CLOSED FILTERS IN CI-ALGEBRAS

Biao Long Meng

Received February 3, 2010

Abstract. In this paper, we first introduce the notion of closed filters in CI-algebras. Next we provide some properties of closed filters. Finally we investigate how to generate a closed filter by a subset in a transitive CI-algebra.

1 Introduction

The study of BCK/BCI-algebras was initiated by K. Iséki in 1966 as a generalization of propositional logic (see[4, 5, 6]). There exist several generalizations of BCK/BCI-algebras, as such BCH-algebras[3], dual BCK-algebras[12], d-algebras[11], etc. Especially, H. S. Kim and Y. H. Kim[7] introduced the notion of BE-algebras which was deeply studied by S. S. Ahn and Y. H. Kim[1], S. S. Ahn and K. S. So[2], H. S. Kim and K. J. Lee [8], A. Walendziak[13] and B.L.Meng[10]. As a generalization of BE-algebras and dual $BCK/BCI/BCH$-algebras (see [12] and [9], respectively), B. L. Meng[9] introduced the notion of CI-algebras and studies its elementary properties. In this paper we continue to sdudy the filter theory of CI-algebras. We first introduce the notion of closed filters in CI-algebras. Next we provide some properties of closed filters. Finally we investigate how to generate a closed filter by a subset in a transitive CI-algebra. In the sequel, let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}$ the set of all integers. The definitions and technologies used in this paper are standard.

2 Preliminaries

Definition 2.1[9]. An algebra $(X; *, 1)$ of type (2,0) is said to be a CI-algebra if it satisfies the following:
(CI1) $x * x = 1$,
(CI2) $1 * x = x$,
(CI3) $x * (y * z) = y * (x * z)$.
The set $B(X) = \{x \in X \ \ x * 1 = 1\}$ is called the BE-part of X.
A CI-algebra X is said to be a BE-algebra if $B(X) = X$[7].
In a CI-algebra, one can introduce a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 1$.

Lemma 2.2[9]. If $(X; *, 1)$ is a CI-algebra, then for all $x, y \in X$
(1) $(x * y) * 1 = (x * 1) * (y * 1)$,
(2) $x * [(x * y) * y] = 1$,
(3) $1 * x = 1$ (or equivalently, $1 \leq x$) implies $x = 1$.

Definition 2.3. A CI-algebra X is said to be *transitive*[1] if for all $x, y, z \in X$,
$(y * z) * [(x * y) * (x * z)] = 1$, or equivalently, $y * z \leq (x * y) * (x * z)$.

Lemma 2.4. If a CI-algebra X is transitive then for all $x, y, z \in X$,

---

2000 *Mathematics Subject Classification.* 06F35; 03G25; 03B52 .
Key words and phrases. CI-algebra, transitive CI-algebra, filter, closed filter .

(1) $y \le z$ implies $x * y \le x * z$,
(2) $y \le z$ implies $z * x \le y * x$.

Proof. It is easy and omitted.

Definition 2.5[9]. Let X be a CI-algebra and F a nonempty subset of X. F is said to be a filter of X if it satisfies: (F1) $1 \in F$, (F2) $x \in F$ and $x * y \in F$ imply $y \in F$. F is said to be a subalgebra of X if for all $x, y \in F$, $x * y \in F$.

Lemma 2.6. Let X be a CI-algebra and F a filter of X. If $x * y = 1$ (or equivalently $x \le y$) and $x \in F$, then $y \in F$.

Proof. Trivial.

3 Closed Filters in CI-algebras

Definition 3.1. A filter F in a CI-algebra X is said to be *closed* if $x \in F$ implies $x * 1 \in F$.

Example 3.2. Let X be the set of all positive real numbers, $\div$ the usual division. Define $x * y = y \div x$. It is easy to verify that $(X; *, 1)$ is a CI-algebra, $F = \{2^n | n \in \mathbb{Z}\}$ is a closed filter of X, $F = \{2^n | n \in \mathbb{N}\}$ is a filter of X but F is not closed.

Proposition 3.3. Every filter of a BE-algebra is closed.

Proof. Trivial.

Proposition 3.4. Every filter of a finite CI-algebra is closed.

Proof. Suppose $(X; *, 1)$ is a finite CI-algebra, $|X| = n$. Let F be any filter of X. Take any $a \in F$, in the following $n + 1$ elements:

$$1,\ a * 1,\ \cdots,\ \underbrace{a * (\cdots * (a}_{n} * 1) \cdots),$$

there are at least two elments to be equal, for instance,

$$\underbrace{a * (\cdots * (a}_{l} * 1) \cdots) = \underbrace{a * (\cdots * (a}_{k} * 1) \cdots)$$

where $0 \le l < k \le n$, when $l = 0$, $\underbrace{a * (\cdots * (a}_{l} * 1) \cdots) = 1$. Hence

$$\underbrace{a * (\cdots * (a}_{k-l} * 1) \cdots) = 1 \in F,$$

and so $a * 1 \in F$. This completes the proof.

Proposition 3.5. A filter of a CI-algebra X is closed if and only if it is a subalgebra of X.

Proof Suppose a filter F of X is closed and $x, y \in F$. Because $x * (y * x) = y * 1 \in F$, it follows from (F2) that $y * x \in F$. This shows that F is a subalgebra of X.

Conversely suppose a filter F of X is a subalgebra of X. For all $x \in F$, it follows from $1 \in F$ that $x * 1 \in F$, so F is closed. This completes the proof.

Proposition 3.6. The BE-part $B(X)$ of a CI-algebra X is a closed filter of X.

Proof Obviously $1 \in B(X)$. If $x * y \in B(X)$ and $x \in B(X)$, then $x * 1 = 1$ and $(x * y) * 1 = 1$. By Lemma 2.2(1) we have

$$y * 1 = 1 * (y * 1) = (x * 1) * (y * 1) = (x * y) * 1 = 1.$$

This shows that $B(X)$ is a filter of X. If $x \in B(X)$, then $x * 1 = 1 \in B(X)$. Therefore $B(X)$ is a closed filter of X.

Corollary 3.7. Let $(X; *', 1)$ be a BE-algebra and $a \notin X$. We define the operation $*$ on $X \cup \{a\}$ as follows

$$x * y = \begin{cases} x *' y & \text{if } x, y \in X, \\ a & \text{if } x = a \text{ and } y \neq a, \\ a & \text{if } x \neq a \text{ and } y = a, \\ 1 & \text{if } x = y = a \end{cases}$$

then $(X \cup \{a\}; *, 1)$ is a CI-algebra[9]. Since $B(X \cup \{a\}) = X$, it follows that X is a closed filter of $X \cup \{a\}$

4 Closed filter generated by a subset

In this section we discuss one of algebraic basic problems: how to generate a closed filter by a subset of a CI-algebra X. For short, denote $x^0 = x * 1$. For any $a_1, \cdots, a_n, x \in X$, we define

$$\prod_{i=1}^{n} a_i * x = a_n * (\cdots * (a_1 * x) \cdots).$$

Obviously, for any $a_1, \cdots, a_n, b_1, \cdots, b_m \in X$ we have

$$\prod_{j=1}^{m} b_j * (\prod_{i=1}^{n} a_i * x) = \prod_{i=1}^{n} a_i * (\prod_{j=1}^{m} b_j * x).$$

Definition 4.1 Let A be a subset of a CI-algebra X. If F is the least closed filter of X containing A, then F is said to be a closed filter generated by A. For simplicity, we denote $F = (A]_c$. For a finite subset $\{a_1, \cdots, a_n\}$, we note $(\{a_1, \cdots, a_n\}]_c = (a_1, \cdots, a_n]_c$.

This definition is well-defined because the intersection of any nonempty family of closed filters of X is a closed filter of X.

Proposition 4.2. Suppose that A and B are subsets of X.
(1) If A is a closed filter of X then $(A]_c = A$.
(2) $A \subseteq B$ implies $(A]_c \subseteq (B]_c$.
(3) $(X]_c = X$.
(4) $(1]_c = \{1\}$.

Proof Trivial.

Proposition 4.3 Let A be a nonempty subset of a transitive CI-algebra X. If a subset F of X is defined as follows:

$$F = \{x \in X \mid \prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x) = 1 \text{ for some } a_1, \cdots, a_n, b_1, \cdots, b_m \in A\},$$

then $I = (A]_c$.

Proof Since $x * x = 1$ for any $x \in A$, it follows that $x \in F$. This shows $A \subseteq F$.
Select $x_0 \in A$. Because $(x_0 * 1) * (x_0 * 1) = 1$, we have $1 \in F$.
If $y * x \in F$ and $y \in F$, then there are

$$a_1, \cdots, a_n, b_1, \cdots, b_m, c_1, \cdots, c_k, d_1, \cdots, d_l \in A$$

such that

$$\prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * (y * x)) = 1,$$

$$\prod_{j=1}^{l} d_j^0 * (\prod_{i=1}^{k} c_i * y) = 1.$$

Thus we have

$$y \leq \prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x).$$

Succesively $*$-multiplying the above inequality on the left-hand side by

$$c_1, \cdots, c_k, d_1^0, \cdots, d_l^0$$

gives the following

$$1 = \prod_{j=1}^{l} d_j^0 * (\prod_{i=1}^{k} c_i * y) \leq \prod_{j=1}^{l} d_j^0 * (\prod_{i=1}^{k} c_i * (\prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x))).$$

By lemma 2.2(3) we have

$$\prod_{j=1}^{l} d_j^0 * (\prod_{i=1}^{k} c_i * (\prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x))) = 1,$$

and so $x \in F$. This proves that F is a filter of X.
If $x \in F$, then there are $a_1, \cdots, a_n, b_1, \cdots, b_m \in A$ such that

$$\prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x) = 1.$$

By $*$-mutiplying the above equality on the right-hand side by 1 we obtain

$$\prod_{j=1}^{m} ((b_j * 1) * 1) * (\prod_{i=1}^{n} a_i^0 * (x * 1)) = 1.$$

Since $b_i \leq (b_i * 1) * 1 (i = 1, \cdots, m)$, it follows that

$$\prod_{j=1}^{m} b_j * (\prod_{i=1}^{n} a_i^0 * (x * 1)) = 1.$$

Hence $x * 1 \in F$, and F is a closed filter of X.

Now let B be any closed filter of X and $A \subseteq B$. For any $x \in F$, there are $a_1, \cdots, a_n, b_1, \cdots, b_m \in A$ such that

$$\prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x) = 1.$$

Observing that $A \subseteq B$ implies $a_1, \cdots, a_n, b_1, \cdots, b_m \in B$, therefore $x \in B$. This proves that $F \subseteq B$, i.e., $F = (A]_c$. The proof is complete.

Corollary 4.4. Let X be a transitive CI-algebra. Then a nonempty subset F of X is a closed filter of X if and only if F satisfies

$(*)$ for all $a_1, \cdots, a_n, b_1, \cdots, b_m \in F$,

$$\prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x) = 1 \text{ implies } x \in F.$$

Proof. Suppose F is a closed filter of X, then $F = (F]_c$. If $x \in X$ satisfies for some $a_1, \cdots, a_n, b_1, \cdots, b_m \in F$,

$$\prod_{j=1}^{m} b_j^0 * (\prod_{i=1}^{n} a_i * x) = 1,$$

it follows from Proposition 4.3 that $x \in (F]_c$. Hence $x \in F$.

Conversely suppose F satisfies $(*)$. It follows from Proposition 4.3 that $(F]_c \subseteq F$. Also, $F \subseteq (F]_c$ is obvious. Hence $(F]_c = F$, and F is a closed filter of X. The proof is complete.

References

[1] S. S. Ahn and Y. H. So, *On ideals and upper sets in BE-algebras,* Sci. Math. Japon. Online **e-2008**(2008), No.2, 279-285.

[2] S. S. Ahn and K. S. So, *On generalized upper sets in BE-algebras*, Bull. Korean Math. Soc. **46**(2009), No.2, 281-287.

[3] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Seminar Notes, **11**(1983), No.2, part 2, 313-320.

[4] K. Iséki, *On BCI-algebras*, Methematics Seminar Notes, **8**(1980), 125-130.

[5] K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japon. **21**(1976) 351-366.

[6] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23**(1978) 1-26.

[7] H. S. Kim and Y. H. Kim, *On BE-algebras,* Sci. Math. Japon. **66(1)**(2007), 113-117.

[8] H. S. Kim and K. J. Lee, *Extended upper sets in BE-algebras,* Submitted.

[9] B. L. Meng, *CI-algebras,* Sci. Math. Japon., **71**, No.1(2010), 11-17; **e2009**, 695-701.

[10] B. L. Meng, *On filter in BE-algebras,* Sci. Math. Japon. online, **e-2010**, 105-111.

[11] J. Negger and H. S. Kim , *On d-algebras,* Math. Slovaca **40**(1999), No.1, 19-26.

[12] K. H. Kim and Y. H. Yon, Dual BCK-algebra and MV-algebra, Sci. Math. Japon., **66**(2007), 247-253.

[13] A. Walendziak, *On commutative BE-algebras*, Sci. Math. Japon. Online **e-2008**, 585-588.

Biao Long MENG:
Department of Basic Courses,
Xian University of Science and Technology,
Xian 710054, P.R.CHINA.
Email address: mengbl_100@yahoo.com.cn