A THEOREM ON THE SUBJECT OF COOK'S INEQUALITY

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Abstract. We show that the span of an arbitrary simple closed curve X does not exceed the span of any starlike curve contained in the closure of the unbounded component of the complement of X.

1. Definitions and auxiliary lemmas

We shall begin by reviewing the definitions introduced by A. Lelek in [6] and [7]. Let X be a connected nonempty metric space. The span $\sigma(X)$ of X is the least upper bound of the set of nonnegative numbers r that satisfy the following condition: there exists a connected space Y and a pair of continuous functions $f, g : Y \to X$ such that $f(Y) = g(Y)$ and $\operatorname{dist}[f(y), g(y)] \geq r$ for every $y \in Y$. To obtain the definition of the semispan $\sigma_0(X)$ of X, the equality $f(Y) = g(Y)$ is relaxed to the inclusion of $f(Y) \supset g(Y)$. Requiring that f be onto gives the definitions of surjective span $\sigma^*(X)$ and surjective semispan $\sigma_0^*(X)$ of X. The last two concepts coincide with the span and semispan, respectively, when X is a simple closed curve.

In general, as was pointed out in [7], $0 \leq \sigma(X) \leq \sigma_0(X) \leq \operatorname{diam}(X)$. Furthermore, it follows from the more general result of A. Lelek [7, Th 2.1, p39] that when X is a continuum then $\sigma_0(X) \leq \varepsilon(X)$, where $\varepsilon(X)$ denotes the infimum of the set of meshes of the chains that cover X. A different, direct proof of this inequality can be found in [1]. The span of an arbitrary simple closed curve X that is a boundary of a convex region has been determined in [5]. It has been proven to be equal to its semispan, the infimum of the set of its directional diameters, called the breadth of X in [8], and $\varepsilon(X)$.

A simple closed curve X is starlike if there is a point Q in the bounded component D of $C \setminus X$ such that for each point $P, P \in X$, the line segment PQ is contained in the closure of D. For prior work on starlike curves related to span see [2] and [3].

The following versions of the Mountain–Climbing Theorem shall be needed (see the work of J. V. Whittaker in [9]).

Lemma 1.1. *Let $0 \leq a < b, c > 0$. Suppose $f : [a, b] \to [0, c]$ is continuous, increasing, and $f(a) = 0$, $f(b) = c$. Suppose also that $g : [a, b] \to [0, c]$ is continuous, piecewise weakly monotone, and $g(a) = 0$, $g(b) = c$. Then, there exists a continuous mapping $\phi : [a, b] \to [a, b]$ such that $\phi(a) = a$, $\phi(b) = b$ and $f(\phi(t)) = g(t)$ for each $t \in [a, b]$.*

Lemma 1.2. *Let $0 \leq a < b$, $c > 0$. Suppose $f : [a, b] \to [0, c]$ is continuous, decreasing, and $f(a) = c$, $f(b) = 0$. Suppose also that $g : [a, b] \to [0, c]$ is continuous, piecewise weakly monotone, and $g(a) = c$, $g(b) = 0$. Then there exists a continuous mapping $\phi : [a, b] \to [a, b]$ such that $\phi(a) = a$, $\phi(b) = b$ and $f(\phi(t)) = g(t)$ for each $t \in [a, b]$.*

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2. The main result

The famous problem of Howard Cook: Do there exist, in the plane, two simple closed curves X and Y, such that X is in the bounded component of the complement of Y and the span of X is greater than the span of Y? [Problem 173 of "A list of problems known as the Houston Problem Book," *Lecture Notes in Pure and Applied Mathematics*, 170, Marcel Dekker, Inc., New York, Basel and Hong Kong, 365–398] has been answered, in the negative, in special cases only. For a survey of related conditions, imposed on either X or Y, or both, that guarantee the negative answer, see [4].

Let h be an arbitrary function with values in $C \setminus \{0\}$. In the following theorem, $\operatorname{Arg} h(t)$ denotes the counterclockwise angle between the positive x–axis and the ray containing the line segment $0h(t)$ connecting the points 0 and $h(t)$. Notice that $\operatorname{Arg} h(t) \in [0, 2\pi)$.

Theorem. *Let X be a simple closed curve in the plane C. If Y is a starlike curve contained in the closure of the unbounded component of $C \setminus X$ then $\sigma(X) \leq \sigma(Y)$.*

Proof. Without loss of generality, we shall assume that 0 lies in the bounded component of $C \setminus X$. Let ε, $\varepsilon > 0$, be an arbitrarily small number. It follows from the definition of span that there exist two continuous functions G_1, $G_2 : [0,1] \to X$ such that $G_1([0,1]) = G_2([0,1]) = X$ and

$$\sigma(X) \geq \inf_{t\in[0,1]} \operatorname{dist}[G_1(t), G_2(t)] > \sigma(X) - \varepsilon/2. \tag{2.1}$$

The Weierstrass Approximation Theorem implies the existence of two polynomials ${}^{\sim}G_1$, ${}^{\sim}G_2$ such that

$$\forall_{t\in[0,1]} |G_i(t) - {}^{\sim}G_i(t)| < \varepsilon/4, \quad i = 1, 2. \tag{2.2}$$

Note that $\operatorname{Arg}{}^{\sim}G_1$, and $\operatorname{Arg}{}^{\sim}G_2$ are not continuous. Let $t_1, \ldots, t_m$ be the points of discontinuity of $\operatorname{Arg}{}^{\sim}G_1$ on $[0, 1]$. Assume, without loss of generality, that $0 < t_1 < \cdots < t_m \leq 1$, and that $\operatorname{Arg}{}^{\sim}G_1(0) = 0$. Furthermore, if $t_m < 1$ put $t_{m+1} = 1$.

We shall also assume, without loss of generality, that Y is a starlike polygonal line with strictly increasing argument. Let $F : [0,1] \to Y$ be the mapping that defines Y. F is one-to-one on $[0, 1)$, and $F(0) = F(1)$. We can also assume, without loss of generality, that $\operatorname{Arg} F(0) = 0$. Let

$$f(t) = \begin{cases} \operatorname{Arg} F(t), & \text{for } t \in [0,1) \\ 2\pi, & \text{for } t = 1. \end{cases}$$

Thus, f is increasing and continuous on $[0, 1]$. Let $t_0 = 0$. Note that for each $n \in N \cup \{0\}$, $0 \leq n \leq m$, $\operatorname{Arg}{}^{\sim}G_1(t_n) = 0$. We shall modify $\operatorname{Arg}{}^{\sim}G_1$ at some of its points of discontinuity, by changing its value from 0 to 2π, so that on every interval $[t_n, t_{n+1}]$ thus modified portion of $\operatorname{Arg}{}^{\sim}G_1$ can be continuous, with values in $[0, 2\pi]$, and piecewise weakly monotone.

There are four cases regarding the behavior of $\operatorname{Arg}{}^{\sim}G_1$ on an arbitrary $[t_n, t_{n+1}]$.

Case 1. The restriction of $\operatorname{Arg}{}^{\sim}G_1$ to $[t_n, t_{n+1}]$ is continuous on $[t_n, t_{n+1})$ only. See Figure 1.

Case 2. The restriction of $\operatorname{Arg}{}^{\sim}G_1$ to $[t_n, t_{n+1}]$ is continuous on $(t_n, t_{n+1}]$ only. See Figure 2.

Notice that, in both case 1 and case 2,

$$\sup_{t\in[t_n,t_{n+1}]} \operatorname{Arg}{}^{\sim}G_1 = 2\pi \quad \text{and} \quad \inf_{t\in[t_n,t_{n+1}]} \operatorname{Arg}{}^{\sim}G_1 = 0.$$

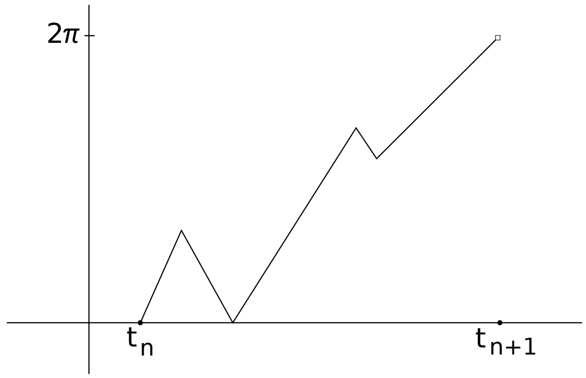

Figure 1

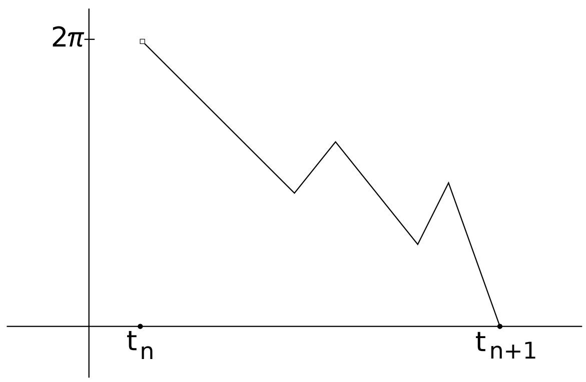

Figure 2

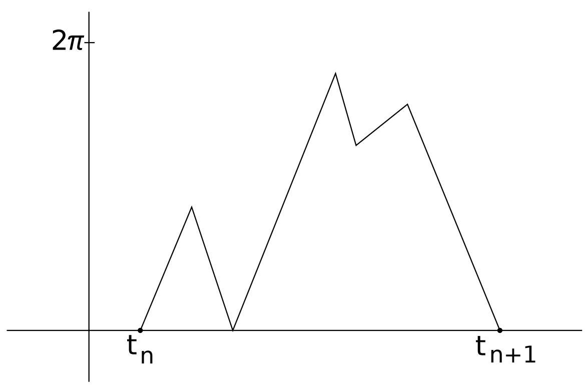

Figure 3

Case 3. The restriction of $\operatorname{Arg}^{\sim} G_1$ to $[t_n, t_{n+1}]$ is continuous. See Figure 3.

Note that in case 3 $\sup\limits_{t\in[t_n,t_{n+1}]} \operatorname{Arg}^{\sim} G_1 < 2\pi$

Case 4. The restriction of $\operatorname{Arg}^{\sim} G_1$ to $[t_n, t_{n+1}]$ is continuous on (t_n, t_{n+1}) only. See Figure 4.

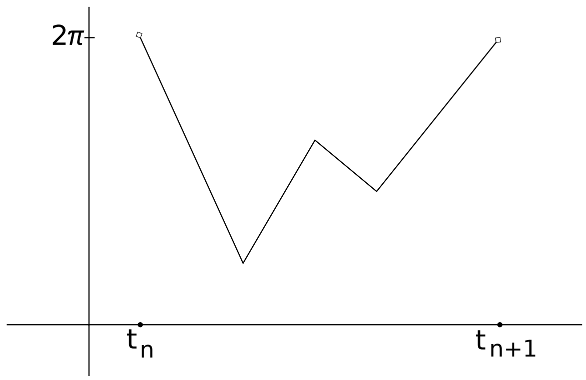

FIGURE 4

In case 1, we define g_1 as follows.

$$g_1(t) = \begin{cases} \operatorname{Arg}^{\sim} G_1(t) & \text{for } t \in [t_n, t_{n+1}) \\ 2\pi & \text{for } t = t_{n+1}. \end{cases}$$

Next, let h_n be an affine mapping from $[t_n, t_{n+1}]$ onto $[0, 1]$ such that $h_n(t_n) = 0$ and $h_n(t_{n+1}) = 1$, and put $f_n(t) = f(h_n(t))$ for all $t \in [t_n, t_{n+1}]$. Since f_n is continuous and increasing on $[t_n, t_{n+1}]$, g_1 is continuous and piecewise weakly monotone on $[t_n, t_{n+1}]$, $f_n(t_n) = g_1(t_n) = 0$ and $f_n(t_{n+1}) = g_1(t_{n+1}) = 2\pi$, by virtue of Lemma 1.1 there exists a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In case 2, we define g_1 as follows.

$$g_1(t) = \begin{cases} 2\pi & \text{for } t = t_n \\ \operatorname{Arg}^{\sim} G_1(t) & \text{for } t \in (t_n, t_{n+1}]. \end{cases}$$

With h_n defined as in case 1, put $f_n(t) = f(h_n(t_{n+1} - (t - t_n)))$. Notice that $f_n(t_n) = f(h_n(t_{n+1})) = 2\pi = g_1(t_n)$, and $f_n(t_{n+1}) = f(h_n(t_n)) = 0 = g_1(t_{n+1})$. Since f_n is decreasing and g_1 is piecewise weakly monotone, by virtue of Lemma 1.2, there exists a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In case 3, put $g_1(t) = \operatorname{Arg}^{\sim} G_1(t)$ for all $t \in [t_n, t_{n+1}]$ and let $c = \sup\limits_{t\in[t_n,t_{n+1}]} g_1(t)$. Furthermore, let t_c be such that $g_1(t_c) = c$ and $g_1(t) < c$ for all $t \in [t_n, t_c)$. Next, with h_n defined as in case 1, put $f_n^{\sim}(t) = f(h_n(t))$ for all $t \in [t_n, t_{n+1}]$. Since $c < 2\pi$ there exists a number t_s, $t_s \in (t_n, t_{n+1})$ such that $f_n^{\sim}(t_s) = c$. If $t_s = t_c$, put $f_n^*(t) = f_n^{\sim}(t)$ for all

$t \in [t_n, t_c]$. If not, let k_n be an affine mapping from $[t_n, t_c]$ onto $[t_n, t_s]$ such that $k_n(t_n) = t_n$ and $k_n(t_c) = t_s$ and put $f_n^*(t) = f_n^\sim(k_n(t))$ for all $t \in [t_n, t_c]$. We define f_n as follows

$$f_n(t) = \begin{cases} f_n^*(t), & \text{when } t \in [t_n, t_c] \\ f_n^*(t_n + (t_c - t_n)(t_{n+1} - t)/(t_{n+1} - t_c)), & t \in [t_c, t_{n+1}]. \end{cases}$$

Notice that $f_n(t_c) = c$, $f_n(t_n) = f_n(t_{n+1}) = 0$, f_n is increasing on $[t_n, t_c]$ and decreasing on $[t_c, t_{n+1}]$. By applying Lemma 1.1 on $[t_n, t_c]$ and Lemma 1.2 on $[t_c, t_{n+1}]$ we obtain a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In case 4, we define g_1 as follows.

$$g_1(t) = \begin{cases} 2\pi & \text{for } t = t_n \\ \operatorname{Arg}{}^\sim G_1(t) & \text{for } t \in (t_n, t_{n+1}) \\ 2\pi & \text{for } t = t_{n+1}. \end{cases}$$

Let $c = \inf_{t \in [t_n, t_{n+1}]} g_1(t)$. Notice that $c \geq 0$. Let t_c be such that $g_1(t_c) = c$ and $g_1(t) > c$ for all $t \in [t_n, t_c)$. We shall define f_n differently depending on whether c is positive or not.

If $c = 0$ then let h_{nc} be an affine mapping from $[t_n, t_c]$ onto $[0, 1]$ such that $h_{nc}(t_n) = 0$ and $h_{nc}(t_c) = 1$, and put $f_n^\sim(t) = f(h_{nc}(t_c - (t - t_n)))$ for all $t \in [t_n, t_c]$. Notice that $f_n^\sim(t_n) = f(h_{nc}(t_c)) = f(1) = 2\pi$, $f_n^\sim(t_c) = f(h_{nc}(t_n)) = f(0) = 0$, and $f_n^\sim$ is decreasing. Next, let h_c be an affine mapping from $[t_c, t_{n+1}]$ onto $[0, 1]$ such that $h_c(t_c) = 0$ and $h_c(t_{n+1}) = 1$, and define f_n as follows

$$f_n(t) = \begin{cases} f_n^\sim(t), & \text{when } t \in [t_n, t_c] \\ f(h_c(t)), & \text{when } t \in [t_c, t_{n+1}]. \end{cases} \tag{2.3}$$

If $c > 0$ then, with h_n defined as in case 1, put $f_n^\sim(t) = f(h_n(t))$ for all $t \in [t_c, t_{n+1}]$. There exists a number t_s, $t_s \in (t_n, t_{n+1})$, such that $f_n^\sim(t_s) = c$. If $t_s = t_c$, put $f_n^*(t) = f_n^\sim(t)$ for all $t \in [t_n, t_c]$. If not, let k_n be an affine mapping from $[t_c, t_{n+1}]$ onto $[t_s, t_{n+1}]$ such that $k_n(t_c) = t_s$ and $k_n(t_n) = t_{n+1}$ and put $f_n^*(t) = f_n^\sim(k_n(t))$ for all $t \in [t_c, t_{n+1}]$. We define f_n as follows

$$f_n(t) = \begin{cases} f_n^*(t_{n+1} - (t - t_n)(t_{n+1} - t_c)/(t_c - t_n)), & t \in [t_n, t_c] \\ f_n^*(t), & \text{when } t \in [t_c, t_{n+1}]. \end{cases} \tag{2.4}$$

Both (2.3) and (2.4) give us f_n that is decreasing on $[t_n, t_c]$ and increasing on $[t_c, t_{n+1}]$. Furthermore, $f_n(t_c) = c$ and $f_n(t_n) = f_n(t_{n+1}) = 2\pi$. We apply Lemma 1.2 on $[t_n, t_c]$ and Lemma 1.1 on $[t_c, t_{n+1}]$ to obtain a continuous mapping $\phi_n : [t_n, t_{n+1}] \to [t_n, t_{n+1}]$ such that $\phi_n(t_n) = t_n$, $\phi_n(t_{n+1}) = t_{n+1}$ and $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$.

In all four cases, $f_n(\phi_n(t)) = g_1(t)$ for all $t \in [t_n, t_{n+1}]$. Furthermore, the principal value of the argument $\operatorname{Arg} g_1(t) = \operatorname{Arg}{}^\sim G_1(t)$ for all $t \in [0, 1]$. We shall now define a mapping $F_1 : [0, 1] \to Y$ in the following manner. For each n, $0 \leq n \leq m$, put $F_1(t_n) = F(0)$ and if $t_m = 1$ then also put $F_1(1) = F(0)$. Suppose $t \in (0, 1)$, $t \neq t_n$, $n = 1, \ldots, m$. Then, $t \in (t_n, t_{n+1})$ for some n, $0 \leq n \leq m$, and $f_n(\phi_n(t)) \in [0, 2\pi)$. If $f_n(\phi_n(t)) = 0$ then put $F_1(t) = F(0)$. If $f_n(\phi_n(t)) \in (0, 2\pi)$ then, since F is 1:1 on $(0, 1)$, there is exactly one value $s \in (0, 1)$ such that $\operatorname{Arg} F(s) = f_n(\phi_n(t))$. Put $F_1(t) = F(s)$. Note that $F_1([0, 1]) = Y$ and

$$\operatorname{Arg} F_1(t) = \operatorname{Arg}{}^\sim G_1(t) \quad \text{for all } t \in [0, 1]. \tag{2.5}$$

Taking analogous steps with respect to ${}^\sim G_2$, we define an onto mapping $F_2 : [0, 1] \to Y$ such that

$$\operatorname{Arg} F_2(t) = \operatorname{Arg}{}^\sim G_2(t) \quad \text{for all } t \in [0, 1]. \tag{2.6}$$

Since Y is starlike, the equalities (2.5) and (2.6) imply that for all $t \in [0,1]$

$$|F_1(t) - F_2(t)| \geq \left|{}^{\sim}G_1(t) - {}^{\sim}G_2(t)\right|. \tag{2.7}$$

Consequently, taking (2.1) and (2.2) into account, it follows that

$$\sigma(Y) \geq \inf_{t\in[0,1]} |F_1(t) - F_2(t)| \geq \inf_{t\in[0,1]} |{}^{\sim}G_1(t) - {}^{\sim}G_2(t)|$$
$$\geq \inf_{t\in[0,1]} |G_1(t) - G_2(t)| - \varepsilon/2 > \sigma(X) - \varepsilon/2 - \varepsilon/2 = \sigma(X) - \varepsilon.$$

Finally, since ε was an arbitrary positive number, we conclude that $\sigma(Y) \geq \sigma(X)$.

□

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