ATOMS IN CI-ALGEBRAS AND SINGULAR CI-ALGEBRAS

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Abstract. In the present paper we continue to study CI-algebras. At first we introduce the notion of atoms in CI-algebras and investigate its elementary properties. Next we introduce the notion of singular CI-algebras and give a number of its properties. Especially we discuss relations between singular CI-algebras and Abelian groups.

1. Introduction.

The study of BCK/BCI-algebras was initiated by K.Iséki in 1966 as a generalization of propositional logic (see[4, 5, 6]). There exist several generalizations of BCK/BCI-algebras, as such BCH-algebras[3], dual BCK-algebras, dual BCI-algebras, d-algebras[12], etc. Especially, H.S.Kim and Y.H.Kim[7] introduced the notion of BE-algebras as another generalization of dual BCK-algebras. They provided an equivalent condition of the filters in BE-algebras. S.S.Ahn and Y.H.Kim in [1], S.S.Ahn and K.S.So in [2] introduced the notion of ideals in BE-algebras and gave several descriptions of ideals. H.S.Kim and K.J.Lee in [8] generalized the notions of upper sets and generalized upper sets and introduced extended upper sets, by using this notion they gave several descriptions of filters in BE-algebras. A. Walendziak in [13] introduced the notion of commutative BE-algebras and discussed some of its properties. Recently we in [9] and [10] introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCI/BCH-algebras, and studied some of its important properties and relations with BE-algebras, especially proved the notion of ideals is equivalent to one of filters in transitive BE-algebras. In [11] we introduced the notion of closed filters in CI-algebras and built elementary theory of closed filter. We give a procedure to generate a closed filter by a nonempty subset of a CI-algebra. In the present paper we continue to study CI-algebras. At first we introduce the notion of atoms in CI-algebras and investigate its important properties. Next we introduce the notion of singular CI-algebras and give a number of its properties. Especially we discuss relations between CI-algebras and Abelian groups. The definitions and terminologies used in this paper are standard.

2. Preliminaries

Definition 2.1[9]. A *CI-algebra* is an algebra $(X; *, 1)$ of type (2,0) satisfying the following axioms: for any $x, y, z \in X$

(CI1) $x * x = 1$;

(CI2) $1 * x = x$;

(CI3) $x * (y * z) = y * (x * z)$.

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In any CI-algebra X one can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 1 \forall x, y \in X$.

Lemma 2.3[9]. *In a CI-algebra X, the following hold: for any $x, y \in X$*

(1) $x * ((x * y) * y) = 1$;

(2) $(x * y) * 1 = (x * 1) * (y * 1)$;

(3) $1 \leq x$ *implies* $x = 1$.

Definition 2.4[9]. Let X be a CI-algebra. A nonempty subset S of X is said to be a *subalgebra* of X if it satisfies: $x, y \in S$ implies $x * y \in S$ for any $x, y \in X$.

A nonempty subset F of X is said to be a *filter* of X if it satisfies

(F1) $1 \in F$;

(F2) for any $x, y \in X$, $x * y \in F$ and $x \in F$ imply $y \in F$.

A filter F of X is said to *closed* if $x \in F$ implies $x * 1 \in F$.

Lemma 2.5[11]. *Let X be a CI-algebra. A filter F of X is closed if and only if F is a subalgebra of X.*

3. Atoms in CI-algebras

In this section we first introduce the notion of atoms in CI-algebras and next study some of its elementary properties.

Definition 3.1. Let a be an element of a CI-algebra X. a is said to be an *atom* in X if for any $x \in X$, $a * x = 1$ implies $a = x$. Denote the set of all atoms in X by $A(X)$, which is called the *singular part* of X.

Obviously $1 \in A(X)$, so $A(X) \neq \emptyset$.

Example 3.2. Let $X = \{1, a, b, c, d\}$ with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	d
b	1	c	1	c	d
c	1	1	b	1	d
d	d	d	d	d	1

1 and d are atoms in X, but a, b, c are not atoms of X.

Proposition 3.3. *Let X be a CI-algebra. Then $a \in X$ is an atom in X if and only if it satisfies for any $x \in X$, $a = (a * x) * x$.*

Proof. Let a be an atom in X and $x \in X$. It follows from $a * ((a * x) * x) = 1$ that $a = (a * x) * x$.

Conversely suppose that $a \in X$ satisfies for any $x \in X$, $a = (a * x) * x$. If $a * x = 1$, then

$$a = (a * x) * x = 1 * x = x,$$

hence a is an atom in X. The proof is complete. □

Proposition 3.4. *Let X be a CI-algebra. If $a, b \in X$ are atoms in X, then the following are true:*

(1) $a = (a * 1) * 1$;

(2) $(a * b) * 1 = b * a$;

(3) $((a * b) * 1) * 1 = a * b$.

Proof. (1) is an immediate consequence of Proposition 3.3.
By Lemma 2.3(2),(CI3) and (1) we have

$$(a * b) * 1 = (a * 1) * (b * 1) = b * ((a * 1) * 1) = b * a.$$

(2) holds.
(3) follows from (2). The proof is complete. □

Proposition 3.5. *Let X be a CI-algebra. If a and b are atoms in X, then the following are true:*

(1) *for any $x \in X$, $(a * x) * (b * x) = b * a$;*

(2) *for any $x \in X$, $(a * x) * b = (b * x) * a$;*

(3) *for any $x \in X$, $(a * x) * (y * b) = (b * x) * (y * a)$.*

Proof. Since $a, b \in A(X)$, it follows from Proposition 3.3 that

$$(a * x) * (b * x) = b * ((a * x) * x) = b * a.$$

(1) holds.
By using (1) and Proposition 3.2 we have

$$(a * x) * b = (a * x) * ((b * x) * x) = (b * x) * ((a * x) * x) = (b * x) * a.$$

(2) holds.
By (2) we have

$$(a * x) * (y * b) = y * [(a * x) * b] = y * [(b * x) * a] = (b * x) * (y * a).$$

The proof is complete. □

Open problem: *Let X be a CI-algebra. Is the set $A(X)$ of all atoms of X a subalgebra of X? When?*

4. Singular CI-algebras

In this section we introduce the notion of singular CI-algebras and give a number of its properties. Especially we discuss relations between CI-algebras and Abelian groups.

Definition 4.1. A CI-algebra X is said to be *singular* if every element of X is an atom of X.

Example 4.2. Let $X = \{1, a, b, c\}$ with the following Cayley table:

*	1	a	b
1	1	a	b
a	b	1	a
b	a	b	1

Then X is a singular CI-algebra. X in the example 3.2 is not singular, because a and b are not atoms.

Proposition 4.3. *Let X be a CI-algebra. Then X is singular if and only if X satisfies the condition*

(D) *for any* $x, y, z \in X$, $(x * y) * z = (z * y) * x$.

Proof. Suppose X is singular. It follows from Proposition 3.4(2) that the condition (D) holds for X.

Conversely suppose that X satisfies the condition (D). If $x * y = 1$, then by (D) we have

$$x = 1 * x = (y * y) * x = (x * y) * y = 1 * y = y.$$

Hence x is an atom of X, and X is singular. The proof is complete. □

Proposition 4.4. *Let* $(X; *, 1)$ *be a singular* CI*-algebra. Define* $x + y := (x * 1) * y$ *for any* $x, y \in X$. *Then* $(X; +, 1)$ *is an Abelian group with identity* 1.

Proof. It follows from the condition (D) that $x + y = y + x$ for any $x, y \in X$, and so the operation + is commutatve.

Because for any $x \in X$, $x + 1 = (x * 1) * 1 = x$, so 1 is identity.

Since for any $x, y, z \in X$,

$$\begin{aligned} (x+y)+z &= \{[(x*1)*y]*1\}*z && \text{by Definition} \\ &= (z*1)*[(x*1)*y] && \text{by (D)} \\ &= (x*1)*[(z*1)*y] && \text{by (CI3)} \\ &= (x*1)*[(y*1)*z] && \text{by (D)} \\ &= x+(y+z), \end{aligned}$$

and so the operation + satisfies associative law.

Because $x + (x * 1) = (x * 1) * (x * 1) = 1$, so $-x = x * 1$ is the inverse of x. Therefore $(X; +, 1)$ is an Abelian group with identity 1. □

The group $(X; +, 1)$ is called the *adjoint group* of CI-algebra $(X; *, 1)$.

Proposition 4.5. *Let* $(X; +, 1)$ *be an Abelian group with identity* 1. *Define* $x*y := y-x$ *for any* $x, y \in X$. *Then* $(X; *, 1)$ *is a singular* CI*-algebra, whose adjoint group is exactly* $(X; +, 1)$.

Proof. Because for any $x \in X$, $x * x = x - x = 1$ and $1 * x = x - 1 = x$, so $(X; *, 1)$ satisfies (CI1) and (CI2).

For any $x, y, z \in X$, we have $x * (y * z) = (z - y) - x = (z - x) - y = y * (x * z)$, hence $(X; *, 1)$ satisfies (CI3). Therefore $(X; *, 1)$ is a CI-algebra.

For any $x, y, z \in X$, we have

$$\begin{aligned} (x*y)*z &= z-(y-x) = z-y+x = x-y+z \\ &= x-(y-z) = (y-z)*x = (z*y)*x. \end{aligned}$$

Hence $(X; *, 1)$ is singular.

For the singular CI-algebra $(X; *, 1)$, define $x \oplus y = (x * 1) * y$ for $x, y \in X$. By Proposition 3.8 we know that $(X; \oplus, 1)$ is the adjoint group of $(X; *, 1)$. Because

$$x \oplus y = (x * 1) * y = y - (x * 1) = y - (1 - x) = y + x = x + y,$$

$(X; \oplus, 1)$ is exactly $(X; +, 1)$. The proof is complete. □

The CI-algebra $(X; *, 1)$ is called the *adjoint algebra* of group $(X; +, 1)$.

Proposition 4.6. *Let* $(X; *, 1)$ *be a singular* CI*-algebra,* $(X; +, 1)$ *the adjoint group of* $(X; *, 1)$. *Then the adjoint algebra of* $(X; +, 1)$ *is exactly* $(X; *, 1)$.

Proof. For the adjoint group $(X;+,1)$, define $x*'y = y-x$ for $x, y \in X$. By Proposition 3.9 we know that $(X;*',1)$ is the adjoint algebra of $(X;+,1)$. Because

$$x*'y = y-x = y+(x*1) = (y*1)*(x*1) = x*((y*1)*1) = x*y,$$

hence $(X;*',1)$ is exactly $(X;*,1)$. The proof is complete. □

Proposition 4.7. *Let $(X;*,1)$ and $(X';*',1')$ be singular CI-algebras. Let $(X;+,1)$ and $(X';+',1')$ be adjoint groups of $(X;*,1)$ and $(X';*',1')$, respectively. Then $(X;*,1)$ is isomorphic to $(X';*',1')$ if and only if $(X;+,1)$ is isomorphic to $(X;*,1)$.*

Proof. Suppose that $(X;*,1)$ is isomorphic to $(X';*',1')$. Let $\varphi:\ X \to X'$ be an isomorphism from $(X;*,1)$ to $(X';*',1')$ such that $\varphi(x) = x'$ for any $x \in X$. Then for any $x, y \in X$,

$$\varphi(x+y) = \varphi((x*1)*y) = (\varphi(x)*'\varphi(1))*'\varphi(y) = \varphi(x)+'\varphi(y),$$

and so $(X;+,1)$ is isomorphic to $(X';+',1')$.

Conversely Suppose that $(X;+,1)$ is isomorphic to $(X';+',1')$. Let $\varphi:\ X \to X'$ be an isomorphism from $(X;+,1)$ to $(X';+',1')$ such that $\varphi(x) = x'$ for any $x \in X$. Then for any $x, y \in X$,

$$\varphi(x*y) = \varphi(y-x) = (\varphi(y) -' \varphi(x) = \varphi(x)*'\varphi(y),$$

and so $(X;*,1)$ is isomorphic to $(X';*',1')$. The proof is complete. □

Proposition 4.8. *Every subalgebra of a singular CI-algebra is a closed filter.*

Proof. Let X be a CI-algebra and F a subalgebra of X. By Lemma 2.6 it suffices to prove that F is a filter of X. Obviously, $1 \in F$. If $x*y \in F$ and $x \in F$, then $1*x \in F$ because F is a subalgebra X. Also $x*y \in F$ implies $y*x = 1*(x*y) \in F$ by Proposition 3.4(2). Thus by Proposition 3.4(3) we have

$$y = 1*y = (x*x)*(1*y) = (y*x)*(1*x) \in F,$$

hence F is a filter of X. □

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