CHARACTERIZATION OF MAXIMAL PRIMITIVE IDEALS OF TOEPLITZ ALGEBRAS

RIZKY ROSJANUARDI AND TAKASHI ITOH

Received January 19, 2010; revised June 16, 2010

Abstract. The upwards-looking topology which was introduced by Adji and Raeburn corresponds to the hull-kernel topology in the primitive ideal space $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ of Toeplitz algebra $\mathcal{T}(\Gamma)$ of totally ordered abelian group Γ. In this paper we discuss the closed sets in $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ with the upwards-looking topology and characterize maximal primitive ideals.

1. Introduction

In [2] Adji and Raeburn studied the ideal structure of the Toeplitz algebras of totally ordered abelian groups. Let Γ be a discrete totally ordered abelian group. An order ideal of a totally ordered abelian group Γ is a subgroup I which is order preserving, in the sense that if $x \in \Gamma^+, y \in I^+$ with $x \le y$ then $x \in I$. The set $\Sigma(\Gamma)$ of order ideals is totally ordered by inclusion. If I is an order ideal of Γ , (I, γ) denotes a character $\gamma \in \hat{I}$.

It was shown in [2] that each irreducible representation of the Toeplitz algebra $\mathcal{T}(\Gamma)$ factors through an irreducible representation of $\mathcal{T}(\Gamma/I)$ for some $I \in \Sigma(\Gamma)$, and that there is a bijective map L of the disjoint union

$$X(\Gamma) := \bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\} = \{(J, \gamma) : J \in \Sigma(\Gamma), \gamma \in \hat{J}\}$$

onto the set $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ of primitive ideals of $\mathcal{T}(\Gamma)$.

Using the bijection L from $X(\Gamma)$ onto $\mathrm{Prim}\,\mathcal{T}(\Gamma)$, Adji and Raeburn describe a topology on $X(\Gamma)$ which corresponds to the hull-kernel topology on $\mathrm{Prim}\,\mathcal{T}(\Gamma)$. The topology on $X(\Gamma)$ is called the upwards-looking topology $X(\Gamma)$ (see Definition 1).

In this paper we firstly discuss some properties of the topology and the relationship between the point wise topology and the relative upwards-looking topology on $\hat{\Gamma}$. Secondly we characterize maximal primitive ideals of Toeplitz algebras which is given in Theorem 11.

2. The Upwards-Looking Topology

Throughout this paper, $(\Gamma, +)$ is a totally ordered discrete abelian group. Let I be an order ideal of Γ. The quotient group Γ/I is defined in the similar manner of usual quotient group, and there is a quotient order on Γ/I defined by

$$x + I \le y + I \Leftrightarrow \exists i \in I \text{ such that } x \le y + i,$$

2000 *Mathematics Subject Classification.* 46L55 .

Key words and phrases. Toeplitz algebras, upwards-looking topology, order ideals, maximal primitive ideals .

This paper was completed when the first author visited Gunma University in 2009 and 2010, supported by Gunma University and Universitas Pendidikan Indonesia (UPI), he would thank to these institutions. Some results in this paper are from the first author's Ph.D thesis (Institut Teknologi Bandung, Indonesia, 2005).

this is also a total order. Moreover, if $q : \Gamma \longrightarrow \Gamma/I$ is the quotient homomorphism then $q(\Gamma^+) = (\Gamma/I)^+$.

Definition 1 (Upwards-Looking Topology)**.** [2, Definition 4.1] The closure $\bar{F}^{ULT}$ of a subset F of $X(\Gamma)$ consists of the pairs (J, γ) where J is an order ideal and $\gamma \in \widehat{J}$ satisfies: for every open neighbourhood N of γ in $\widehat{J}$, there exist $I \in \Sigma(\Gamma)$ and $\chi \in N$ such that $I \subset J$ and $(I, \chi|_I) \in F$.

It was verified in [2, Lemma 4.2] that the closure operation in the definition above satisfies the Kuratowski closure axiom, hence it generates a topology in $X(\Gamma)$. Thus the definition above describes a topology for $X(\Gamma)$. In this paper we are dealing with many topologies, we need to state our convention. We write $\bar{F}^{ULT}$ to denote the closure of a subset $F \subset X(\Gamma)$ in the upwards-looking topology for $X(\Gamma)$, and we write $\bar{F}^{PWT}$ to denote the closure of a subset $F \subset \hat{I}$ in the point wise topology for $\hat{I}$. For $I, J \in \Sigma(\Gamma)$ with $J \subset I$, the map $\hat{I} \ni \gamma \mapsto \gamma|_J \in \hat{J}$ is an open map with respect to the point wise topology.

The next theorem gives the set of basis of open sets in the upwards-looking topology.

Theorem 2. [7, Lemma IV.18] *Suppose* Γ *is a totally ordered abelian group. For* $I \in \Sigma(\Gamma)$ *and open set* $M \subset \hat{I}$ *we define* $\mathcal{O}_{I,M} := \{(J, \gamma|_J) : J \subset I, \gamma \in M\}$. *Then*

$$\{\mathcal{O}_{I,M} : I \in \Sigma(\Gamma), M \text{ is open in the point wise topology on } \hat{I}\} \tag{2.1}$$

is an open basis for the topology upwards-looking topology described in Definition 1 .

Proof. To see $\mathcal{O}_{I,M}$ is open, we show that its complement is closed. For this, let $(K, \gamma) \in \overline{X(\Gamma) \setminus \mathcal{O}_{I,M}}$. We want $(K, \gamma) \in X(\Gamma) \setminus \mathcal{O}_{I,M}$, i.e $K \subset I$ and $\gamma \notin \{\alpha|_K : \alpha \in M\}$, or $K \supsetneqq I$. Assume that $K \subset I$ and $\gamma \in \{\alpha|_K : \alpha \in M\}$. We claim that $K \supsetneqq I$. That $K = I$ is impossible. To see this, take M as an open neighbourhood of γ in $\hat{I}$ and there is $\chi \in M$, $J \in \Sigma(\Gamma)$ such that $J \subset I$ and $(J, \chi|_J) \in X(\Gamma) \setminus \mathcal{O}_{I,M}$. But this implies that $J \supsetneqq I$, or, $J \subset I$ and $\chi|_J \notin \{\alpha|_K : \alpha \in M\}$, which both give contradictions. That $K \subsetneqq I$ is also impossible. Because for the open neighbourhood $N = \{\alpha|_K : \alpha \in M\}$ of γ in $\hat{K}$, there is $\chi \in N$, $L \in \Sigma(\Gamma)$ such that $L \subset K$ and $(L, \chi|_L) \in X(\Gamma) \setminus \mathcal{O}_{I,M}$ which implies that $L \supsetneqq I$, or, $L \subset I$ and $\chi|_L \notin \{\alpha|_L : \alpha \in M\}$, which again both of them give contradictions. Therefore $\mathcal{O}_{I,M}$ is open.

We next show that (2.1) is a basis of open sets for the topology described in Definition 1. Given $(J, \gamma) \in X(\Gamma)$ and a closed set F such that $(J, \gamma) \notin F$. Definition 1 implies that there is a neighbourhood N of γ in $\hat{J}$ such that $(I, \tau|_I) \notin F$ for all $\tau \in N$, $I \subset J$. So $(J, \gamma) \in \mathcal{O}_{J,N} \subset X(\Gamma) \setminus F$. ☐

Example 3. We are going to discuss some description of sets in $X(\Gamma)$ by considering a specific case of Γ. An observation on $\Gamma := \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ gives interesting results. Let I be the ideal $\{(0, n) : n \in \mathbb{Z}\}$, since I is the only ideal, we have $X(\Gamma) = \hat{0} \sqcup \hat{I} \sqcup \hat{\Gamma}$. Suppose λ_0 is a character in $\hat{I}$ defined by $(0, n) \mapsto e^{2\pi i n}$, and let $F = \{\lambda_0\}$. Next we consider a character γ in $\hat{\Gamma}$ defined by $(m, n) \mapsto e^{2\pi i(m+n)}$. It is clear that $\gamma|_I = \lambda_0$. Then $\gamma \in \bar{F}^{ULT}$, because every open neighbourhood N of γ in $\hat{\Gamma}$ contains an element λ (which is nothing but γ it self) such that its restriction on I gives a character in F. It is clear that $\gamma \notin F$, hence F is not closed in the upwards-looking topology for $X(\Gamma)$.

The example above implies that any closed set in the point wise topology is not necessarily closed in the upwards-looking topology. Similarly, when we apply to any complement F^C of a set F, we arrive to a conclusion that any open set in the point wise topology, is not necessarily open in the upwards-looking topology. This observation gives a similar result for more general cases.

Proposition 4. *Let Γ be a totally ordered abelian group which has a nontrivial order ideal. Suppose $I \in \Sigma(\Gamma)$ such that $I \neq \Gamma$. Then any subset $F \subset \hat{I}$ is not closed in $X(\Gamma)$. Moreover $X(\Gamma)$ is not a T_1 space.*

Proof. Let $\lambda \in F$, and let χ be an extension of λ to a character in $\hat{\Gamma}$. We are going to show that $\chi \in \bar{F}$, and hence F is not closed. Let N be any open neighbourhood of χ in $\hat{\Gamma}$. Since $\chi \in N$ and $\chi|_I = \lambda \in F$, the character χ is in the closure $\bar{F}^{ULT}$ of F. But $\chi \notin F$, hence F is not closed.

For the second assertion, let $\lambda \in \hat{I}$ and apply the first assertion to the singleton set $\{\lambda\}$. □

Remark 5. A sequence in $X(\Gamma)$ could converge to more than one point. Indeed, let $\{(J, \gamma)\} \subset \hat{J}$, and $\gamma_n := \gamma \; \forall n \in \mathbb{N}$. Let $K \in \Sigma(\Gamma)$, such that $K \supset J$ and consider a character $(K, \chi) \in \hat{K}$ such that $\chi|_J = \gamma$. We show that $\gamma_n \to \chi$ in the upwards-looking topology for $X(\Gamma)$. Let $\mathcal{O}_{M,I}$ be any open neighbourhood of (K, χ) in $X(\Gamma)$, then $I \supset J$ and M is an open set in the point wise topology for $\hat{I}$ such that the set $M|_K := \{(K, \gamma|_K) : \gamma \in M\}$ contains (K, χ). Then $(J, \gamma) \in \mathcal{O}_{I,M}$, hence $\gamma_n \to \chi$ in the upwards-looking topology for $X(\Gamma)$. It is clear that we also have $\gamma_n \to \gamma$. Therefore there is more than one point of convergence of the sequence $\{\gamma_n\}$.

We found that a closed subset in the point wise topology on $\hat{\Gamma}$ is also closed in the upwards-looking topology on $X(\Gamma)$.

Proposition 6. *Suppose $F \subset \hat{\Gamma}$. If F is closed in the point wise topology on $\hat{\Gamma}$ then F is closed in the upwards-looking topology for $X(\Gamma)$.*

Proof. Let $\gamma \notin F$, we are going to show that $\gamma \notin \bar{F}^{ULT}$. We next consider two cases, i.e. when $\gamma \notin \hat{\Gamma}$ and when $\gamma \in \hat{\Gamma}$. If $\gamma \notin \hat{\Gamma}$, it is clear that $\gamma \notin \bar{F}^{ULT}$, because the restriction of every character in any open neighbourhood of γ will not give any character in $F \subset \hat{\Gamma}$. For the case $\gamma \in \hat{\Gamma}$, take $N = F^c$ as an open neighbourhood of γ in $\hat{\Gamma}$. Since $N \cap F = \emptyset$, then $\gamma \notin \bar{F}^{ULT}$. □

Corollary 7. *Let γ be any character in $\hat{\Gamma}$. Then the set $\{\gamma\}$ is closed in the upwards-looking topology for $X(\Gamma)$.*

For every order ideal $I \in \Sigma(\Gamma)$, we have $\hat{I} \subset X(\Gamma)$. Then it will be interesting to discuss the relative topology on every $\hat{I}$, and find out the relation with the point wise topology on it. We will write $\bar{F}^{RULT}$ for the closure of a subset $F \subset \hat{\Gamma}$ in the relative upwards-looking topology on $\hat{\Gamma}$, i.e

$$\bar{F}^{RULT} := \bar{F}^{ULT} \cap \hat{\Gamma}.$$

Theorem 8. *Suppose that Γ is a totally ordered abelian group. Then the point wise topology on $\hat{\Gamma}$ and the relative upward-looking topology on $\hat{\Gamma}$ are coincide.*

Proof. We firstly show that the point wise topology on $\hat{\Gamma}$ is stronger than the relative upwards-looking topology on $\hat{\Gamma}$. To show this, let $F \subset \hat{\Gamma}$ be closed under the relative upwards-looking topology on $\hat{\Gamma}$, i.e

$$\bar{F}^{RULT} = F.$$

We want to show that $\bar{F}^{PWT} = F$. It is clear that $F \subset \bar{F}^{PWT}$. Now let $\gamma \notin F$. An argumentation like in the proof of Proposition 6 implies that either $\gamma \notin \bar{F}^{ULT}$ or $\gamma \notin \hat{\Gamma}$. But $\gamma \in \hat{\Gamma}$, thus $\gamma \notin \bar{F}^{ULT}$. Therefore there is a neighbourhood N of γ such that for every $\chi \in N$, $(\Gamma, \chi) \notin F$. Hence $N \cap F = \emptyset$, which concludes that $\gamma \notin \bar{F}^{PWT}$.

Secondly we show that the relative upwards-looking topology in $\hat{\Gamma}$ is stronger than the point wise topology in $\hat{\Gamma}$. For this, let F be a closed subset in the point wise topology of $\hat{\Gamma}$, we want to show that $F = \bar{F}^{RULT}$. It is clear that $F \subset \bar{F}^{RULT}$. To complete the proof, let $\gamma \notin F$ and we show that $\gamma \notin \bar{F}^{ULT}$, which then implies that $\gamma \notin \bar{F}^{ULT} \cap \hat{\Gamma}$. If $\gamma \in \bar{F}^{ULT}$, then for every open neighbourhood $N(\gamma)$ in $\hat{\Gamma}$ of γ, there exists $\chi \in N(\gamma)$, an order ideal $J \subset \Gamma$ such that $(J, \chi|_J) \in F$. Since $F \subset \hat{\Gamma}$, it is clear that $J = \Gamma$. Thus $N(\gamma) \cap F \neq \emptyset$, which then implies that $\gamma \in \bar{F}^{PWT} = F$. This contradicts the assumption that $\gamma \notin F$.

□

3. Characterization of Maximal Primitive Ideals

Let Γ be a totally ordered abelian group. The Toeplitz algebra $\mathcal{T}(\Gamma)$ of Γ is the C*-subalgebra of $B(\ell^2(\Gamma^+))$ generated by the isometries $\{T_x = T_x^\Gamma : x \in \Gamma^+\}$ which are defined in terms of the usual basis by $T_x(e_y) = e_{y+x}$. This algebra is universal for isometric representation of Γ^+ [5, Theorem 2.9].

Let I be an order ideal of Γ. Then the map $x \longmapsto T_{x+I}^{\Gamma/I}$ is an isometric representation of Γ^+ in $\mathcal{T}(\Gamma/I)$. Therefore by the universality of $\mathcal{T}(\Gamma)$, there is a homomorphism $Q_I : \mathcal{T}(\Gamma) \longrightarrow \mathcal{T}(\Gamma/I)$ such that $Q_I(T_x) = T_{x+I}^{\Gamma/I}$, and that Q_I is surjective. Suppose $\mathcal{C}(\Gamma, I)$ denotes the ideal in $\mathcal{T}(\Gamma)$ generated by $\{T_uT_u^* - T_vT_v^* : v - u \in I^+\}$ and $\mathrm{Ind}_{I^\perp}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I})$ is the closed subalgebra of $C(\hat{\Gamma}, \mathcal{T}(\Gamma/I))$ satisfying $f(xh) = {\alpha_h^{\Gamma/I}}^{-1}(f(x))$ for $x \in \hat{\Gamma}, h \in I^\perp$. It was proved in [1, Theorem 3.1] that there is a short exact sequence of C^*-algebras:

$$0 \to \mathcal{C}(\Gamma, I) \to \mathcal{T}(\Gamma) \xrightarrow{\phi_I} \mathrm{Ind}_{I^\perp}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I}) \to 0. \tag{3.1}$$

in which $\phi_I(a)(\gamma) = Q_I \circ (\alpha_\gamma^\Gamma)^{-1}(a)$ for $a \in \mathcal{T}(\Gamma)$, $\gamma \in \hat{\Gamma}$, and α_γ is dual action of $\hat{\Gamma}$ on $\mathcal{T}(\Gamma)$ characterized by $\alpha_\gamma^\Gamma(T_x) = \gamma(x)T_x$. The identity representation $T^{\Gamma/I}$ of $\mathcal{T}(\Gamma/I)$ is irreducible [5], it follows from [6, Proposition 6.16] that $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ is a primitive ideal of $\mathcal{T}(\Gamma)$. Moreover since

$$Q_I \circ \alpha_\chi^\Gamma = \alpha_\chi^{\Gamma/I} \circ Q_I \text{ for } \chi \in I^\perp = \widehat{\Gamma/I}, \tag{3.2}$$

the map $\gamma \longmapsto \ker Q_I \circ \alpha_\gamma^{-1}$ is constant on $I^\perp$ cosets in $\hat{\Gamma}$. Therefore it induces a well defined map L of $\hat{I} = \hat{\Gamma}/I^\perp$ into $\mathrm{Prim}\,\mathcal{T}(\Gamma)$. So that

$$L(I, \gamma) := \ker Q_I \circ {\alpha_\nu^\Gamma}^{-1} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma, \tag{3.3}$$

and then it was proved in Theorem 3.1 that L is a bijection of $X(\Gamma)$ onto $\mathrm{Prim}\,\mathcal{T}(\Gamma)$. Using the bijection $L : (I, \gamma) \in X(\Gamma) \longmapsto \ker Q_I \circ {\alpha_\nu^\Gamma}^{-1} \in \mathrm{Prim}\,\mathcal{T}(\Gamma)$, Adji and Raeburn describe a topology on $X(\Gamma)$ which corresponds to the hull-kernel topology on $\mathrm{Prim}\,\mathcal{T}(\Gamma)$.

Adji and Raeburn [2] proved that for any totally ordered abelian group Γ, every primitive ideal of $\mathcal{T}(\Gamma)$ has the form $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ for some $I \in \Sigma(\Gamma)$ and $\gamma \in \hat{\Gamma}$. In this section we are going to apply our findings and a theorem of Dixmier [4, 3.1.4], which states that a singleton subset $\{\mathcal{I}\}$ of the set of primitive ideals is closed if and only if $\mathcal{I}$ is a maximal ideal, to identify which ones are maximal ideals and which ones are not.

Theorem 4.7 of [2] says that for a totally ordered abelian group Γ such that $\Sigma(\Gamma)$ is isomorphic to a subset of $\mathbb{N} \cup \{\infty\}$, the topology of $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ is described by the upwards-looking topology on the disjoint union $X(\Gamma) := \bigsqcup\{\widehat{I} : I \in \Sigma(\Gamma)\}$ by specifying the closure operation. More recent result in [3, Theorem 3.1] shows that the topology on $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ is given by the upwards-looking topology on $X(\Gamma)$, if and only if the set $\Sigma(\Gamma)$ of order ideals in Γ is well-ordered by inclusion.

Lemma 9. *Suppose that* Γ *is a totally ordered abelian group such that the chain* $\Sigma(\Gamma)$ *is well-ordered. If* $I \in \Sigma(\Gamma)$ *such that* $I \neq \Gamma$*, then for any* $\gamma \in \hat{\Gamma}$*, the primitive ideal*

$$\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$$

is not maximal.

Proof. The identification of primitive ideals in $\mathcal{T}(\Gamma)$ given in (3.3), implies that for any $I \in \Sigma(\Gamma)$ and $\gamma \in \hat{\Gamma}$, the singleton set $\{\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}\}$ corresponds to the singleton set $\{\lambda\}$ for some $\lambda \in \hat{I}$ such that $\gamma|_I = \lambda$. From Proposition 4, the set $\{(I, \lambda)\}$ is not closed, hence the set $\{\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}\}$ is not closed. The theorem of Dixmier hence implies that $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ is not a maximal ideal. □

Lemma 10. *Suppose that* Γ *is a totally ordered abelian group such that the chain* $\Sigma(\Gamma)$ *is well-ordered. For any* $\gamma \in \hat{\Gamma}$*, the primitive ideal*

$$\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}$$

is maximal.

Proof. Since $\{\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}\}$ corresponds to $\{\gamma\}$ which is closed by Corollary 7, then $\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}$ is maximal by the theorem of Dixmier. □

Theorem 11. *Suppose that* Γ *is a totally ordered abelian group such that the chain* $\Sigma(\Gamma)$ *is well-ordered. Every maximal ideal in* $\operatorname{Prim} \mathcal{T}(\Gamma)$ *is of the form* $\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}$.

Proof. By [2, Corollary 3.4], every primitive ideal in $\mathcal{T}(\Gamma)$ is of the form $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ for some $I \in \Sigma(\Gamma)$ and $\gamma \in \hat{\Gamma}$. The result then follows from Lemma 9 and Lemma 10. □

References

[1] S. Adji, *Semigroup crossed products and the structure of Toeplitz algebras*, J. Op. Theory 44 (2000), 139–150.

[2] S. Adji and I. Raeburn, *The ideal structure of Toeplitz algebra*, Integral Equations Operator Theory 48 (2004), 281-293.

[3] S. Adji, I. Raeburn and R. Rosjanuardi, *Group extensions and the primitive ideal spaces of Toeplitz algebras*, Glasgow Mathematical Journal 49 (2007), pp. 81-92.

[4] J. Dixmier, *C*-Algebras*, North Holland Publ. Comp., Amsterdam, New York, Oxford (1982).

[5] G.J. Murphy, *Ordered groups and Toeplitz algebras*, J. Operator Theory 18 (1987), 303-326.

[6] D. P William and I. Raeburn, *Morita equivalence and continuous-trace C^*-algebras.* Amer. Math. Soc., Providence, RI (1998).

[7] R. Rosjanuardi, *Structure theory of twisted Toeplitz algebras and ideal structure of the (untwisted) Toeplitz algebras of ordered groups*, Dissertation, ITB, Indonesia (2005).

Rizky ROSJANUARDI
Department of Mathematics Education
Universitas Pendidikan Indonesia (UPI)
Jl. Dr Setia Budhi 229 Bandung 40154
Indonesia
e-mail: rizky@upi.edu

Takashi ITOH
Department of Mathematics
Gunma University
4-2 Aramaki, Maebashi, Gunma 371-8510
Japan
e-mail: itoh@edu.gunma-u.ac.jp