defined by $\varepsilon : (X, \leq_X) \to (F_X, \preceq) \mid x \to (x)$.
Then the pair $((F_X, *, \preceq), \varepsilon)$ is a free ordered semigroup over $(X, \leq_X)$. In fact, if $(S, ., \leq_S)$ is an ordered semigroup and $f : (X, \leq_X) \to (S, \leq_S)$ an isotone mapping, then the mapping

$$\varphi : (F_X, *, \preceq) \to (S, ., \leq_S) \mid (x_1, x_2, ..., x_n) \to f(x_1)f(x_2).....f(x_n)$$

is the unique homomorphism of $(F_X, *, \preceq)$ into $(S, ., \leq_S)$ such that $\varphi \circ \varepsilon = f$. $\square$

Remark 2.3. Using the technics of Construction 2.2, one can easily prove that the concept of free ordered semigroups defined in this paper generalizes the concept of free semigroup (without order). In fact, if $(F, .)$ is a free semigroup over the alphabet X, $\leq_X := \{(x, y) \mid x = y\}$ the equality relation on X, $\leq_F$ the equality relation on F defined by

$$(x_1, x_2, ..., x_n) \leq_F (y_1, y_2, ..., y_n) \Longleftrightarrow n = m \text{ and } x_i = y_i \ \forall\, i = 1, 2, ..., n$$

and ε the isotone mapping of $(X, \leq_X)$ into $(F, \leq_F)$ defined by $\varepsilon(x) := (x)$, then the "universal" condition of Definition 2.1 is satisfied, and the pair $((F, ., \leq_F), \varepsilon)$ is a free ordered semigroup over $(X, \leq_X)$.

Throughout the paper we denote by $((F_X,*,\preceq),\varepsilon)$ or simply by F_X the free ordered semigroup considered in Construction 2.2. We use the same symbol F_X for the free semigroup (without order) over the alphabet X as well.

Theorem 2.4. *Let $(X,\leq_X)$ be an ordered set and $((F,.,\leq_F),\varepsilon)$ a free ordered semigroup over $(X,\leq_X)$. Then the mapping*

$$\varepsilon : (X,\leq_X) \to (F,\leq_F)$$

is reverse isotone.

Proof. Let $x,y\in X$, $\varepsilon(x)\leq_F \varepsilon(y)$. Then $x\leq_X y$. In fact: Suppose $x\not\leq_X y$. We consider the sets:
$A := \{z\in X \mid x\leq_X z\}$
$B := \{w\in X \mid y\leq_X w\}$
$C := A\cup B$.
Clearly, $x\in A$, $y\in B$, and $y\notin A$. Moreover, we have

$$\begin{aligned} X &= C\cup(X\backslash C) = (A\cup B)\cup(X\backslash C)\\ &= A\cup((B\backslash\{y\})\cup\{y\})\cup(X\backslash C)\\ &= (A\cup(B\backslash\{y\}))\cup((X\backslash C)\cup\{y\}). \end{aligned}$$

$$\begin{aligned} &(A\cup(B\backslash\{y\}))\cap((X\backslash C)\cup\{y\}) =\\ =\ &(A\cap(X\backslash C))\cup(A\cap\{y\})\cup((B\backslash\{y\})\cap(X\backslash C))\cup((B\backslash\{y\})\cap\{y\})\\ =\ &\emptyset\ \ (\text{since } A,B\subseteq C, y\notin A). \end{aligned}$$

Let $(Z,+,\leq)$ be the ordered semigroup of integers with the usual operation, order. We consider the mapping:

$$f:(X,\leq_X)\to(Z,\leq)\mid z\to\begin{cases} 1 & if\ \ z\in A\cup(B\backslash\{y\})\\ 0 & if\ \ z\in(X\backslash C)\cup\{y\}.\end{cases}$$

The mapping f is clearly well defined.

The mapping f is isotone: Let $a,b\in X$, $a\leq_X b$. Then $f(a)\leq f(b)$. Indeed:

1. Let $a\in A\cup(B\backslash\{y\})$. Then $f(a):=1$, $a\in A$ or $a\in B\backslash\{y\}$.

1.1. If $a\in A$, then $x\leq_X a$. Since $x\leq_X a$, $a\leq_X b$, we have $x\leq_X b$, then $b\in A\subseteq A\cup(B\backslash\{y\})$, and $f(b):=1$, so $f(a)\leq f(b)$.

1.2. Let $a\in B\backslash\{y\}$. Then $a\in B$, $a\neq y$. Since $a\in B$, we have $y\leq_X a$. Since $y\leq_X a$, $a\leq_X b$, we have $y\leq_X b$, then $b\in B$. If $b=y$, then $a\leq_X b=y$. Since $a\leq_X y$, $y\leq_X a$, we have $a=y$ which is impossible. Thus $b\neq y$. Since $b\in B\backslash\{y\}\subseteq A\cup(B\backslash\{y\})$, we have $f(b):=1$, so $f(a)\leq f(b)$.

2. Let $a\in(X\backslash C)\cup\{y\}$. Then $f(a):=0$. Since $b\in X$, we have $f(b)\in\{0,1\}$, so $0\leq f(b)$, and $f(a)\leq f(b)$.

Since $((F,.,\leq_F),\varepsilon)$ is a free ordered semigroup over $(X,\leq_X)$, $(Z,+,\leq)$ an ordered semigroup and $f:(X,\leq_X)\leq(Z,\leq)$ an isotone mapping, from the "universal" condition, there exists a (unique) homomorphism

$$\varphi:(F,.,\leq_F)\to(Z,+,\leq)$$

such that $\varphi\circ\varepsilon=f$. Since $\varepsilon(x)\leq_F\varepsilon(y)$ and φ is isotone, we obtain

$$f(x)=(\varphi\circ\varepsilon)(x)=\varphi(\varepsilon(x))\leq\varphi(\varepsilon(y))=(\varphi\circ\varepsilon)(y)=f(y).$$

Since $x \in A \subseteq A \cup (B \backslash \{y\})$, we have $f(x) := 1$. Since $y \in (X \backslash C) \cup \{y\}$, we have $f(y) = 0$. We get a contradiction. □

Proposition 2.5. *Let $((F, ., \leq_F), \varepsilon)$ be a free ordered semigroup over the ordered set $(X, \leq_X$ $)$. Suppose*

$$h : (F, ., \leq_F) \to (F, ., \leq_F)$$

is a homomorphism such that $h \circ \varepsilon = \varepsilon$. Then $h = 1_F$.

Proof. Since $((F, ., \leq_F), \varepsilon)$ is a free ordered semigroup over $(X, \leq_X)$, $(F, ., \leq_F)$ an ordered semigroup and $\varepsilon : (X, \leq_X) \to (F, \leq_F)$ an isotone mapping, from the "universal" condition, there exists a unique homomorphism $\varphi : (F, ., \leq_F) \to (F, ., \leq_F)$ such that $\varphi \circ \varepsilon = \varepsilon$. Since the mapping h and the identity mapping 1_F on F are such homomorphisms, we have $h = 1_F$. □

Theorem 2.6. *Let $((F_1, ., \leq_1), \varepsilon_1)$, $((F_2, *, \leq_2), \varepsilon_2)$ be free ordered semigroups over the ordered set $(X, \leq_X)$. Then there exists an isomorphism*

$$g : (F_1, ., \leq_1) \to (F_2, *, \leq_2)$$

such that $g \circ \varepsilon_1 = \varepsilon_2$.

Proof. Since $((F_1, ., \leq_1), \varepsilon_1)$ is a free ordered semigroup over $(X, \leq_X)$, $(F_2, *, \leq_2)$ an ordered semigroup and $\varepsilon_2 : (X, \leq_X) \to (F_2, \leq_2)$ an isotone mapping, there exists a homomorphism

$$g : (F_1, ., \leq_1) \to (F_2, *, \leq_2)$$

such that $g \circ \varepsilon_1 = \varepsilon_2$. In a similar way, there exists a homomorphism

$$f : (F_2, *, \leq_2) \to (F_1, ., \leq_1)$$

such that $f \circ \varepsilon_2 = \varepsilon_1$.

The mapping $f \circ g : (F_1, ., \leq_1) \to (F_1, ., \leq_1)$ is a homomorphism (as f and g are so), and

$$(f \circ g) \circ \varepsilon_1 = f \circ (g \circ \varepsilon_1) = f \circ \varepsilon_2 = \varepsilon_1.$$

Since $((F_1, ., \leq_1), \varepsilon_1)$ is a free ordered semigroup over $(X, \leq_X)$, and the mapping $f \circ g$ is a homomorphism of $(F_1, ., \leq_1)$ into $(F_1, ., \leq_1)$ such that $(f \circ g) \circ \varepsilon_1 = \varepsilon_1$, by Proposition 2.5, we have $f \circ g = 1_{F_1}$. In a similar way we get $g \circ f = 1_{F_2}$. Since $(F_2, \leq_2)$ and $(F_1, \leq_1)$ are ordered sets, f an isotone mapping of $(F_2, \leq_2)$ into $(F_1, \leq_1)$ and g a mapping of $(F_1, \leq_1)$ into $(F_2, \leq_2)$ such that $f \circ g = 1_{F_1}$, by Lemma 1.1, g is reverse isotone. Since $(F_1, \leq_1)$ and $(F_2, \leq_2)$ are ordered sets, g an isotone mapping of $(F_1, \leq_1)$ into $(F_2, \leq_2)$ and f a mapping of $(F_2, \leq_2)$ into $(F_1, \leq_1)$ such that $g \circ f = 1_{F_2}$, by Lemma 1.1, the mapping g is onto. Since g is an onto reverse isotone homomorphism, it is an isomorphism. □

Theorem 2.7. *Let $((F_1, ., \leq_1), \varepsilon_1)$, $((F_2, *, \leq_2), \varepsilon_2)$ be free ordered semigroups over the ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ respectively. Suppose*

$$\pi : (X, \leq_X) \to (Y, \leq_Y)$$

is an isotone, reverse isotone and onto mapping. Then there exists an isomorphism

$$g : (F_1, ., \leq_1) \to (F_2, *, \leq_2)$$

such that $g \circ \varepsilon_1 = \varepsilon_2 \circ \pi$.

Proof. Since $((F_1,.,\leq_1),\varepsilon_1)$ is a free ordered semigroup over $(X,\leq_X)$, $(F_2,*,\leq_2)$ an ordered semigroup and the mapping $\varepsilon_2\circ\pi:(X,\leq_X)\to(F_2,\leq_2)$ is isotone (as ε and π are so), there exists a homomorphism

$$g:(F_1,.,\leq_1)\to(F_2,*,\leq_2)$$

such that $g\circ\varepsilon_1=\varepsilon_2\circ\pi$. Since the mapping π is an isomorphism, the mapping $\pi^{-1}:(Y,\leq_Y)\to(X,\leq_X)$ is well defined and it is isotone. Since $((F_2,*,\leq_2),\varepsilon_2)$ is a free ordered semigroup over $(Y,\leq_Y)$, $(F_1,.,\leq_1)$ an ordered semigroup and the mapping $\varepsilon_1\circ\pi^{-1}:(Y,\leq_Y)\to(F_1,\leq_1)$ is isotone, there exists a homomorphism

$$f:(F_2,*,\leq_2)\to(F_1,.,\leq_1)$$

such that $f\circ\varepsilon_2=\varepsilon_1\circ\pi^{-1}$. The mapping

$$f\circ g:(F_1,.,\leq_1)\to(F_1,.,\leq_1)$$

is a homomorphism (as f and g are so) and

$$\begin{aligned}(f\circ g)\circ\varepsilon_1 &= f\circ(g\circ\varepsilon_1)=f\circ(\varepsilon_2\circ\pi)=(f\circ\varepsilon_2)\circ\pi\\ &= (\varepsilon_1\circ\pi^{-1})\circ\pi=\varepsilon_1\circ(\pi^{-1}\circ\pi)=\varepsilon_1\circ 1_X=\varepsilon_1.\end{aligned}$$

By Proposition 2.5, we have $f\circ g=1_{F_1}$. In a similar way we obtain $g\circ f=1_{F_2}$. On the other hand, f is an isotone mapping of $(F_2,\leq_2)$ into $(F_1,\leq_1)$ and g is a mapping of $(F_1,\leq_1)$ into $(F_2,\leq_2)$ such that $f\circ g=1_{F_1}$. By Lemma 1.1, the mapping g is reverse isotone. Moreover, since g is an isotone mapping of $(F_1,\leq_1)$ into $(F_2,\leq_2)$ and f is a mapping of $(F_2,\leq_2)$ into $(F_1,\leq_1)$ such that $g\circ f=1_{F_2}$, by Lemma 1.1, the mapping g is onto. We have already seen that g is a homomorphism, hence g is an isomorphism. □

According to Construction 2.2, each $u\in F_X$ can be written in a unique way as $u=(x_1,x_2,...,x_n)$, where $n\in N$ and $x_i\in X$, $i=1,2,...,n$. As in free semigroups (without order), the number n, denoted by $l(u)$, is the length of u.

Order plays no role in Propositions 2.8, 2.9, 2.10 below, so they hold for both free semigroups (without order) and free ordered semigroups.

Proposition 2.8. For $m\in N$ and $u_i\in F_X$, $i=1,2,...,m$, we have
$l(u_1*u_2*...*u_n)=l(u_1)+l(u_2)+...+l(u_n)$.

Proposition 2.9. *If $u\in F_X^2:=F_X*F_X$, then $l(u)\geq 2$. Conversely, if $u\in F_X$ such that $l(u)\geq 2$, then $u\in F_X^2$.*

Proof. Let $u=a*b$ for some $a,b\in F_X$. Then

$$l(u)=l(a*b)=l(a)+l(b)\geq 2 \text{ (since } l(a),l(b)\in N).$$

Conversely, let $l(u)\geq 2$. Since $u\in F_X$, there exist $x_1,x_2,....,x_{l(u)}\in X$ such that $u=(x_1,x_2,....,x_{l(u)})$. Then we have

$$\begin{aligned}u &= (x_1,x_2,...,x_{l(u)})=(x_1)*(x_2)*.....*(x_{l(u)})\\ &= (x_1)*((x_2)*.....*(x_{l(u)}) \text{ (since } l(u)\geq 2)\\ &= (x_1)*(x_2,...,x_{l(u)})\in F_X*F_X:=F_X^2.\end{aligned}$$

□

Proposition 2.10. *Let $((F_X,*,\preceq),\varepsilon)$ be the free ordered semigroup constructed in Construction 2.2. Then*

$$\varepsilon(X)=F_X\backslash F_X^2.$$

Proof. Let $a\in\varepsilon(X)$. Then there exists $x\in X$ such that $a=\varepsilon(x)=(x)$ (by the definition of ε). Then $l(x)=1$ and $a\in F_X$. If $a\in F_X^2$, then by Proposition 2.9, $l(a)\geq 2$ which is impossible. Thus we have $a\in F_X\backslash F_X^2$. Let now $b\in F_X\backslash F_X^2$. Then $b\in\varepsilon(X)$. Indeed: Since $b\in F_X$, there exists $x_1,x_2,...,x_{l(b)}\in X$ such that $b=(x_1,x_2,...,x_{l(b)})$. If $l(b)\geq 2$ then, by Proposition 2.9, we have $b\in F_X^2$ which is impossible. Since $l(b)$ is a natural number such that $l(b)<2$, we have $l(b)=1$, so $b=(x_1)$. Since $x_1\in X$, we have $\varepsilon(x_1):=(x_1)$. Thus we have $b=\varepsilon(x_1)\in\varepsilon(X)$. □

In the rest of the paper, we use the following: For any two sets A,B, a mapping $f:A\to B$ and $C\subseteq A$, we have $f(A)\backslash f(C)\subseteq f(A\backslash C)$. In particular, if the mapping f is (1–1), then $f(A)\backslash f(C)=f(A\backslash C)$.

Proposition 2.11. *Let $((F,.,\leq_F),\overline{\varepsilon})$ be a free ordered semigroup over the ordered set $(X,\leq_X)$. Then*

$$\overline{\varepsilon}(X)=F\backslash F^2.$$

Proof. We consider the free ordered semigroup $((F_X,*,\preceq),\varepsilon)$ considered in Construction 2.2. By hypothesis, $((F,.,\leq_F),\overline{\varepsilon})$ is also a free ordered semigroup over $(X,\leq_X)$. By Theorem 2.6, there exists an isomorphism

$$g:(F_X,*,\preceq)\to(F,.,\leq_F)$$

such that $g\circ\varepsilon=\overline{\varepsilon}$. Then we have

$$\begin{aligned}\overline{\varepsilon}(X) &= (g\circ\varepsilon)(X)=g(\varepsilon(X))=g(F_X\backslash F_X^2)\text{ (by Proposition 2.10)}\\ &= g(F_X)\backslash g(F_X^2)\text{ (since } g \text{ is } (1-1))\\ &= g(F_X)\backslash(g(F_X)g(F_X))\text{ (since } g\text{ is a homomorphism)}\\ &= F\backslash F^2\text{ (since } g \text{ is onto).}\end{aligned}$$

□

Theorem 2.12. *Let $((F_1,.,\leq_1),\varepsilon_1)$, $((F_2,*,\leq_2),\varepsilon_2)$ be free ordered semigroups over the ordered sets $(X,\leq_X)$ and $(Y,\leq_Y)$ respectively. Suppose*

$$\varphi:(F_1,.,\leq_1)\to(F_2,*,\leq_2)$$

is an isomorphism. Then the mapping

$$\pi:(X,\leq_X)\to(Y,\leq_Y)\mid x\to y,\ \textit{where } y\in Y \textit{ such that } (\varphi\circ\varepsilon_1)(x)=\varepsilon_2(y)$$

is an isomorphism as well.

Proof. We have $(\varphi\circ\varepsilon_1)(X)=\varepsilon_2(Y)$ $\qquad(*)$
In fact: Since $((F_1,.,\leq_1),\varepsilon_1)$, $((F_2,*,\leq_2),\varepsilon_2)$ are free ordered semigroups over $(X,\leq_X)$ and $(Y,\leq_Y)$ respectively, by Proposition 2.11, we have $\varepsilon_1(X)=F_1\backslash F_1^2$ and $\varepsilon_2(Y)=F_2\backslash F_2^2$. Then we have

$$\begin{aligned}(\varphi\circ\varepsilon_1)(X) &= \varphi(\varepsilon_1(X))=\varphi(F_1\backslash F_1^2)\\ &= \varphi(F_1)\backslash\varphi(F_1^2)\text{ (since } g \text{ is } (1-1))\\ &= \varphi(F_1)\backslash(\varphi(F_1)*\varphi(F_1))\text{ (since } \varphi \text{ is a homomorphism)}\\ &= F_2\backslash(F_2*F_2)\text{ (since } \varphi \text{ is onto)}\\ &= F_2\backslash F_2^2=\varepsilon_2(Y).\end{aligned}$$

1. The mapping π is well defined: If $x \in X$ then, by $(*)$, there exists $y \in Y$ such that $(\varphi \circ \varepsilon_1)(x) = \varepsilon_2(y)$. If $x, z \in X$, $x = z$ and $y, w \in Y$ such that $(\varphi \circ \varepsilon_1)(x) = \varepsilon_2(y)$ and $(\varphi \circ \varepsilon_1)(z) = \varepsilon_2(w)$, then we have

$$\varepsilon_2(y) = (\varphi \circ \varepsilon_1)(x) = (\varphi \circ \varepsilon_1)(z) = \varepsilon_2(w).$$

Since $((F_2, *, \leq_2), \varepsilon_2)$ is a free ordered semigroup over $(Y, \leq_Y)$, by Theorem 2.4, the mapping ε_2 is reverse isotone, so it is a (1–1) mapping. Thus we have $y = w$.

2. The mapping π is isotone: Let $x \leq_X z$. Since $x, z \in X$, we have
$\pi(x) := y$ for some $y \in Y$ such that $(\varphi \circ \varepsilon_1)(x) = \varepsilon_2(y)$ and
$\pi(z) := w$ for some $w \in Y$ such that $(\varphi \circ \varepsilon_1)(z) = \varepsilon_2(w)$.
Since $((F_1, ., \leq_1), \varepsilon_1)$ is a free ordered semigroup over $(X, \leq_X)$, the mapping

$$\varepsilon_1 : (X, \leq_X) \to (F_1, \leq_1)$$

is isotone. Since $x \leq_X z$, we have $\varepsilon_1(x) \leq_1 \varepsilon_1(z)$. Since φ is isotone, we have $\varphi(\varepsilon_1(x)) \leq_2 \varphi(\varepsilon_1(z))$. Then we have

$$\varepsilon_2(y) = (\varphi \circ \varepsilon_1)(x) = \varphi(\varepsilon_1(x)) \leq_2 \varphi(\varepsilon_1(z)) = (\varphi \circ \varepsilon_1)(z) = \varepsilon_2(w).$$

Since ε_2 is reverse isotone, we have $y \leq_Y w$, that is $\pi(x) \leq_Y \pi(z)$.

3. The mapping π is reverse isotone: Let $x, z \in X$ such that $\pi(x) \leq_Y \pi(z)$. Then $x \leq_X z$. In fact: We have
$\pi(x) := y$ for some $y \in Y$ such that $(\varphi \circ \varepsilon_1)(x) = \varepsilon_2(y)$ and
$\pi(z) := w$ for some $w \in Y$ such that $(\varphi \circ \varepsilon_1)(z) = \varepsilon_2(w)$.
Since $\pi(x) \leq_Y \pi(z)$, we have $y \leq_Y w$. Since $((F_2, *, \leq_2), \varepsilon_2)$ is a free ordered semigroup over $(Y, \leq_Y)$, the mapping $\varepsilon_2 : (Y, \leq_Y) \to (F_2, \leq_2)$ is isotone. Since $y \leq_Y w$, we have $\varepsilon_2(y) \leq_2 \varepsilon_2(w)$, then $(\varphi \circ \varepsilon_1)(x) \leq_2 (\varphi \circ \varepsilon_1)(z)$, that is $\varphi(\varepsilon_1(x)) \leq_2 \varphi(\varepsilon_1(z))$. Since the mapping φ is reverse isotone, we get $\varepsilon_1(x) \leq_1 \varepsilon_1(z)$. Since $((F_1, ., \leq_1), \varepsilon_1)$ is a free ordered semigroup over $(X, \leq_X)$, by Theorem 2.4, the mapping ε_1 is reverse isotone. Hence we have $x \leq_X z$.

4. The mapping π is onto: Let $y \in S$. Since $\varepsilon_2(y) \in \varepsilon_2(Y) = (\varphi \circ \varepsilon_1)(X)$, there exists $x \in X$ such that $(\varphi \circ \varepsilon_1)(x) = \varepsilon_2(y)$. By the definition of π, we have $\pi(x) = y$. $\square$
By Theorems 2.7 and 2.12, we have the following

Corollary 2.13. (see also [3; Proposition 2.3]) *If $(F_1, ., \leq_1)$ and $(F_2, *, \leq_2)$ are free ordered semigroups over the ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ respectively, then we have*

$$(X, \leq_X) \cong (Y, \leq_Y) \Longleftrightarrow (F_1, ., \leq_1) \cong (F_2, *, \leq_2).$$

Remark 2.14. For semigroups (without order) the free semigroups on a set X are isomorphic. Exactly as in semigroups, the free ordered semigroups over an ordered set $(X, \leq)$ are isomorphic. However, order plays an essential role for free ordered semigroups. If we have two different orders over the same set X, that is, if $(X, \leq)$ and $(X, \leq_X)$ are ordered sets with $\leq \neq \leq_X$, the free ordered semigroup over $(X, \leq)$ and the free ordered semigroup over $(X, \leq_X)$ are not isomorphic, in general. As an example, consider the set N of natural numbers with the equality relation "=" and the usual relation " $\leq$ " on N. Let now $(F_1, ., \leq_1)$ be a free ordered semigroup over the ordered set $(N, =)$ and $(F_2, *, \leq_1)$ a free ordered semigroup over the ordered set $(N, \leq)$ (according to Construction 2.2 such free ordered semigroups exist). If $(F_1, ., \leq_1) \cong (F_2, *, \leq_2)$ then, by Corollary 2.13, $(N, =) \cong (N, \leq)$, that is, there exists a mapping $\pi : (N, =) \to (N, \leq)$ which is isotone, reverse isotone and onto. As $2, 3 \in (N, \leq)$, there exist $x, y \in (N, =)$ such that $\pi(x) = 2$, $\pi(y) = 3$. Since $\pi(x) \leq \pi(y)$ and π is reverse isotone, we have $x = y$, then $\pi(x) = \pi(y)$, that is $2 = 3$ which is impossible. Hence we have $(F_1, ., \leq_1) \ncong (F_2, *, \leq_2)$. $\square$

Remark 2.15. If $u, v \in F_X$ such that $u \preceq v$, then $l(u) = l(v)$. In fact, let $u = (x_1, x_2, ..., x_n)$, $v = (y_1, y_2, ..., y_m)$ for some $n, m \in N$, $x_i, y_j \in X$, $i = 1, ..., n$, $j = 1, ..., m$. Since $u \preceq v$, we have $m = n$ and $x_i \leq_X y_i$ $\forall$ $i = 1, ..., n$. Since $m = n$, we have $l(u) = l(v)$.

If $(S, ., \leq)$ is an ordered semigroup and $M \subseteq S$, we denote by $(M]$, $[M)$ the subsets of S defined by

$$(M] := \{t \in S \text{ such that } t \leq a \text{ for some } a \in M\},$$

$$[M) := \{t \in S \text{ such that } t \geq a \text{ for some } a \in M\}.$$

Proposition 2.16. *If $(X, \leq_X)$ is an ordered set and $u \in F_X$, then we have the following:*
(1) $u \in (F_X^2] \Longleftrightarrow l(u) \geq 2$.
(2) $u \in [F_X^2) \Longleftrightarrow l(u) \geq 2$.
(3) $(F_X^2] = F_X^2 = [F_X^2)$.

Proof. (1) $\Longrightarrow$. Let $u \preceq a$ for some $a \in F_X^2$. Since $u, a \in F_X$, $u \preceq a$, by Remark 2.15, we have $l(u) = l(a)$. Since $a \in F_X^2$, by Proposition 2.9, we get $l(a) \geq 2$, so $l(u) \geq 2$.
$\Longleftarrow$. If $l(u) \geq 2$ then, by Proposition 2.9, $u \in F_X^2 \subseteq (F_X^2]$.
The proof of (2) is similar.
(3) If $u \in (F_X^2]$ then, by (1), $l(u) \geq 2$. Since $u \in F_X$, $l(u) \geq 2$, by Proposition 2.9, we have $u \in F_X^2$. On the other hand, $F_X^2 \subseteq (F_X^2]$, so $(F_X^2] = F_X^2$. Similarly we prove that $[F_X^2) = F_X^2$. $\square$

Lemma 2.17. *Let $(A, \leq_A)$, $(B, \leq_B)$ be ordered sets, $f : (A, \leq_A) \to (B, \leq_B)$ an onto, isotone and reverse isotone mapping (that is, $(A, \leq_A)$ is isomorphic to $(B, \leq_B)$ under f), and $M \subseteq A$. Then the following statements hold true:*
(1) $f((M]) = (f(M)]$.
(2) $f([M)) = [f(M))$.

Proof. (1) Let $y \in f((M])$. Then $y = f(b)$ for some $b \in (M]$. Since $b \in (M]$, $b \leq_A t$ for some $t \in M$. Since f is isotone, we have $y = f(b) \leq_B f(t) \in f(M)$, so $y \in (f(M)]$.
Let $y \in (f(M)]$. Then $y \leq_B t$ for some $t \in f(M)$. Let $a \in M$ such that $t = f(a)$. Since $y \in B$ and f is onto, there exists $b \in A$ such that $y = f(b)$. Since $f(b) \leq_B f(a)$ and f is reverse isotone, we get $b \leq_A a$. Since $B \ni b \leq_A a \in M$, we have $b \in (M]$. Then $y = f(b) \in f((M])$. The proof of (2) is similar. $\square$

Proposition 2.18. *Let $(X, \leq_X)$ be an ordered set and $((F, ., \leq_F), \overline{\varepsilon})$ a free ordered semigroup over $(X, \leq_X)$. Then we have*

$$(F^2] = F^2 = [F^2).$$

Proof. Since $((F_X, *, \preceq), \varepsilon)$ and $((F, ., \leq_F), \overline{\varepsilon})$ are free ordered semigroups over $(X, \leq_X)$, by Theorem 2.6, there exists an isomorphism

$$g : (F_X, *, \preceq) \to (F, ., \leq_F).$$

We have $(F^2] = F^2$. In fact,

$$\begin{aligned} (F^2] &= (FF] = (g(F_X)g(F_X)] \text{ (since } g \text{ is onto)} \\ &= (g(F_X * F_X)] \text{ (since } g \text{ is a homomorphism).} \end{aligned}$$

Since $(F_X, \preceq)$, $(F, \leq_F)$ are ordered sets, $g : (F_X, \preceq) \to (F, \leq_F)$ an isomorphism and $F_X * F_X \subseteq F_X$, by Lemma 2.17(1), we have

$$g((F_X * F_X]) = (g(F_X * F_X)].$$

On the other hand, by Proposition 2.16(3), we have $(F_X * F_X] = F_X * F_X$.
Therefore we have

$$\begin{aligned}
(F^2] &= g((F_X * F_X]) = g(F_X * F_X) \\
&= g(F_X)g(F_X) \text{ (since } g \text{ is a homomorphism)} \\
&= FF \text{ (since } g \text{ is onto)} \\
&= F^2.
\end{aligned}$$

In a similar way we prove that $F^2 = [F^2)$. $\square$

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