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ON SOME REGULAR SUBSEMIGROUPS OF SEMIGROUPS

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Abstract. For a subsemigroup T of a semigroup S, $Reg(T)$ denotes the set of regular elements of T, $LReg(T)$ the set of left regular elements of T and $reg(T)$ the set of elements of T which are regular in S. Characterizations of a semigroup S for which $reg(Se) = Reg(Se)$ for each idempotent element e of S have been given in [3]. This type of semigroups is the semigroups S in which each element of the subsemigroup Se of S which is regular in S is a left regular element of Se for every idempotent element e of S. Moreover, this type of semigroups is the semigroups S in which the regular elements are left regular, equivalently the sets of regular and completely regular elements coincide [3]. In the present paper we prove that the type of semigroups mentioned above is actually the semigroups in which $reg(Sa) = reg(Sa)$ for every $a \in S$.

1. Introduction and prerequisites.

If S is a semigroup, an element a of S is called regular if there exists $x \in S$ such that $a = axa$ [1], it is called completely regular if there exists $x \in S$ such that $a = a^2xa^2$ [4]. Keeping the notation given in [3], for a subsemigroup T of S, $Reg(T)$ denotes the set of regular elements of T, $LReg(T)$ (resp. $RReg(T)$) the set of left (resp. right) regular elements of T, $reg(T)$ the set of elements of T which are regular in S, and $Gr(T)$ the set of completely regular elements of T. As usual, $E(S)$ denotes the set of idempotent elements of S. The aim in [3] was to characterize the semigroups S such that $reg(Se) = Reg(Se)$ for every idempotent element e of S (cf. [3; p. 357]) and the characterization is given in the Theorem and the Corollary of the paper mentioned below. The right analogue of the results in [3] also hold.

Theorem. For a semigroup S the following conditions are equivalent:
(1) $reg(Se) = Gr(Se)\ \forall\, e \in E(S)$
(2) $reg(Se) = Reg(Se)\ \forall\, e \in E(S)$
(3) $reg(Se) \subseteq LReg(Se)\ \forall\, e \in E(S)$
(4) $Reg(S) \subseteq LReg(S)$
(5) $Reg(S) = Gr(S)$.

Corollary. Each of the following conditions on a semigroup S is equivalent to the above conditions (1)–(5):
(6) $reg(eSf) = Gr(eSf)\ \forall\, e, f \in E(S)$
(7) $reg(eSf) = Reg(eSf)\ \forall\, e, f \in E(S)$
(8) $reg(eSf) \subseteq LReg(eSf)\ \forall\, e, f \in E(S)$.

According to the Theorem and the Corollary above, the paper in [3] investigates regular subsets of semigroups related to their idempotents.

In the present note we characterize the semigroups S in which $reg(Sa) = Reg(Sa)$ for every $a \in S$ and show that the type of semigroups related with their idempotents considered

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in [3] is actually the type of semigroups in which $reg(Sa) = Reg(Sa)$ for every $a \in S$. The right analogue of Theorem 1 below also holds. Combining the Theorem 1 of the present note with the Theorem in [3], we obtain the following:

(1) $reg(Se) = Reg(Se)\ \forall\ e \in E(S) \Longleftrightarrow reg(Sa) = Reg(Sa)\ \forall\ a \in S$

(2) $reg(Se) \subseteq LReg(Se)\ \forall\ e \in E(S) \Longleftrightarrow reg(Sa) \subseteq LReg(Sa)\ \forall\ a \in S$

(3) $reg(Se) = Gr(Se)\ \forall\ e \in E(S) \Longleftrightarrow reg(Sa) = Gr(Sa)\ \forall\ a \in S$.

Moreover, the Theorem in [3] together with the Theorem 1 of the present paper give 10 equivalent conditions regarding to regularity. As far as the Corollary in [3] is concerned, we remark that taking into account the Theorem 2 of the present paper we obtain the following:

(4) $reg(eSf) = Gr(eSf)\ \forall\ e, f \in E(S) \Longleftrightarrow reg(aSb) = Gr(aSb)\ \forall\ a, b \in S$

(5) $reg(eSf) = Reg(eSf)\ \forall\ e, f \in E(S) \Longleftrightarrow reg(aSb) = Reg(aSb)\ \forall\ a, b \in S$

(6) $reg(eSf) \subseteq LReg(eSf)\ \forall\ e, f \in E(S) \Longleftrightarrow reg(aSb) \subseteq LReg(aSb)\ \forall\ a, b \in S$.

The Theorem 2 of this paper adds 8 additional conditions to the 10 conditions of regularity mentioned above.

2. Main results

Theorem 1. *In a semigroup S, the following are equivalent:*

(1) $reg(Sa) = Gr(Sa)\ \forall\ a \in S$

(2) $reg(Sa) = Reg(Sa)\ \forall\ a \in S$

(3) $reg(Sa) \subseteq LReg(Sa)\ \forall\ a \in S$

(4) $Reg(S) \subseteq LReg(S)$

(5) $Reg(S) = Gr(S)$.

For the proof of Theorem 1 we need the following Lemma which shows that the Lemma 1 in [3] holds for any element a and not only for idempotent elements e of S. Its proof is directly by definitions and no use of the $\mathcal{H}$-classes of S is needed.

Lemma. *If S is a semigroup then, for every element $a \in S$, we have*

$$Gr(Sa) = Gr(S) \cap Sa.$$

Proof. Let $a \in S$. As one can easily see, for any subsemigroup T of S, we have $Gr(T) \subseteq Gr(S) \cap T$. Since Sa is a subsemigroup of S, we have $Gr(Sa) \subseteq Gr(S) \cap Sa$.
Let now $b \in Gr(S) \cap Sa$. Since $b \in Gr(S)$, we have $b = b^2sb^2$ for some $s \in S$. Since $b \in Sa$, we get $b = ta$ for some $t \in S$. Therefore we have

$b = b^2sb^2 = b^2s(b^2sb^2)b = b^2(sb^2sb)b^2 = b^2(sb^2sta)b^2$.

Then, since $b \in Sa$ and $sb^2sta \in Sa$, we obtain $b \in Gr(Sa)$. □

Proof of Theorem 1. (1) $\Longrightarrow$ (2). Let $a \in S$. Since Sa is a subsemigroup of S, we have $Gr(Sa) \subseteq Reg(Sa) \subseteq reg(Sa)$. Then, by (1), $reg(Sa) = Reg(Sa)$.

(2) $\Longrightarrow$ (3). Let $a \in S$ and $b \in reg(Sa)$. Then by (2), $b \in Reg(Sa)$, that is $b \in Sa$ and $b = bxb$ for some $x \in Sa$. Since $b \in S$ and $b = bxb$, $x \in S$, we have $b \in Reg(S)$. On the other hand, $b \in Sxb$, so $b \in Sxb \cap Reg(S)$. Since $xb \in S$, Sxb is a subsemigroup of S, so $reg(Sxb) := Sxb \cap Reg(S)$, hence $b \in reg(Sxb)$. Since $xb \in S$, by (2), $reg(Sxb) = Reg(Sxb)$, so $b \in Reg(Sxb)$. Then $b \in Sxb$ and $b = byb$ for some $y \in Sxb$. Then $y = sxb$ for some $s \in S$ and $b = b(sxb)b = bsxb^2$. Since $x \in Sa$, we have $x = ta$ for some $t \in S$. Thus we have $b = (bsta)b^2$. Since $b \in Sa$, $b = (bsta)b^2$ and $bsta \in Sa$, we obtain $b \in LReg(Sa)$.

(3) $\Longrightarrow$ (4). Let $b \in Reg(S)$. Then $b \in S$ and $b = bxb$ for some $x \in S$. As $b \in Sxb$, we have $b \in Sxb \cap Reg(S)$. Since Sxb is a subsemigroup of S, $reg(Sxb) := Sxb \cap Reg(S)$, so $b \in reg(Sxb)$. Then, by (3), $b \in LReg(Sxb)$, that is $b \in Sxb$ and $b = zb^2$ for some $z \in Sxb$. Since $b \in S$ and $b = zb^2$; $z \in S$, we have $b \in LReg(S)$.

(4) $\Longrightarrow$ (5). Cf. (iv)$\Longrightarrow$ (v) in [3].

(5) $\Longrightarrow$ (1). Let $a \in S$. Since Sa is a subsemigroup of S, $reg(Sa) := Sa \cap Reg(S)$. Then, by (5), $reg(Sa) = Sa \cap Gr(S)$. By the Lemma, $Sa \cap Gr(S) = Gr(S)$, thus we have $reg(Sa) = Gr(Sa)$. □

Theorem 2. *For a semigroup S, the following are equivalent:*

(1) $reg(aSb) = Gr(aSb)\ \forall\, a, b \in S$
(2) $reg(aSa) = Gr(aSa)\ \forall\, a \in S$
(3) $reg(aSb) = Reg(aSb)\ \forall\, a, b \in S$
(4) $reg(aSa) = Reg(aSa)\ \forall\, a \in S$
(5) $reg(aSb) \subseteq LReg(aSb)$ (resp. $reg(aSb) \subseteq RReg(aSb)$) $\forall\, a, b \in S$
(6) $reg(aSa) \subseteq LReg(aSa)$ (resp. $reg(aSa) \subseteq RReg(aSa)$) $\forall\, a \in S$
(7) $reg(S) \subseteq LReg(S)$ (resp. $reg(S) \subseteq RReg(S)$) $\forall\, a \in S$
(8) $Reg(S) = Gr(S)$.

Proof. The implications (1) $\Longrightarrow$ (2), (3) $\Longrightarrow$ (4) and (5) $\Longrightarrow$ (6) are obvious. For the implication (7) $\Longrightarrow$ (8) we refer to [3].

(2) $\Longrightarrow$ (3). Let $a, b \in S$ and $c \in reg(aSb)$. Since aSb is a subsemigroup of S, we have $reg(aSb) := aSb \cap Reg(S)$. Since $c \in Reg(S)$, we get $c = cxc$ for some $x \in S$, so

$$c \in cSc \cap Reg(S) := reg(cSc) = Gr(cSc)$$

by (2). That is, $c = c^2yc^2$ for some $y \in cSc$. On the other hand, $c \in aSb$ implies $c = azb$ for some $z \in S$. Thus we have $c = c(azb)y(azb)c = c(azbyazb)c$. Since $c \in aSb$ and $c = c(azbyazb)c$; $azbyazb \in aSb$, we have $c \in Reg(aSb)$.

(4) $\Longrightarrow$ (5). Let $a, b \in S$ and $c \in reg(aSb) := aSb \cap Reg(S)$. Since $c \in Reg(S)$, we have $c = cxc$ for some $x \in S$. Then $c \in cSc \cap Reg(S) := reg(cSc) = Reg(cSc)$ by (4). Since $c \in Reg(cSc)$, we have $c = cyc$ for some $y \in cSc$. Since $y \in cSc$, we get $y = czc$ for some $z \in S$. Then

$$c = c(czc)c = c^2zc^2 = c^2z(c^2zc^2)c = (c^2zc^2zc)c^2.$$

Since $c \in aSb$, we get $c = atb$ for some $t \in S$. Thus we have

$$c^2zc^2zc = (atb)czc^2z(atb) = a(tbczc^2zat)b \in aSb.$$

Since $c \in aSb$ and $c = (c^2zc^2zc)c^2$, with $c^2zc^2zc \in aSb$, we have $c \in LReg(aSb)$. Similarly we obtain $c \in RReg(aSb)$.

(6) $\Longrightarrow$ (7). Suppose $reg(aSa) \subseteq LReg(aSa)$ for each $a \in S$. Let now $b \in Reg(S)$. Since $b = bxb$ for some $x \in S$ and $b \in bSb$, we have $b \in bSb \cap Reg(S) := reg(bSb)$. By hypothesis, $reg(bSb) \subseteq LReg(bSb)$, so $b \in LReg(bSb) \subseteq LReg(S)$. The rest of the proof is similar.

(8) $\Longrightarrow$ (1). Let $a, b \in S$ and $c \in reg(aSb) := aSb \cap Reg(S)$. Then $c \in Reg(S) = Gr(S)$ by (8), so $c = c^2xc^2$ for some $x \in S$. Hence we have

$$c = c^2xc^2 = c(c^2xc^2)x(c^2xc^2)c = c^2(cxc^2xc^2xc)c^2.$$

Since $c \in aSb$, we have $c = ayb$ for some $y \in S$. Then we obtain

$$cxc^2xc^2xc = (ayb)xc^2xc^2x(ayb) = a(ybxc^2xc^2xay)b \in aSb.$$

Since $c \in aSb$, $c = c^2(cxc^2xc^2xc)c^2$, where $cxc^2xc^2xc \in aSb$, we have $c \in Gr(aSb)$. The inclusion $Gr(aSb) \subseteq Reg(aSb)$ is obvious, and the proof of the theorem is complete. □

Remark. The right analogue of the Theorem 1 and Corollary 2 also hold. For Theorem 1, for example, its right analogue reads as follows: In a semigroup S the following are equivalent: (1) $reg(aS) = Gr(aS)\ \forall\ a \in S$. (2) $reg(aS) = Reg(aS)\ \forall\ a \in S$. (3) $reg(aS) \subseteq RReg(aS)\ \forall\ a \in S$. (4) $Reg(S) \subseteq RReg(S)$ (5) $Reg(S) = Gr(S)$.

Note. As far as the case of ordered semigroups is concerned, keeping the notation and terminology given in [2], one gets the following and their right analogue which add some additional conditions in the results given in [2].

Theorem 3. *Let S be an ordered semigroup. We consider the statements:*
(1) $reg(Sa] = Gr(Sa]\ \forall\ a \in S$
(2) $reg(Sa] = Reg(Sa]\ \forall\ a \in S$
(3) $reg(Sa] \subseteq LReg(Sa]\ \forall\ a \in S$
(4) $Reg(S) \subseteq LReg(S)$
(5) $Reg(S) = Gr(S)$.
Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) *and* (5) $\Longrightarrow$ (1).
It remains as an open problem if (4) $\Longrightarrow$ (5).

Theorem 4. *For a semigroup S, the following are equivalent:*
(1) $reg(aSb] = Gr(aSb]\ \forall\ a, b \in S$
(2) $reg(aSa] = Gr(aSa]\ \forall\ a \in S$
(3) $reg(aSb] = Reg(aSb]\ \forall\ a, b \in S$
(4) $reg(aSa] = Reg(aSa]\ \forall\ a \in S$
(5) $reg(aSb] \subseteq LReg(aSb]$ (resp. $reg(aSb] \subseteq RReg(aSb]$) $\forall\ a, b \in S$
(6) $reg(aSa] \subseteq LReg(aSa]$ (resp. $reg(aSa] \subseteq RReg(aSa]$) $\forall\ a \in S$
(7) $reg(S) \subseteq LReg(S)$ (resp. $reg(S) \subseteq RReg(S)$) $\forall\ a \in S$
(8) $Reg(S) = Gr(S)$.

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