ON VAGUE SUBALGEBRA OF d-ALGEBRA

KUNG HO KIM

Received April 13, 2010

Abstract. In this paper, the notion of vague subalgebra of a d-algebra is introduced and some properties are investigated. Using subsets with some conditions, vague subalgebras are constructed and also using a given vague subalgebra, a new vague subalgebra is established.

1. **Introduction.** Y. Imai and Iseki [3, 4] introduced two classes of abstract: BCK-algebras and BCI-algebras. J. Negges [5] introduced the class of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. The notion of vague set theory is introduced by Gau and Buehrer [2]. Using the vague set in the sense of Gau and Buehrer, Biawas[1] studied vague groups. In this paper, the notion of vague subalgebra of a d-algebra is introduced and some properties are investigated. Using subsets with some conditions, vague subalgebras are constructed and also using a given vague subalgebra, a new vague subalgebra is established.

2. **Preliminaries.** In this section we include some elementary aspects that are necessary for this paper.

A d-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:
(a1) $x * x = 0$,
(a2) $0 * x = 0$,
(a3) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.

Let S be a nonempty subset of a d-algebra X. Then S is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$. A map f from a d-algebra X to a d-algebra Y is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Definition 2.1. ([1]) A *vague set* A in the universe of discourse U is characterized by two membership functions given by :

1. A true membership function

$$t_A : U \to [0, 1]$$

and

2. A false membership function

$$f_A : U \to [0, 1],$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and $t_A(u) + f_A(u) \leq 1$.

2000 *Mathematics Subject Classification.* 06F35, 03G25, 03E72.
Key words and phrases. d-algebra, subalgebra, vague set. vague subalgebra, vague-cut.

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1-f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership of u is $\mu(u)$, then $t_A(u) \leq \mu(u) \leq 1-f_A(u)$. The vague set A is written as $A = \{\langle u, [t_A(u), f_A(u)]\rangle \mid u \in U\}$, where the interval $[t_A(u), 1-f_A(u)]$ of $[0,1]$ is called the *vague value* of u in A, denoted by $V_A(u)$. For $\alpha, \beta \in [0,1]$, we now define (α,β)-cut and α-cut of a vague set.

Definition 2.2. ([1]) A vague set A of a set U is called

1. the *zero vague set* of U if $t_A(u) = 0$ and $f_A(u) = 1$ for all $u \in U$,
2. the *unit vague set* of U if $t_A(u) = 1$ and $f_A(u) = 0$ for all $u \in U$,
3. the α*-vague set* of U if $t_A(u) = \alpha$ and $f_A(u) = 1-\alpha$ for all $u \in U$, where $\alpha \in [0,1]$.

For $\alpha, \beta \in [0,1]$ we define (α,β)-cut and α-cut of a vague set.

Definition 2.3. ([1]) Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The (α,β)*-cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \geq [\alpha,\beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α,β)-cuts are also *vague-cuts* of the vague set A.

Definition 2.4. ([1]) The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\alpha,\beta)} = A_\alpha$. Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

We shall use the following notations, which are given in [3], on interval arithmetic.

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. if $I_1 = [a_1, b_1]$ and $I_2 = [a_2 b_2]$ are two elements of $[0,1]$, we call $I_1 \geq I_2$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly we understand the relations $I_1 \leq I_2$ and $I_1 = I_2$. Clearly the relation $I_1 \geq I_2$ does not necessarily imply that $I_1 \geq I_2$ and conversely. We define the term *"imax"* to mean the maximum of two intervals as

$$imax(I_1, I_2) = [\max(a_1,a_2), \max(b_1,b_2)].$$

Similarly we define *"imin"*. could be extended to define *"isup"* and *"iinf"* of infinite number of elements of $I[0,1]$.

It is clear that $L = \{I[0,1], isup, iinf, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ (see [1]).

3. **Vague subalgebra of d-algebra.** In what follows, we use X to denote a d-algebra unless otherwise specified.

Definition 3.1. A vague set A of a d-algebra X is called a *vague subalgebra* of X if

$$(\forall x, y \in R) \quad (V_A(x*y) \geq \min\{V_A(x), V_A(y)\},$$

that is,

$$t_A(x*y) \geq \min\{t_A(x), t_A(y)\}, \quad 1-f_A(x*y) \geq \min\{1-f_A(x), 1-f_A(y)\},$$

for all $x, y \in X$.

Example 3.1. Consider a d-algebra $X = \{0, 1, 2\}$ having the following Cayley table:

$*$	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.7, 0.3]\rangle, \langle 1, [0.2, 0.5]\rangle, \langle 2, [0.2, 0.5]\rangle\}.$$

It is routine to verify that A is a vague subalgebra of d-algebra X.

Lemma 3.2. *Every vague subalgebra A of X satisfies*

$$(\forall x \in R) \quad (V_A(0) \geq V_A(x)),$$

that is, $t_A(0) \geq t_A(x)$ and $1 - f_A(0) \geq 1 - f_A(x)$ for all $x \in X$.

Proof. Let $x \in X$. Then $t_A(0) = t_A(x * x) \geq \min\{t_A(x), t_A(x)\} = t_A(x)$ and $1 - f_A(0) = 1 - f_A(x * x) \geq \min\{1 - f_A(x), 1 - f_A(x)\} = 1 - f_A(x)$. Hence we have $V_A(0) \geq V_A(x)$. □

Clearly the following proposition is straightforward.

Proposition 3.3. *The necessary and sufficient condition for a vague set $A = (x, t_A, f_A)$ of X to be a vague subalgebra of X is that t_A and $1 - f_A$ are fuzzy subalgebras of X.*

Proposition 3.4. *If A is an vague subalgebra of X, then the set*

$$T := \{x \in X \mid V_A(x) = V_A(0)\}$$

is a subalgebra of X.

Proof. Let $x, y \in T$. Then we have $V_A(x) = V_A(y) = V_A(0)$, and so $V_A(x*y) \geq \min\{V_A(x), V_A(y)\} = \min\{V_A(0), V_A(0)\} = V_A(0)$. Hence by Lemma 3.3, we get $V_A(x * y) = V_A(0)$, i. e., $x * y \in T$. Hence T is a subalgebra of X. □

Theorem 3.5. *Let A be a vague subalgebra of X. Then for $\alpha \in [0, 1]$, the α-cut A_α is a crisp subalgebra of X.*

Proof. Let $x, y \in A_\alpha$. Then we get $t_A(x) \geq \alpha$ and $t_A(y) \geq \alpha$, and so $t_A(x * y) \geq \min\{t_A(x), t_A(y)\} = \min\{\alpha, \alpha\} = \alpha$. Thus $x * y \in A_\alpha$. Therefore A_α is a crisp subalgebra of X. □

Theorem 3.6. *Let A be a vague subalgebra of X. Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ is a crisp subalgebra of X.*

Proof. Let $x, y \in A_{(\alpha,\beta)}$. Then $t_A(x) \geq \alpha$, $t_A(y) \geq \alpha$, $1 - f_A(x) \geq \beta$ and $1 - f_A(y) \geq \beta$. Thus $t_A(x*y) \geq \min\{t_A(x), t_A(y)\} \geq \min\{\alpha, \alpha\} = \alpha$ and $1 - f_A(x*y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} = \min\{\beta, \beta\} = \beta$. Therefore $x * y \in A_{(\alpha,\beta)}$. This completes the proof. □

This subalgebras like $A_{(\alpha,\beta)}$ are also called *vague-cut subalgebras* of X. Clearly we have the following result.

Proposition 3.7. *Let A be a vague subalgebra of X. Two vague-cut subalgebras $A_{(\alpha,\beta)}$ and $A_{(\omega,\gamma)}$ with $[\alpha, \beta] < [\omega, \gamma]$ are equal if and only if there is no $x \in X$ such that*

$$[\alpha, \beta] \leq V_A(x) \leq [\omega, \gamma].$$

Theorem 3.8. *Let X be a finite and let A be a vague subalgebra of X. Consider the set $V(A)$ given by*

$$V(A) := \{V_A(x) \mid x \in X\}.$$

Then A_i are the only vague-cut subalgebras of X, where $i \in V(A)$.

Proof. Let $[a_1, a_2] \in I[0,1]$ with $[a_1, a_2] \notin V_A$. If $[\alpha, \beta] < [a_1, a_2] < [\omega, \gamma]$ with $[\alpha, \beta], [\omega, \gamma] \in V_A$, then we have $A_{(\alpha,\beta)} = A_{(a_1,a_1)} = A_{(\omega,\gamma)}$. If $[a_1, a_2] < [a_1, a_3]$ where

$$[a_1, a_3] = imin\{x \mid x \in V(A)\},$$

then $A_{(a_1,a_3)}t = X = A_{(a_1,a_2)}$. Hence for any $[a_1, a_2] \in I[0,1]$, the vague-cut subalgebra $A_{(a_1,a_2)}$ is one of A_i for $i \in V(A)$. This completes the proof. □

Theorem 3.9. *Any subalgebra Q of a near-ring X is a vague-cut subalgebra of some vague subalgebra of X.*

Proof. Consider the vague set A of X given by

$$V_A(x) := \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in Q \\ [0,\ 0] & \text{if } x \notin Q, \end{cases}$$

where $\alpha_1, \alpha_2 \in (0, 1]$ with $\alpha_1 < \alpha_2$. It is clear that $Q = A_{(\alpha,\alpha_2)}$. We will show that A is an vague subalgebra of X. Let $x, y \in X$. If $x, y \in Q$, then $x * y \in Q$, and so $t_A(x * y) = \alpha_1 = \min\{\alpha_1, \alpha_2\}$ and $1 - f_A(x * y) = \alpha_2 = \min\{1 - f_A(x), 1 - f_A(y)\}$. If $x, y \notin Q$, then $t_A(x*y) \geq 0 = \min\{t_A(x), t_A(y)\}$ and $1-f_A(x*y) \geq 0 = \min\{1-f_A(x), 1-f_A(y)\}$. If $x \in Q$ and $y \notin Q$ (or $x \notin Q$ and $y \in Q$), then $t_A(x * y) \geq 0 = \min\{\alpha_1, 0\} = \min\{t_A(x), t_A(y)\}$ and $1 - f_A(x * y) \geq 0 = \min\{0, \alpha_2\} = \min\{1 - f_A(x), 1 - f_A(y)\}$. Therefore A is a vague subalgebra of X. □

Definition 3.10. If A is a vague set of X and θ is a map from X into itself, we define a maps $t_A^\theta : X \to [0, 1]$ and $f_A^\theta : X \to [0, 1]$ given by, respectively,

(1) $(\forall x \in X)\ \ t_A^\theta(x) = t_A(\theta(x))$ and

(2) $(\forall x \in X)\ \ f_A^\theta(x) = f_A(\theta(x))$.

In such cases, we write $V_A^\theta(x) = V_A(\theta(x))$ for all $x \in X$.

Theorem 3.11. *If A is a vague subalgebra of X and θ is a homomorphism of X, then the vague set A^θ of X given by*

$$A^\theta = \{\langle x, [t_A^\theta(x), f_A^\theta(x)]\rangle \mid x \in X\},$$

is also a vague subalgebra of X.

Proof. For every $x, y \in X$, we have

$$\begin{aligned} t_A^\theta(x * y) =& t_A(\theta(x * y)) = t_A(\theta(x) - \theta(y)) \\ &\geq \min\{t_A(\theta(x)), t_A(\theta(y))\} = \min\{t_A^\theta(x), t_A^\theta(y)\} \end{aligned}$$

and

$$\begin{aligned} 1 - f_A^\theta(x * y) =& 1 - f_A(\theta(x * y)) = 1 - f_A(\theta(x) - \theta(y)) \\ &\geq \min\{1 - f_A(\theta(x)), 1 - f_A(\theta(y))\} \\ &= \min\{1 - f_A^\theta(x), 1 - f_A^\theta(y)\} \end{aligned}$$

Therefore, V_{A^θ} is a vague subalgebra of X. □

Department of Mathematics, Chungk National University, Chungk, Chungking 380-702, Korea
E-mail address: ghkim@cjnu.ac.kr