A NOTE ON BE-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of essence in BE-algebras and related properties are investigated. Also, we discuss relations among subalgebras, filters and essences. Finally, we consider the homomorphic image and inverse image of an essence.

1. **Introduction.** Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generation of a BCK-algebras. In this paper, we introduce the notion of essence in BE-algebras and related properties are investigated. Also, we discuss relations among subalgebras, filters and essences. Finally, we consider the homomorphic image and inverse image of an essence.

2. **Preliminaries.** In what follows, let X denote an BE-algebra unless otherwise specified.

By a *BE-algebra* we mean an algebra $(X; *, 1)$ of type $(2, 0)$ with a single binary operation "$*$" that satisfies the following identities: for any $x, y, z \in X$,

(BE1) $x * x = 1$ for all $x \in X$,
(BE2) $x * 1 = 1$ for all $x \in X$,
(BE3) $1 * x = x$ for all $x \in X$,
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

We introduce a relation "$\leq$" on X by $x \leq y$ imply $x * y = 1$. An BE-algebra $(X, *, 1)$ is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of an BE-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$.

In an BE-algebra, the following identities are true:

(p1) $x * (y * x) = 1$.
(p2) $x * ((x * y) * y)) = 1$.

Definition 2.1. Let $(X, *, 1)$ be an BE-algebra and F a non-empty subset of X. Then F is said to be a *filter* of X if

(F1) $1 \in F$,
(F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

Example 2.1. Let $X = \{1, a, b, c, d\}$ in which "$*$" is defined by

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$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to know that X is a BE-algebra, and $F_1 = \{1, a\}, F_2 = \{1, b\}, F_3 = \{1, c\}, F_4 = \{1, a, b\}$ are filters of X.

Lemma 2.2. *If F is a filter of an BE-algebra X, then F is a subalgebra of X.*

Proof. If $x, y \in F$, then we have $y*(x*y) = 1 \in F$, and so $x*y \in F$ by (F2). This completes the proof. □

Definition 2.3. A non-empty subset I of X is called an *ideal* of X if
(I1) If $x \in X$ and $a \in I$, then $x * a \in I$, i.e., $X * I \subseteq I$,
(I2) If $x \in X$ and $a, b \in I$, then $(a * (b * x)) * x \in I$.

The following Lemma 2.5 is well-known in BE-algebras(see [6]).

Lemma 2.4. *Let X be an BE-algebra. Then*
(1) *Every ideal of X contains* 1,
(2) *If I is an ideal of X, then $(a * x) * x \in I$ for all $a \in I$ and $x \in X$.*

Let I be an ideal of X. Define I_w by

$$I_w = \{x \in X \mid w * x \in I\}$$

for any $w \in X$.

Proposition 2.5. *Let X be a self-distributive BE-algebra and I an ideal of X. Then I_w is a subalgebra of an BE-algebra X.*

Proof. Let $a, b \in I_w$. Then $w * a \in I$ and $w * b \in I$, and so $w * (a * b) = (w * a) * (w * b) \subseteq I * I \subseteq X * I \subseteq I$. This implies $a * b \in I_w$. □

Proposition 2.6. *Let X be a self-distributive BE-algebra and I an ideal of X. Then I_w is an ideal of an BE-algebra X.*

Proof. Let $x \in X$ and $a \in I_w$. Then we have $w*a \in I$, and so $w*(x*a) = (w*x)*(w*a) \in X * I \subseteq I$ from (I1). This implies $x * a \in I_w$. Now let $a, b \in I_w$ and $x \in X$. Then we obtain $w*a \in I$ and $w*b \in I$. Thus we get $w*((a*(b*x))*x) = (w*((a*(b*x))))*(w*x) = ((w*a)*(w*(b*x)))*(w*x) = ((w*a)*((w*b)*(w*x)))*(w*x) = ((w*a)*((w*b)*(w*x)))*(w*x) \in I$ by (I2). This implies $(a * (b * x)) * x \in I_w$. This completes the proof. □

Proposition 2.7. *Let X be a self-distributive BE-algebra and I an ideal of X. If $a \in I_w$ and $a \leq b$, than $b \in I_w$.*

Proof. Let $a \in I_w$ and $a \leq b$. Then we have $w * a \in I$ and $a * b = 1$. Hence we get

$$\begin{aligned} w * b &= w * (1 * b) \\ &= w * ((a * b) * b) \\ &= (w * (a * b)) * (w * b) \\ &= ((w * a) * (w * b)) * (w * b) \in I, \end{aligned}$$

from Lemma 2.5(2). This implies $b \in I_w$. □

Let X be an BE-algebra and $x, y \in X$. Define $A(x, y)$ by

$$A(x, y) = \{z \in X \mid x * (y * z) = 1\}.$$

We call $A(x, y)$ an *upper set* of x and y. It is easy to see that $1, x, y \in A(x, y)$ for all $x, y \in X$.

Proposition 2.8. *Let X be an BE-algebra. If $(X; *, 1)$ is a self distributive BE-algebra, then $A(x, y)$ is a subalgebra of X.*

Proof. Let $m, n \in A(x, y)$. Then we have $x * (y * m) = 1$ and $x * (y * n) = 1$, and so $x * (y * (m * n)) = x * ((y * m) * (y * n)) = (x * (y * m)) * (x * (y * n)) = 1 * 1 = 1$. Thus $m * n \in A(x * y)$. This completes the proof. □

Definition 2.9. Let X be an BE-algebra and $a \in X$. Define $A(a)$ by

$$A(a) = \{x \in X \mid a \leq x\}.$$

Then we call $A(a)$ the *initial section* of the element a.

Example 2.2. Let $X = \{1, a, b, c\}$ in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is an BE-algebra. Also, we have $A(1) = \{1\}, A(a) = \{1, a\}, A(b) = \{1, a, b\}$ and $A(c) = \{1, a, c\}$.

Lemma 2.10. *Let X be an BE-algebra and $x \leq y$. If $x \in A(a)$, then $y \in A(a)$.*

Proof. Since $x \in A(a)$, we have $a \leq x$. Hence $a \leq x \leq y$, that is, $a \leq y$. This implies $y \in A(a)$. □

Proposition 2.11. *Let X be a self-distributive BE-algebra and $a \in X$. Then $A(a)$ is a filter of X.*

Proof. Clearly $1 \in A(a)$ because $a * a = 1$. Let $x \in A(a)$ and $x * y \in A(a)$. Then we have $a \leq x$ and $a \leq x * y$. Hence we have $a * x = 1$ and $a * (x * y) = 1$, and so $a * (x * y) = (a * x) * (a * y) = 1 * (a * y) = a * y = 1$, that is, $y \in A(a)$. This completes the proof. □

Proposition 2.12. *Let X be a self-distributive BE-algebra and $x, y, z \in X$. If $z \leq x * y$ and $z \leq x$, then $z \leq y$.*

Proof. Suppose that $z \leq x * y$ and $z \leq x$ for all $x, y, z \in X$. Then we have $x * y \in A(z)$ and $x \in A(z)$. Since $A(z)$ is a filter, it follows that $y \in A(z)$ or $z \leq y$. This completes the proof. □

Theorem 2.13. *Let X be an BE-algebra, F a filter and $x \in F$. Then $A(x) \subset F$.*

Proof. If $y \in A(x)$, then we have $x \leq y$. Hence $x * y = 1$. Since F is a filter of X and $x \in X$, we obtain $y \in F$. Therefore $A(x) \subset F$. □

3. **Essences.** Let X be an BE-algebra. For any subsets A and B of X, we define

$$A * B =:= \{a * b \mid a \in A, y \in B\}.$$

We use the notation $A * b$ (resp. $a * B$) instead of $A * \{b\}$ (resp. $\{a\} * B$) Note that $A * B = \bigcup_{a \in A} (a * B) = \bigcup_{b \in B} (A * b)$.

Lemma 3.1. *For any subsets A, B and E of an BE-algebra X, we have*

(1) $A \subseteq B \Rightarrow A * E \subseteq B * E, \quad E * A \subseteq E * B$,
(2) $(A \cap B) * E \subseteq (A * E) \cap (B * E)$,
(3) $E * (A \cap B) \subseteq (E * A) \cap (E * B)$,
(4) $(A \cup B) * E = (A * E) \cup (B * E)$,
(5) $E * (A \cup B) = (E * A) \cup (E * B)$.

Proof. (1) Let $x \in A * E$. Then $x = a * e$ for some $a \in A$ and $e \in E$. Since $A \subseteq B$, it follows that $x = a * e$ for some $a \in B$ and $e \in E$ so that $x \in B * E$. Therefore $A * E \subseteq B * E$. Similarly, we obtain $E * A \subseteq E * B$.

(2) Since $A \cap B \subseteq A, B$, it follows from (1) that $(A \cap B) * E \subseteq A * E$ and $(A \cap B) * E \subseteq B * E$ so that $(A \cap B) * E \subseteq (A * E) \cap (B * E)$. Similarly, (3) is valid.

(4) Since $A, B \subseteq A \cup B$, we get $A * E \subseteq (A \cup B) * E$ and $B * E \subseteq (A \cup B) * E$ by (1), and so $(A * E) \cup (B * E) \subseteq (A \cup B) * E$. If $x \in A \cup B) * E$, then $x = y * e$ for some $y \in A \cup B$ and $e \in E$. It follows that $x = y * e$ for some $y \in A$ and $e \in E$; or $x = y * e$ for some $y \in B$ and $e \in E$ so that $x = y * e \in A * E$ or $x = y * e \in B * E$. Hence $x \in (A * E) \cup (B * E)$, which shows that $(A \cup B) * E \subseteq (A * E) \cup (B * E)$. Therefore (4) is valid. Similarly, we can prove that (5) is valid. □

Definition 3.2. If a non-empty subset A of an BE-algebra X satisfies the following equality

$$X * A = A,$$

then we say that A is an *essence* of X. Note that $\{1\}$ and X itself are essences of X.

Example 3.1. Let $X = \{1, a, b, c\}$ in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	1
c	1	a	c	1

Then X is an BE-algebra. It is easy to check that $A_1 = \{1, a\}, A_2 = \{1, c\}, A_3 = \{1, a, c\}$ are an essence of X but $A_4 = \{1, b\}, A_5 = \{1, a, b\}$ are not essences of X.

Proposition 3.3. *Every essence contains the constant* 1.

Proof. Let A be an essence of X. Then $\phi \neq A = X * A$, and so there exists $a \in A$ such that and thus $1 = a * a \in X * A = A$. This completes the proof. □

Theorem 3.4. *Every essence is a subalgebra of X.*

Proof. Let A be an essence of X and $x, y \in A$. Then

$$x * y \in A * A \subseteq X * A = A$$

by Lemma 3.1(1), and so A is a subalgebra of X. □

The converse of Theorem 3.5 is not true. For Example, the set $D := \{1, b\}$ in Example 3.3 is a subalgebra which is not an essence of X.

Theorem 3.5. *Every filter is an essence of* X.

Proof. Let F be a filter of X. Then $1 \in F$, and so $F \neq \phi$. Since $d \leq b * d$ for all $d \in F$ and $b \in X$, we have $b * d \in F$. Thus $X * F \subseteq F$. Obviously, $F = \{1\} * F \subseteq X * F$ by Lemma 3.1(1). Therefore $X * F = F$, i.e., F is an essence of X. □

The converse of Theorem 3.6 may not be true. For example, the set $C = \{1, a, c\}$ in Example 2.11 is an essence which is not a filter of X since $a * c = a \in C$ and $c \notin C$.

Theorem 3.6. *If* F *is a filter of an BE-algebra* X, *then* $A * F$ *is an essence of* X *for every non-empty subset* A *of* X.

Proof. Let A be a non-empty subset of X and F a filter of X. Then F is an essence of X (see Theorem 3.6). Using (BE4), we have

$$X * (A * F) = A * (X * F) = A * F,$$

and hence $A * F$ is an essence of X. □

Corollary 3.7. *If* A *is a non-empty proper subset of an BE-algebra* X, *then* $A * X$ *is an essence of* X.

Theorem 3.8. *Let* A *and* B *be essences of* X. *Then* $A \cap B$ *and* $A \cup B$ *are essences of* X.

Proof. Let $K = A \cap B$. Then

$$K = 1 * K \subseteq X * K = X * (A \cap B) \subseteq (X * A) \cap (X * B) = A \cap B = K,$$

and so $X * K = K$, that is, $K = A \cap B$ is an essence of X. Now let $L = A \cup B$. Then

$$L = 1 * L \subseteq X * L = X * (A \cup B) = (X * A) \cup (X * B) = A \cup B = K,$$

and thus $X * L = L$, that is, $L = A \cup B$ is an essence of X. □

Generally, we have the following the results.

Theorem 3.9. *If* $\{A_i \mid i \in \Lambda \subseteq \mathbf{N}\}$ *is a family of an BE-algebra* X, *then* $\bigcup_{i\in\Lambda}$ *and* $\bigcap_{i\in\Lambda}$ *are essences of* X.

In general, the union of two filters of BE-algebras X may not be a filter of X. For example, In Example 2.2, $F_1 = \{1, b\}$ and $F_2 = \{1, c\}$ are filters, but $F_1 \cup F_2$ is not a filter of X. But we know that the following result is derived from Theorem 3.6 and 3.9.

Corollary 3.10. *The union of two filters of an BE-algebra* X *is an essence of* X.

Let A be an essence and B a subalgebra of an BE-algebra X. Then $A \cup B$ is not an essence of X in general as seen in the following example.

Example 3.2. In Example 2.2, it is easy to check that $A = \{1, a, c\}$ is an essence of X and $B = \{1, d\}$ is a subalgebra of X. But $A \cup B = \{1, a, c, d\}$ is not an essence of X.

Theorem 3.11. *Let* X *be an BE-algebra. If* A *is an essence of* X *and* B *is a subalgebra of* X, *then* $A \cup B$ *is an essence of* B.

Proof. Using Lemma 3.1 (3), we have

$$B * (A \cap B) \subseteq (B * A) \cap (B * B) \subseteq (X * A) \cap B = A \cap B \subseteq B * (A \cap B),$$

and so $B * (A \cap B) = A \cap B$. Therefore $A \cap B$ is an essence of B. □

Proposition 3.12. *Let* A *be an essence of an BE-algebra of* X. *If* $1 \in B \subseteq X$, *then* $B * A = A$.

Proof. Let A be an essence of an BE-algebra of X. Then

$$A = 1 * A \subseteq B * A \subseteq X * A = A.$$

□

Let X and Y be BE-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that $f(1) = 1$.

Lemma 3.13. *Let X be an BE-algebra. If $1 \in A \subseteq X$, then B is contained in $A * B$ for every subset B of X.*

Proof. Let $b \in B$. Then $b = 1 * b \in A * B$, and so B is contained in $A * B$. □

Theorem 3.14. *Let $f : X \to Y$ be a homomorphism of BE-algebras.*

(1) *If f is onto and A is an essence of X, then $f(A)$ is an essence of Y.*
(2) *If B is an essence of Y, then $f^{-1}(B)$ is an essence of X.*

Proof. Suppose that f is onto and A is an essence of X. Using Lemma 3.1(2) and 3.15, we have $f(A) \subseteq Y * f(A)$. Let $b \in f(A)$ and $y \in Y$. Then $b = f(a)$ and $y = f(x)$ for some $a \in A$ and $x \in X$. Thus

$$y * b = f(x) * f(a) = f(x * a) \in f(X * A) = f(A),$$

and so $Y * f(A) \subseteq f(A)$. Therefore $f(A)$ is an essence of Y.

(ii) Using Lemma 3.1(1), we have $f^{-1}(B) \subseteq X * f^{-1}(B)$. Let $a \in f^{-1}(B)$ and $x \in X$. Then $f(a) \in B$ and $f(x) \in Y$. It follows that

$$f(x * a) = f(x) * f(a) \in Y * B = B$$

so that $x * a \in f^{-1}(B)$, i.e., $X * f^{-1}(B) \subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is an essence of X. □

Corollary 3.15. *If $f : X \to Y$ is a homomorphism of BE-algebras, then f^{-1} is an essence of X.*

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