(WEAK) COMPACTNESS AND LOCAL (WEAK) COMPACTNESS IN FUZZY PREUNIFORM CONVERGENCE SPACES

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Dedicated to the memory of Jun-iti Nagata

Abstract. Using fuzzy filters in the sense of P. Eklund and W. Gähler [2] in fuzzy preuniform convergence spaces as introduced in [11] compactness and weak compactness are presented by means of convergence and preconvergence of fuzzy ultrafilters respectively. The generalization of these concepts by localization leads to bicoreflective subconstructs of the construct of **FPUConv** of fuzzy preuniform convergence spaces which are additionally cartesian closed. These results are even new in the non-fuzzy case (i.e. in the realm of preuniform convergence spaces) which is included and improve the situation for topological spaces.

0 Introduction

Fuzzy preuniform convergence spaces introduced in [11] form a strong topological universe, denoted by **FPUConv**, i.e. a cartesian closed topological construct which is extensional and in which arbitrary products of quotients are quotients (cf. [12]). Its full subconstruct of fuzzy semiuniform convergence spaces is also a strong topological universe which is mainly studied in the realm of fuzzy convenient topology whereas fuzzy preuniform convergence spaces are a good candidate for non-symmetric fuzzy convenient topology (cf. [12]). As in the non-fuzzy case for the definition of these spaces a filter concept is needed. We use here fuzzy filters in the sence of P. Eklund and W. Gähler [2] which fuzzificate the membership of filter elements too. Sometimes they are also called tight stratified L-filters (cf. [7]) where L is a frame. It makes sense to define compactness (via fuzzy ultrafilter convergence) whenever convergence of fuzzy filters can be defined. W. Gähler has done this for fuzzy limit spaces in [4]. Since fuzzy preuniform convergence spaces induce two different fuzzy generalized convergence spaces, there are two possibilities to define convergence of fuzzy filters, called preconvergence and convergence. In certain subconstructs of **FPUConv** both convergence concepts coincide, e.g. in the subconstruct **FULim** of fuzzy uniform limit spaces or in the subconstruct **FConv** of fuzzy convergence spaces both introduced in this paper. But in general it is needed to study two concepts of compactness, namely compactness and weak compactness, according to the convergence and preconvergence of fuzzy ultrafilters respectively. For both concepts a product theorem (Tychonoff theorem) is valid. By localization of these concepts one obtains local compactness and local weak compactness in fuzzy preuniform convergence spaces and it turns out that both are closed under formation of final structures in **FPUConv**. Since the construct **FGConv** of fuzzy generalized convergence spaces can be bicoreflectively embedded into **FPUConv** (cf. [11]) we obtain that quotients of locally compact fuzzy generalized convergence spaces are locally

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compact. This is hightly remarkable since in the subconstruct **FTop** of fuzzy topological spaces such a result is not true even in the non-fuzzy case. Furthermore, the constructs **LC-FPUConv** and **LWC-FPUConv** of locally compact fuzzy preuniform convergence spaces and locally weakly compact fuzzy preuniform convergence spaces respectively are cartesian closed. From this result can be derived that the construct **LC-FGConv** of locally compact fuzzy generalized convergence spaces is cartesian closed too and also the construct **LCT$_2$-FGConv** of locally compact T_2 fuzzy generalized convergence spaces is cartesian closed, where T_2 means uniqueness of fuzzy filter convergence. This improves the situation for locally compact T_2 fuzzy topological spaces, namely the construct **LCT$_2$-FTop** of locally compact T_2 fuzzy topological spaces is a subconstruct of **LCT$_2$-FGConv** and not even cartesian closed in the non-fuzzy case.

Last but not least in fuzzy (quasi) uniform spaces as introduced in [5] there is no difference between compactness and local compactness.

All results of this paper are valid for the non-fuzzy case, i.e. in case $L = \{0, 1\}$. They are new when applied to preuniform convergence spaces.

The terminology of this paper corresponds to [9].

1 Preliminaries

In the following let L be a frame (i.e. a complete lattice such that for each $l \in L$ and each $M \subset L$ the infinite distributive law $l \wedge \bigvee M = \bigvee\{l \wedge m : m \in M\}$ holds) with different least element 0 and greatest element 1, e.g. $L = \{0, 1\}$ or $L = [0, 1]$ (= closed unit interval).

1.1 Remark. For each set X, L^X can be endowed with a partial order $\leq$ defined as follows: $f \leq g$ iff $f(x) \leq g(x)$ for each $x \in X$.
As in L, for infima and suprema in L^X the symbols $\wedge$ and $\bigwedge$ as well as $\vee$ or $\bigvee$ are used respectively, e.g. for each pair $(f, g) \in L^X \times L^X$ and each $x \in X$, $(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$.

1.2 Definition. An *L-fuzzy filter* (shortly: a *fuzzy filter*) on a non-empty set X is a map $\mathcal{F} : L^X \to L$ such that the following are satisfied:

$FFil_1$) $\mathcal{F}(\bar{l}) = l$ for each $l \in L$, where $\bar{l} : X \to L$ is defined by $\bar{l}(x) = l$ for each $x \in X$.

$FFil_2$) $\mathcal{F}(f \wedge g) = \mathcal{F}(f) \wedge \mathcal{F}(g)$ for all $f, g \in L^X$.

The set of all fuzzy filters on X is denoted by $F_L(X)$, where $F_L(\emptyset) = \emptyset$.

1.3 Remark. 1) If $\mathcal{F}$ is a fuzzy filter on X, then $\mathcal{F}(f) \leq \mathcal{F}(g)$ whenever $f \leq g$. Furthermore for each $f \in L^X$, $\mathcal{F}(f) \leq supf = sup\{f(x) : x \in X\}$.

2) For each $x \in X$, there is a fuzzy filter $\dot{x} : L^X \to L$ defined by $\dot{x}(f) = f(x)$ for each $f \in L^X$. In case $X = \{x\}$, $\dot{x}$ is the unique fuzzy filter on X.

3) The elements of L^X are called $L-$ *fuzzy subsets* of X or shortly *fuzzy sets*. Usually, a subset A of X is identified with its *characteristic function* $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

Thus, in case $L = \{0, 1\}$, L^X may be identified with the powerset of X, and a $\{0, 1\}$-fuzzy filter is an ordinary filter (up to identification).

1.4 Definition. A *fuzzy filter base* on a non-empty set X is a non-empty subset $\mathcal{B}$ of L^X such that the following are satisfied:

FB_1) $\bar{l} \in \mathcal{B}$ for each $l \in L$.

FB_2) For each $(f, g) \in \mathcal{B} \times \mathcal{B}$ there is some $h \in \mathcal{B}$ such that $h \leq f \wedge g$ and $\sup h = \sup f \wedge \sup g$.

1.5 Proposition. (cf. [4]). *Each fuzzy filter base $\mathcal{B}$ on X generates a fuzzy filter $\mathcal{F}$ on X defined by*

$$\mathcal{F}(f) = \bigvee_{g \leq f, g \in \mathcal{B}} \sup g \quad \text{for each } f \in L^X.$$

Conversely, each fuzzy filter $\mathcal{F}$ can be generated by a fuzzy filter base on X, even a greatest one, denoted by base $\mathcal{F}$, *where* base $\mathcal{F} = \{f \in L^X : \mathcal{F}(f) = \sup f\}$.

1.6 Proposition. (cf. [3; 2.11]). *There is a one-to-one correspondence between fuzzy filters $\mathcal{F}$ on X and the subsets $\mathcal{B}$ of L^X which fulfill the following conditions:*

1. $\bar{l} \in \mathcal{B}$ *for each* $l \in L$.
2. *For all* $f, g \in \mathcal{B}$, $f \wedge g \in \mathcal{B}$ *and* $\sup(f \wedge g) = \sup f \wedge \sup g$.
3. $\bigvee_{f \in F} f \in \mathcal{B}$ *for each* $F \subset \mathcal{B}$.
4. $f \in \mathcal{B}, f \leq g$, *and* $\sup f = \sup g$ *imply* $g \in \mathcal{B}$.

This correspondence is given by:
$\mathcal{B} =$ base $\mathcal{F}$ *and* $\mathcal{F}(f) = \bigvee_{g \in \mathcal{B}, g \leq f} \sup g$ *for each* $f \in L^X$.

1.7 Definition. Let $\mathcal{F}$ and $\mathcal{G}$ be L-fuzzy filters on X. Then $\mathcal{F}$ is called *coarser* than $\mathcal{G}$ (or $\mathcal{G}$ is called *finer* than $\mathcal{F}$), denoted by $\mathcal{F} \subset \mathcal{G}$, iff $\mathcal{F}(f) \leq \mathcal{G}(f)$ for each $f \in L^X$.

1.8 Proposition. (cf. [4]). *Let $\mathcal{F}$ and $\mathcal{G}$ be fuzzy filters generated by fuzzy filter bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. If $\mathcal{B} \subset \mathcal{B}'$, then $\mathcal{F} \subset \mathcal{G}$. The inverse implication is true whenever $\mathcal{B} =$* base $\mathcal{F}$ *and $\mathcal{B}' =$* base $\mathcal{G}$.

1.9 Proposition. *Let $f : X \to Y$ be a map, $\mathcal{F}$ a fuzzy filter on X, and $\mathcal{B}$ a base of $\mathcal{F}$. Define for each $g \in L^X$, $f[g] \in L^Y$ by $f[g](y) = \bigvee_{x \in f^{-1}(y)} g(x)$ for each $y \in Y$. Then $\{f[g] : g \in \mathcal{B}\} \cup \{\bar{l} : l \in L\}$ is a base of the fuzzy filter $f(\mathcal{F})$, defined by $f(\mathcal{F})(h) = \mathcal{F}(h \circ f)$ for each $h \in L^Y$, where $f(\mathcal{F})$ is called the image of $\mathcal{F}$ under f. If f is surjective, then $\{f[g] : g \in \mathcal{B}\}$ is a base of $f(\mathcal{F})$.*

1.10 Definition. Let $f : X \to Y$ be a map and $\mathcal{F}$ a fuzzy filter on Y. Then the *inverse image of $\mathcal{F}$ under f* is *the coarsest fuzzy filter $\mathcal{G}$* on X such that $\mathcal{F} \subset f(\mathcal{G})$ provided that it exists. Usually, we write $f^{-1}(\mathcal{F})$ instead of $\mathcal{G}$. If $X \subset Y$ and $i : X \to Y$ denotes the inclusion map, then $i^{-1}(\mathcal{F})$ is called the *trace* of $\mathcal{F}$ on X.

1.11 Proposition. (cf. [4; proposition 9]). *Let $f : X \to Y$ be a map, $\mathcal{F}$ a fuzzy filter on Y, and $\mathcal{B}$ a base of $\mathcal{F}$. Then $f^{-1}(\mathcal{F})$ exists iff $\sup g = \sup(g \circ f)$ for each $g \in \mathcal{B}$. If $f^{-1}(\mathcal{F})$ exists, then $\{g \circ f : g \in \mathcal{B}\}$ is a base of $f^{-1}(\mathcal{F})$.*

1.12 Proposition. *Let $\mathcal{F}$ be a fuzzy filter on X, A a non-empty subset of X, and $i_A : A \to X$ the inclusion map. If $\chi_A \in$* base $\mathcal{F}$, *then $i_A^{-1}(\mathcal{F})$ exists and $i_A(i_A^{-1}(\mathcal{F}) = \mathcal{F}$.*

Proof. 1. $i_A^{-1}(\mathcal{F})$ exists: Let $f \in \text{base } \mathcal{F}$. Then $\sup f \circ i_A = \sup f|_A = \sup(f \wedge \chi_A) = \sup f \wedge \sup \chi_A = \sup f$, since $\sup \chi_A = 1$ because $A \neq \emptyset$.

2. (a) $i_A(i_A^{-1}(\mathcal{F}) \supset \mathcal{F}$ is always valid by definition.
   (b) $\mathcal{B} = \{i_A[f|_A] : f \in \text{base } \mathcal{F}\} \cup \{\bar{l} : l \in L\}$ is a base of $i_A(i_A^{-1}(\mathcal{F}))$. Since for each $f \in \text{base } \mathcal{F}$, $i_A[f|_A] = f \wedge \chi_A \in \text{base } \mathcal{F}$, it follows $\mathcal{B} \subset \text{base } \mathcal{F}$, which implies $i_A(i_A^{-1}(\mathcal{F})) \subset \mathcal{F}$.

1.13 Definition. Let M be a non-empty set of fuzzy filters on X. Then a fuzzy filter $\bigcap_{\mathcal{F}\in M} \mathcal{F}$, called the *intersection* of all $\mathcal{F} \in M$, is defined by $\bigcap_{\mathcal{F}\in M} \mathcal{F}(f) = \bigwedge_{\mathcal{F}\in M} \mathcal{F}(f)$ for each $f \in L^X$.

1.14 Proposition. *Let M be a non-empty set of fuzzy filters on X.*

1. *The infimum of M in $(F_L(X), \subset)$ exists and is equal to* $\bigcap_{\mathcal{F}\in M} \mathcal{F}$.
2. (a) *The supremum of M in $(F_L(X), \subset)$ exists iff for each non-empty finite subset N of M the following condition is satisfied:*

$$\sup(f_1 \wedge \cdots \wedge f_n) = \sup f_1 \wedge \cdots \wedge \sup f_n$$

   for all $f_1 \in$ base $\mathcal{F}_1, \ldots, f_n \in$ base $\mathcal{F}_n$, where $N = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ (cf. [4; proposition 8])

   (b) *If the supremum $\mathcal{S}$ of $M = \{\mathcal{F}_i : i \in I\}$ in $(F_L(X), \subset)$ exists and $\mathcal{B}_i$ is a base of $\mathcal{F}_i$ for each $i \in I$, then $\mathcal{B} = \{f_{i_1} \wedge \cdots \wedge f_{i_n} : \{i_1, \ldots, i_n\} \subset I$ finite and $f_i \in \mathcal{B}_i$ for each $i \in I\}$ is a base of $\mathcal{S}$.*

1.15 Definition. Let $(X_i)_{i\in I}$ be a non-empty family of non-empty sets and $\mathcal{F}_i$ a fuzzy filter on X_i for each $i \in I$. If $p_i : \prod_{i\in I} X_i \to X_i$ denotes the i-th projection, then the coarsest fuzzy filter $\mathcal{F}$ on $\prod_{i\in I} X_i$ such that $p_i(\mathcal{F}) = \mathcal{F}_i$ for each $i \in I$ is called the *product* of $(\mathcal{F}_i)_{i\in I}$ where $\prod_{i\in I} \mathcal{F}_i$ is written instead of $\mathcal{F}$ or $\mathcal{F}_1 \times \mathcal{F}_2$ in case $I = \{1, 2\}$.

1.16 Proposition. (cf. [3; proposition 3.10] and [4; proposition 19]). *If I is a non-empty set and for each $i \in I$, $\mathcal{F}_i$ is a fuzzy filter on X_i, then the product $\prod_{i\in I} \mathcal{F}_i$ exists and is equal to the supremum of $\{p_i^{-1}(\mathcal{F}_i) : i \in I\}$ in $(F_L(X), \subset)$. In particular, if $\mathcal{B}_i$ is a base of $\mathcal{F}_i$ for each $i \in I$, then*

$$\mathcal{B} = \{\bigwedge_{j\in J} f_j \circ p_j : J \subset I \text{ finite and } f_j \in \mathcal{B}_j \text{ for each } j \in J\}$$

is a base of $\prod_{i\in I} \mathcal{F}_i$.

1.17 Definition. A fuzzy filter $\mathcal{U}$ on X is called a *fuzzy ultrafilter* iff it is a maximal element in $(F_L(X), \subset)$.

1.18 Proposition. *For each $\mathcal{F} \in F_L(X)$ there is a fuzzy ultrafilter $\mathcal{U} \in F_L(X)$ such that $\mathcal{F} \subset \mathcal{U}$.*

Proof. By means of Zorn's lemma it suffices to prove that each chain M in $(F_L(X), \subset)$ has an upper bound. Indeed we prove that each chain M in $(F_L(X), \subset)$ has a supremum. If M is non-empty and $N = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\} \subset M$ is non-empty and finite, then we may assume, without loss of generality, that $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$ since M is a chain. For all $f_1 \in$ base $\mathcal{F}_1, \ldots, f_n \in$ base $\mathcal{F}_n$, $f_i \in$ base $\mathcal{F}_n$ for each $i \in \{1, \ldots, n\}$. Consequently, $\sup f_1 \wedge \cdots \wedge f_n = \sup f_1 \wedge \cdots \wedge \sup f_n$. Thus, by 1.14.2.a), M has a supremum. If $M = \emptyset$, we have nothing to prove since $\sup M$ as the least element of $F_L(X)$ always exists (note: $\sup M = \bigcap_{x \in X} \dot{x} \subset \mathcal{F}$ for each $\mathcal{F} \in F_L(X)$ since $f \in$ base $\bigcap_{x \in X} \dot{x} = \bigcap_{x \in X}$ base $\dot{x}$, i.e. $f(x) = \sup f$ for each $x \in X$ or equivalently $f = \overline{\sup f}$, implies $\mathcal{F}(f) = \mathcal{F}(\overline{\sup f}) = \sup f$ for each $\mathcal{F} \in F_L(X)$, i.e. $f \in$ base $\mathcal{F}$ for each $\mathcal{F} \in F_L(X)$).

1.19 Proposition. (cf. [1; proposition 3]). *$\mathcal{U} \in F_L(X)$ is a fuzzy ultrafilter iff each $f \in L^X$, for which $\sup(f \wedge g) = \sup f \wedge \sup g$ for each $g \in$ base $\mathcal{U}$, belongs to base $\mathcal{U}$.*

1.20 Proposition. *Let $f : X \to Y$ be a map between the sets X and Y. If $\mathcal{U}$ is a fuzzy ultrafilter on X, then $f(\mathcal{U})$ is a fuzzy ultrafilter on Y.*

Proof. Let $h \in L^Y$ such that $\sup(h \wedge k) = \sup h \wedge \sup k$ for each $k \in$ base $f(\mathcal{U})$. Hence, $f^{-1}[h] = h \circ f \in L^X$. Furthermore, let $u \in$ base $\mathcal{U}$. This implies $f[u] \in$ base $f(\mathcal{U})$ and by assumption

$$(1)\ \sup(h \wedge f[u]) = \sup h \wedge \sup f[u] = \sup h \wedge \sup u$$

On the other hand,

$$\begin{aligned}(2)\ \sup(h \wedge f[u]) &= \bigvee_{y \in f[X]} (h(y) \wedge \bigvee_{x \in f^{-1}(y)} u(x)) = \bigvee_{y \in f[X]} h(y) \wedge \bigvee_{x \in f^{-1}(y)} u(x) \\ &= \bigvee_{y \in f[X]} (\bigvee_{x \in f^{-1}(y)} (h(y) \wedge u(x))) = \bigvee_{x \in X} h(f(x)) \wedge u(x) \\ &= \sup(u \wedge (h \circ f))\end{aligned}$$

Now (1) and (2) imply
(3) $\sup(u \wedge (h \circ f)) = \sup h \wedge \sup u = \sup u \wedge \sup(h \circ f)$, since it follows from $f[\chi_X] = \chi_{f[X]} \in$ base $f(\mathcal{U})$, $\sup(h \circ f) = \sup(h \wedge \chi_{f[X]}) = \sup h \wedge \sup \chi_{f[X]} = \sup h$. Consequently, $h \circ f \in$ base $\mathcal{U}$ because $\mathcal{U}$ is a fuzzy ultrafilter on X. Thus, $f(\mathcal{U})(h) = \mathcal{U}(h \circ f) = \sup h \circ f = \sup h$, i.e. $h \in$ base $f(\mathcal{U})$. By 1.19, $f(\mathcal{U})$ is a fuzzy ultrafilter.

1.21 Corollary. *Let X be a non-empty set. For each $x \in X$, $\dot{x}$ is a fuzzy ultrafilter on X.*

Proof. Let $x \in X$ and let $i : \{x\} \to X$ be the inclusion map. Since $\dot{x}$ is the unique fuzzy filter on $\{x\}$, it is a fuzzy ultrafilter by 1.18. By 1.20, $i(\dot{x}) = \dot{i(x)}$ is a fuzzy ultrafilter on X.

1.22 Definition. Let X be a non-empty set and $\mathcal{F}, \mathcal{G}$ fuzzy filters on $X \times X$. Then

1. a fuzzy filter $\mathcal{F}^{-1}$ on $X \times X$ is defined by $\mathcal{F}^{-1}(f) = \mathcal{F}(f^{-1})$ for each $f \in L^{X \times X}$, where $f^{-1}(x, y) = f(y, x)$ for each $(x, y) \in X \times X$,
2. a *composition* $\mathcal{F} \circ \mathcal{G}$ is defined by $\mathcal{F} \circ \mathcal{G}(h) = \bigvee_{f \circ g \leq h} \mathcal{F}(f) \wedge \mathcal{G}(g)$ for each $h \in L^{X \times X}$ where $f \circ g(x, y) = \bigvee_{z \in X} g(x, z) \wedge f(z, y)$ for each $(x, y) \in X \times X$.

1.23 Proposition. *Let $\mathcal{F},\mathcal{G} \in F_L(X\times X)$.*
Then the following are equivalent:

1. $\mathcal{F}\circ\mathcal{G} \in F_L(X\times X)$.
2. *For each $\alpha \in L$ and all $f,g \in L^{X\times X}$ the following is satisfied: $f\circ g \le \bar{\alpha}$ implies $\mathcal{F}(f)\wedge\mathcal{G}(g) \le \alpha$.*
3. $\mathcal{F}\circ\mathcal{G}(\bar{\alpha}) \le \alpha$ *for each* $\alpha \in L$.

Proof. The equivalence of (2) and (3) is obvious.
(1) $\Rightarrow$ (3). This is obvious since even $\mathcal{F}\circ\mathcal{G}(\bar{\alpha}) = \alpha$ for each $\alpha \in L$.
(3) $\Rightarrow$ (1). $FFil_1$). For each $\alpha \in L, \bar{\alpha}\circ\bar{\alpha} = \bar{\alpha}$. Thus, for each $\alpha \in L$, $\alpha = \mathcal{F}(\bar{\alpha})\wedge\mathcal{G}(\bar{\alpha}) \le \bigvee_{f\circ g\le\bar{\alpha}} \mathcal{F}(f)\wedge\mathcal{G}(g) = \mathcal{F}\circ\mathcal{G}(\bar{\alpha})$,which implies, by assumption, $\mathcal{F}\circ\mathcal{G}(\bar{\alpha}) = \alpha$ for each $\alpha \in L$.
$FFil_2$)

a) Let $u,v \in L^{X\times X}$ and $u \le v$. Then $\mathcal{F}\circ\mathcal{G}(u) = \bigvee_{f\circ g\le u} \mathcal{F}(f)\wedge\mathcal{G}(g) \le \bigvee_{f\circ g\le v} \mathcal{F}(f)\wedge\mathcal{G}(g) = \mathcal{F}\circ\mathcal{G}(v)$.

b) Let $k_1,k_2 \in L^{X\times X}$. Then, $f_1\circ g_1 \le k_1$ and $f_2\circ g_2 \le k_2$ imply $f\circ g \le k_1\wedge k_2$ where $f = f_1\wedge f_2$ and $g = g_1\wedge g_2$. Hence, $\mathcal{F}\circ\mathcal{G}(k_1)\wedge\mathcal{F}\circ\mathcal{G}(k_2) = (\bigvee_{f_1\circ g_1\le k_1} \mathcal{F}(f_1)\wedge\mathcal{G}(g_1))\wedge(\bigvee_{f_2\circ g_2\le k_2} \mathcal{F}(f_2)\wedge\mathcal{G}(g_2))$
$\le \bigvee_{f\circ g\le k_1\wedge k_2} \mathcal{F}(f)\wedge\mathcal{G}(g) = \mathcal{F}\circ\mathcal{G}(k_1\wedge k_2)$.

c) By means of a) and b), $FFil_2$) is fulfilled for $\mathcal{F}\circ\mathcal{G}$.

1.24 Proposition. *Let $\mathcal{F},\mathcal{G} \in F_L(X\times X)$. If there are $x,y,z \in X$ such that $\mathcal{G} \subset (x,y)^{\cdot}$ and $\mathcal{F} \subset (y,z)^{\cdot}$, then $\mathcal{F}\circ\mathcal{G} \in F_L(X\times X)$.*

Proof. Let $\alpha \in L$ and $f,g \in L^{X\times X}$ such that $f\circ g \le \bar{\alpha}$. By assumption there are $x,y,z \in X$ with $\mathcal{F}(f) \le f(y,z)$ and $\mathcal{G}(g) \le g(x,y)$. Furthermore, $g(x,y)\wedge f(y,z) \le f\circ g(x,z)) = \bigvee_{a\in X} g(x,a)\wedge f(a,z) \le \alpha$. Consequently, $\mathcal{F}(f)\wedge\mathcal{G}(g) \le \alpha$. By 1.23. $\mathcal{F}\circ\mathcal{G} \in F_L(X\times X)$.

2 Fuzzy compact spaces

2.1 Definition. 1. (a) A fuzzy *generalized convergence space* is a pair (X,q) where X is a set and $q \subset F_L(X)\times X$ such that the following are satisfied

FC_1) $(\dot{x},x) \in q$ for each $x \in X$,
FC_2) $(\mathcal{F},x) \in q$ whenever $(\mathcal{G},x) \in q$ and $\mathcal{G} \subset \mathcal{F}$.

If (X,q) is a fuzzy generalized convergence space, then we say $\mathcal{F}$ *converges to* x instead of $(\mathcal{F},x) \in q$ and write sometimes $\mathcal{F} \xrightarrow{q} x$ or shortly $\mathcal{F} \to x$.

(b) A fuzzy generalized convergence space (X,q) is called

α) a *fuzzy Kent convergence space* provided that the following is statisfied:
FC_3) $(\mathcal{F}\cap\dot{x},x) \in q$ whenever $(\mathcal{F},x) \in q$, and

β) *symmetric* provided that the following is satisfied:
FS) $(\mathcal{F},x) \in q$ and $\mathcal{F} \subset \dot{y}$ imply $(\mathcal{F},y) \in q$.

2. A map $f:(X,q)\to(X',q')$ between fuzzy generalized convergence spaces is called *fuzzy continuous* iff $(f(\mathcal{F}),f(x))\in q'$ for each $(\mathcal{F},x)\in q$.

3. A fuzzy generalized convergence space (X,q) is called *compact* provided that each fuzzy ultrafilter $\mathcal{U}$ on X converges to some $x\in X$.

2.2 Proposition. *A fuzzy generalized convergence space (X,q) is compact iff each fuzzy filter $\mathcal{F}$ on X has an adherence point $x\in X$ (i.e. there is a fuzzy filter $\mathcal{F}'$ on X finer than $\mathcal{F}$ which converges to x).*

Proof. Because of 1.18 the proof is similar to the classical case.

2.3 Proposition. *Let $f:(X,q)\to(X',q')$ be a surjective fuzzy continuous map between fuzzy generalized convergence spaces. If (X,q) is compact, then (X',q') is compact too.*

Proof. If $\mathcal{F}\in F_L(X')$, then $f^{-1}(\mathcal{F})\in F_L(X)$ because f is surjective. By assumption, there is some $(\mathcal{G},x)\in q$ with $\mathcal{G}\supset f^{-1}(\mathcal{F})$. Consequently, $f(\mathcal{G})\xrightarrow{q'}f(x)$ since f is fuzzy continuous and $f(\mathcal{G})\supset f(f^{-1}(\mathcal{F})\supset\mathcal{F}$, i.e. $f(x)$ is an adherence point of $\mathcal{F}$. By 2.2., (X',q') is compact.

2.4 Remark. The category **FGConv** of fuzzy generalized convergence spaces (and fuzzy continuous maps) is a topological construct, where the initial **FGConv**-structures are formed as follows: Let X be a set, $((X_i,q_i))_{i\in I}$ a family of fuzzy generalized convergence spaces, and $(f_i:X\to X_i)_{i\in I}$ a family of maps. Then $q=\{(\mathcal{F},x)\in F_L(X)\times X: (f_i(\mathcal{F}),f_i(x))\in q_i$ for each $i\in I\}$ is the initial **FGConv**-structure X w.r.t. the given data.

2.5 Theorem. *Let $((X_i,q_i))_{i\in I}$ be a family of non-empty fuzzy generalized convergence spaces. Then the product space $\prod_{i\in I}(X_i,q_i)$ of this family is compact iff (X_i,q_i) is compact for each $i\in I$.*

Proof. "⇒." Apply 2.3. to the projections $p_i:\prod X_i\to X_i$ which are surjective and fuzzy continuous.
"⇐." Let $\mathcal{U}$ be a fuzzy ultrafilter on $\prod X_i$. Then for each $i\in I$, $p_i(\mathcal{U})$ is a fuzzy ultrafilter on X_i converging to some $x_i\in X_i$, since (X_i,q_i) is compact. Thus $\mathcal{U}$ converges to $x=(x_i)$. Consequently, $\prod_{i\in I}(X_i,q_i)$ is compact.

2.6 Definition. Let (X,q) be a fuzzy generalized convergence space, $A\subset X$, and $i_A: A\to X$ the inclusion map.

1. *The closure of A w.r.t q*, denoted by cl_qA, is defined by $cl_qA=\{x\in X:$ there is some $\mathcal{F}\in F_L(A)$ with $i_A(\mathcal{F})\xrightarrow{q}x\}$. If $A=cl_qA$, A is called *closed*.

2. A is called *compact* iff the subspace (A,q_A) of (X,q) is compact.

2.7 Proposition. *Let (X,q) be a compact fuzzy generalized convergence space and $A\subset X$ a closed subset. Then A is compact.*

Proof. Let $\mathcal{U}$ be a fuzzy ultrafilter on A. Then $i_A(\mathcal{U})$ is a fuzzy ultrafilter on X where $i_A: A\to X$ denotes the inclusion map. By assumption, $i_A(\mathcal{U})$ converges to some $x\in X$. Since A is closed, $x\in A$. Thus, $\mathcal{U}$ converges to x in the subspace (A,q_A) of (X,q). Consequently, A is compact.

2.8 Definition. 1. (a) A *fuzzy preuniform convergence space* is a pair (X, FJ_X) where X is a set and FJ_X a set of fuzzy filters on $X \times X$ such that the following are satisfied:

$FC_1)$ $(\dot{x,x}) \in FJ_X$ for each $x \in X$.

$FC_2)$ $\mathcal{F} \in FJ_X$ whenever $\mathcal{G} \in FJ_X$ and $\mathcal{G} \subset \mathcal{F}$.

(b) A fuzzy preuniform convergence space (X, FJ_X) is called a *fuzzy semiuniform convergence space* provided that the following is satisfied:

$FC_3)$ $\mathcal{F} \in FJ_X$ implies $\mathcal{F}^{-1} \in FJ_X$.

2. A map $f : (X, FJ_X) \to (Y, FJ_Y)$ between fuzzy preuniform convergence spaces is called *uniformly continuous* iff $(f \times f)(\mathcal{F}) \in FJ_Y$ for each $\mathcal{F} \in FJ_X$.

2.9 Remark. A fuzzy preuniform convergence space (X, FJ_X) has two canonical underlying fuzzy generalized convergence spaces, namely
$1°(X, q_{FJ_X})$ and $2°(X, q_{\gamma_{FJ_X}})$ defined as follows:

$1°$ $(\mathcal{F}, x) \in q_{FJ_X}$ iff $\dot{x} \times \mathcal{F} \in FJ_X$,

$2°$ $(\mathcal{F}, x) \in q_{\gamma_{FJ_X}}$ iff $(\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}) \in FJ_X$.

Fuzzy filter convergence in (X, q_{FJ_X}) and in $(X, q_{\gamma_{FJ_X}})$ are called *preconvergence* in (X, FJ_X) and *convergence* in (X, FJ_X) respectively. In particular, $(X, q_{\gamma_{FJ_X}})$ is a symmetric Kent convergence space. Obviously, convergence implies preconvergence whereas the inverse implication is not even true in case $L = \{0, 1\}$ (cf. [10]).

2.10 Definition. 1. A fuzzy preuniform convergence space (X, FJ_X) is called *compact* (resp. *weakly compact*) iff each fuzzy ultrafilter $\mathcal{U}$ on X converges (resp. preconverges) to some $x \in X$, i.e. iff $(X, q_{\gamma_{FJ_X}})$(resp. (X, q_{FJ_X})) is compact.

2. If (X, FJ_X) is a fuzzy preuniform convergence space then a subset A of X is called (*weakly*) *compact* iff the subspace (A, FJ_A) of (X, FJ_X) is (weakly) compact.

2.11 Remark. 1. Obviously, every compact fuzzy preuniform convergence space is weakly compact.

2. By [11] the category **FPUConv** of fuzzy preuniform convergence spaces (and fuzzy uniformly continuous maps) is a topological construct and its full subcategory **FSUConv** of fuzzy semiuniform convergence spaces is bireflective and bicoreflective (in **FPUConv**), i.e. a topological construct too where initial and final structures in **FSUCoonv** are formed as in **FPUConv**. The initial structures in **FPUConv** are formed as follows: Let X be a set, $((X_i, FJ_{X_i}))_{i \in I}$ a family of fuzzy preuniform convergence spaces and $(f_i : X \to X_i)_{i \in I}$ a family of maps. Then $FJ_X = \{\mathcal{F} \in F_L(X \times X) : (f_i \times f_i)(\mathcal{F}) \in FJ_{X_i}$ for each $i \in I\}$ is the initial **FPUConv**-structure on X w.r.t. the given data. It has been proved in [11] that the construct **FGConv** of fuzzy generalized convergence spaces is bicoreflectively embedded in **FPUConv**. Furthermore the construct **FKConv**$_S$ of symmetric fuzzy Kent convergence spaces can be bicoreflectively embedded into **FSUConv** and thus in **FPUConv** (cf. [11] and [12]). Since a bicoreflector preserves initial structures we obtain the following corollaries 2.12 and 2.14.

2.12 Corollary. *Let $((X_i, FJ_{X_i}))_{i\in I}$ be a family of non-empty fuzzy preuniform convergence spaces. Then the product space $\prod_{i\in I}(X_i, FJ_{X_i})$ is (weakly) compact iff (X_i, FJ_{X_i}) is (weakly) compact for each $i \in I$.*

Proof. Apply 2.5.

2.13 Definition. A subset A of a fuzzy preuniform convergence space (X, FJ_X) is called *closed (resp. preclosed)* iff it is closed in $(X, q_{\gamma_{FJ_X}})$ (resp. (X, q_{FJ_X})).

2.14 Corollary. *Let (X, FJ_X) be a (weakly) compact fuzzy preuniform convergence space and $A \subset X$ (pre) closed. Then A is (weakly) compact as a subspace of (X, FJ_X).*

Proof. Apply 2.7.

2.15 Corollary. *Let $f : (X, FJ_X) \to (Y, FJ_Y)$ be a surjective fuzzy uniformly continuous maps between fuzzy preuniform convergence spaces. If (X, FJ_X) is (weakly) compact, then (Y, FJ_Y) is (weakly) compact.*

Proof. Apply 2.3 and note that each fuzzy uniformly continuous map is continuous.

2.16 Remark. Since initial structures in **FSUConv** are formed as in **FPUConv**, 2.14 and 2.15 are also valid in **FSUConv**.

3 Fuzzy locally compact spaces

3.1 Definitions.

1. A fuzzy generalized convergence space (X, q) is called *locally compact* provided that for each $(\mathcal{F}, x) \in q$ there is a non-empty compact subset K of X such that $\chi_K \in$ base $\mathcal{F}$.
2. A fuzzy preuniform convergence space (X, FJ_X) is called
   (a) *locally compact (locally weakly compact)* iff for each $\mathcal{F} \in FJ_X$ there is a non-empty compact (weakly compact) subset K of the product space $(X, FJ_X) \times (X, FJ_X)$ such that $\chi_K \in$ base $\mathcal{F}$,
   (b) *diagonal* iff $\bigcap_{x\in X} \dot{x} \times \dot{x} \in FJ_X$.

3.2 Remark. Every fuzzy (quasi) uniform space as studied in [12] is diagonal if it is regarded as a fuzzy preuniform convergence space.

3.3 Proposition.

1. *Every (weakly) compact fuzzy preuniform convergence space is locally (weakly) compact.*
2. *Every diagonal fuzzy preuniform convergence space, which is locally (weakly) compact, is (weakly) compact.*

Proof.

1. Let $(X, FJ_X) \in |\mathbf{FPUConv}|$ be (weakly) compact and $X \neq \emptyset$ (the case $X = \emptyset$ is trivial). For each $\mathcal{F} \in FJ_X$, $\chi_X = \bar{1} \in$ base $\mathcal{F}$. Thus, (X, FJ_X) is locally (weakly) compact.
2. Let $(X, FJ_X) \in |\mathbf{FPUConv}|$ be diagonal and locally (weakly) compact. Since $\bigcap_{x\in X} \dot{x}\times \dot{x} \in FJ_X$, there is some non-empty (weakly) compact $K \subset X \times X$ such that $\chi_K \in$

base $\bigcap_{x\in X} \dot{x}\times\dot{x} = \bigcap_{x\in X}$ base $\dot{x}\times\dot{x}$, i.e. $\chi_K(x,x) = 1$ for each $x \in X$. Thus, $\Delta = \{(x,x) : x \in X\} \subset K$. If $p_1 : X \times X \to X$ denotes the first projection, then $X = p_1[\Delta] \subset p_1[K] \subset X$, i.e. $X = p_1[K]$. By 2.15, X is (weakly) compact.

3.4 Definitions. 1. A fuzzy semiuniform convergence space (X, FJ_X) is called a *fuzzy uniform limit space* provided that the following are satisfied:

FUC_4) $\mathcal{F} \in FJ_X$ and $\mathcal{G} \in FJ_X$ imply $\mathcal{F} \cap \mathcal{G} \in FJ_X$,

FUC_5) $\mathcal{F} \in FJ_X$ and $\mathcal{G} \in FJ_X$ imply $\mathcal{F} \circ \mathcal{G} \in FJ_X$ whenever $\mathcal{F} \circ \mathcal{G}$ exists as a fuzzy filter (cf. 1.23).

2. A fuzzy uniform limit space (X, FJ_X) is called a *principal fuzzy uniform limit space* iff there is some $\mathcal{U} \in F_L(X \times X)$ such that $FJ_X = [\mathcal{U}]$, where $[\mathcal{U}] = \{\mathcal{F} \in F_L(X \times X) : \mathcal{F} \supset \mathcal{U}\}$.

3.5 Remark. If X is a non-empty set and $\mathcal{U}$ is a fuzzy filter on $X \times X$ then $\mathcal{U}$ is a fuzzy uniformity on X (i.e. 1° $\mathcal{U} \subset (\dot{x,x})$ for each $x \in X$, 2° $\mathcal{U} = \mathcal{U}^{-1}$, and 3° $\mathcal{U} \subset \mathcal{U} \circ \mathcal{U}$) iff $(X, [U])$ is a (principal) fuzzy uniform limit space (note: $\mathcal{U} \circ \mathcal{U}$ exists by 1.24 since 1° is satisfied). As in the non-fuzzy space *we need not distinguish between principal fuzzy uniform limit spaces and fuzzy uniform spaces* where a fuzzy uniform space is a set endowed with a fuzzy uniformity (a fuzzy uniformity $\mathcal{U}$ on the empty set is defined as a map $\mathcal{U} : L^{\emptyset} \to L$ such that $\mathcal{U}(\emptyset) = 1$ where $L^{\emptyset} = \{\emptyset\}$). Furthermore, *in fuzzy uniform limit spaces there is no difference between convergence and preconvergence of fuzzy filters* since for each $\mathcal{F} \in F_L(X)$ and each $x \in X$ the following formular is valid: $\mathcal{F} \cap \dot{x} \times \mathcal{F} \cap \dot{x} = \mathcal{F} \times \mathcal{F} \cap \mathcal{F} \times \dot{x} \cap \dot{x} \times \mathcal{F} \cap \dot{x} \times \dot{x}$, where $\mathcal{F} \times \mathcal{F} = \dot{x} \times \mathcal{F} \circ \mathcal{F} \times \dot{x}$ (cf. [6; 2.3.(2)] and [8; 5.1]). Consequently, in fuzzy uniform limit spaces there is no difference between compactness and weak compactness as well as between local compactness and local weak compactness. Obviously, principal fuzzy uniform limit spaces (= fuzzy uniform spaces) are diagonal. Thus, we obtain the following corollary.

3.6 Corollary. *A principal fuzzy uniform limit space (= fuzzy uniform space) is locally compact iff it is compact.*

3.7 Proposition. *Let (X, FJ_X) be a locally compact (resp. locally weakly compact) fuzzy preuniform convergence space. Then the underlying symmetric fuzzy Kent convergence space $(X, q_{\gamma_{FJ_X}})$ (resp. the underlying fuzzy generalized convergence space (X, q_{FJ_X})) is locally compact.*

Proof. 1. Let $(X, FJ_X) \in |\mathbf{FPUConv}|$ be locally compact and $(\mathcal{F}, x) \in q_{\gamma_{FJ_X}}$. Hence, $\mathcal{F} \cap \dot{x} \times \mathcal{F} \cap \dot{x} \in FJ_X$ and therefore $\mathcal{F} \times \mathcal{F} \in FJ_X$. By assumption, there is some non-empty compact $K \subset X \times X$ such that $\chi_K \in \mathcal{F} \times \mathcal{F}$. Consequently, $p_1[\chi_K] = \chi_{p_1[K]} \in$ base $p_1(\mathcal{F} \times \mathcal{F}) =$ base $\mathcal{F}$,where $p_1 : X \times X \to X$ denotes the first projection. Since $p_1[K] = (p_1|K)'[K]$ is compact ($(p_1|K)' : K \to p_1[K]$ defined by $(p_1|K)'(z) = p_1(z)$ for each $z \in K$ is surjective and fuzzy continuous!), $(X, q_{\gamma_{FJ_X}})$ is locally compact.

2. Let $(X, FJ_X) \in |\mathbf{FPUConv}|$ be locally weakly compact and $(\mathcal{F}, x) \in q_{\gamma_{FJ_X}}$, i.e. $\dot{x} \times \mathcal{F} \in FJ_X$. Then there is a non-empty compact K in $(X, q_{\gamma_{FJ_X}}) \times (X, q_{\gamma_{FJ_X}})$ such that $\chi_K \in \dot{x} \times \mathcal{F}$. Hence, $p_2[\chi_K] = \chi_{p_2[K]} \in$ base $p_2(\dot{x} \times \mathcal{F}) =$ base $\mathcal{F}$ where $p_2 : X \times X \to X$ denotes the second projection. Since $p_2[K]$ is compact, $(X, q_{\gamma_{FJ_X}})$ is locally compact.

3.8 Definition. A fuzzy preuniform convergence space (X, FJ_X) is called

1. a *fuzzy convergence space* iff
   $FJ_X = \{\mathcal{F} \in F_L(X \times X) :$ there is some $(\mathcal{G}, x) \in q_{\gamma_{FJ_X}}$ with $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$, i.e. iff (X, FJ_X) is 'generated' by its convergent fuzzy filters,

2. a *fuzzy preconvergence space* iff
   $FJ_X = \{\mathcal{F} \in F_L(X \times X) :$ there is some $(\mathcal{G}, x) \in q_{FJ_X}$ with $\dot{x} \times \mathcal{G} \subset \mathcal{F}\}$, i.e. iff (X, FJ_X) is 'generated' by its preconvergent fuzzy filters.

3.9 Remark. 1. *In fuzzy convergence spaces there is no difference between convergence and preconvergence* (note: If (X, FJ_X) is a fuzzy convergence space and $(\mathcal{F}, x) \in q_{FJ_X}$, i.e. there is some $(\mathcal{G}, y) \in q_{\gamma_{FJ_X}}$ with $\mathcal{G} \times \mathcal{G} \subset \dot{x} \times \mathcal{F}$, then $\mathcal{G} \subset \dot{x}$ and $\mathcal{G} \subset \mathcal{F}$, which implies $(\mathcal{G}, x) \in q_{\gamma_{FJ_X}}$ and thus $(\mathcal{F}, x) \in q_{\gamma_{FJ_X}}$.).

2. The construct **FConv** of fuzzy convergence spaces (and fuzzy uniformly continuous maps) is concretely isomorphic to $\mathbf{FKConv}_s$ (similarly to the non-fuzzy case) and the construct **FPConv** is concretely isomorphic to **FGConv** (cf. [11]).

3.10 Proposition. *Let (X, FJ_X) be a fuzzy convergence space and $(X, q_{\gamma_{FJ_X}})$ its underlying fuzzy Kent convergence space. Then the following are equivalent:*

1. *(X, FJ_X) is locally compact.*

2. *$(X, q_{\gamma_{FJ_X}})$ is locally compact.*

Proof. (1)⇒(2). See 3.7.
(2)⇒(1). Let $\mathcal{H} \in FJ_X$. Then there is some $(\mathcal{F}, x) \in q_{\gamma_{FJ_X}}$ such that $\mathcal{F} \times \mathcal{F} \subset \mathcal{H}$. By assumption, there exists some non-empty compact K in (X, FJ_X) with $\chi_K \in$ base $\mathcal{F}$. By 2.12, $K \times K$ is compact in $(X, FJ_X) \times (X, FJ_X)$. Furthermore, $\chi_{K \times K} = \chi_K \circ p_1 \wedge \chi_K \circ p_2 \in$ base $\mathcal{F} \times \mathcal{F}$, where for each $i \in \{1, 2\}$, $p_i : X \times X \to X$ denotes the i-th projection. Consequently, $\chi_{K \times K} \in$ base $\mathcal{H}$ and (X, FJ_X) is locally compact.

3.11 Remark. In fuzzy preconvergence spaces preconvergence is more interesting than convergence since in each $(X, FJ_X) \in |\mathbf{FPConv}|$, $(\mathcal{F}, x) \in q_{\gamma_{FJ_X}}$ iff $\mathcal{F} = \dot{x}$. Thus, we study in these spaces rather weak compactness and local weak compactness than compactness and local compactness.

3.12 Proposition. *Let (X, FJ_X) be a fuzzy preconvergence space and (X, q_{FJ_X}) its underlying fuzzy generalized convergence space. Then the following are equivalent:*

1. *(X, FJ_X) is locally weakly compact.*

2. *(X, q_{FJ_X}) is locally compact.*

Proof. (1)⇒(2). See 3.7.
(2)⇒(1). Let $\mathcal{H} \in FJ_X$, i.e. there is some $(\mathcal{F}, x) \in q_{FJ_X}$ such that $\dot{x} \times \mathcal{F} \subset \mathcal{H}$. By assumption, there is some non-empty weakly compact K in (X, FJ_X) with $\chi_K \in$ base $\mathcal{F}$. Since $\chi_{\{x\}} \in$ base $\dot{x}$, $\chi_{\{x\}} \circ p_1 \wedge \chi_K \circ p_2 = \chi_{\{x\} \times K} \in$ base $\dot{x} \times \mathcal{F}$ which implies $\chi_{\{x\} \times K} \in$ base $\mathcal{H}$ where $\{x\} \times K$ is weakly compact in (X, FJ_X) by 2.12. Thus, (X, FJ_X) is locally weakly compact.

3.13 Proposition. *The construct* **LC-FPUConv** *(resp.* **LWC-FPUConv***) of locally compact (resp. locally weakly compact) fuzzy preuniform convergence spaces (and uniformly continuous maps) is bicoreflective in* **FPUConv**.

Proof. Let $(X, FJ_X) \in |\mathbf{FPUConv}|$ and $FJ_X^* = \{\mathcal{F} \in FJ_X$: there is a non-empty (weakly) compact subset K of $(X, FJ_X) \times (X, FJ_X)$ with $\chi_K \in$ base $\mathcal{F}\}$. Then (X, FJ_X^*) is a fuzzy preuniform convergence space (note that if $X \neq \emptyset$ then $\dot{x} \times \dot{x} = (x, x)^{\cdot} \in FJ_X^*$ for each $x \in X$ since $\{x\} \times \{x\}$ is a (weakly) compact subset of $(X, FJ_X) \times (X, FJ_X)$ and $\chi_{\{x\}\times\{x\}} = \chi_{\{x\}} \circ p_1 \wedge \chi_{\{x\}} \circ p_2 \in$ base $\dot{x} \times \dot{x}$ because $\chi_{\{x\}} \in$ base $\dot{x}$) which is locally (weakly) compact: Let $\mathcal{F} \in FJ_X^*$. Then $\mathcal{F} \in FJ_X$ and there is a non-empty (weakly) compact subset K of $(X, FJ_X) \times (X, FJ_X)$ with $\chi_K \in$ base $\mathcal{F}$. But K is also a (weakly) compact subset of $(X, FJ_X^*) \times (X, FJ_X^*)$, namely $FJ_X^* \times FJ_X^* = (FJ_X \times FJ_X)^*$, where $\times$ stands for forming the product structure, and it follows from the last part of this proof that the inclusion map $i : (K, (FJ_X \times FJ_X)_K) \to (X \times X, (FJ_X \times FJ_X)^*)$ is fuzzy uniformly continuous and thus $i[K] = K$ is a (weakly) compact subset of $(X \times X, (FJ_X \times FJ_X)^*) = (X, FJ_X^*) \times (X, FJ_X^*)$. Consequently, (X, FJ_X^*) is locally (weakly) compact. Now $1_X : (X, FJ_X^*) \to (X, FJ_X)$ is the desired bicoreflection: Let $(Y, FJ_Y) \in |\mathbf{LC\text{-}FPUConv}|$ (resp. $|\mathbf{LWC\text{-}FPUConv}|$) and let $f : (Y, FJ_Y) \to (X, FJ_X)$ be a fuzzy uniformly continuous map. If $\mathcal{F} \in FJ_Y$ then there is a non-empty (weakly) compact subset K of $(Y, FJ_Y) \times (Y, FJ_Y)$ with $\chi_K \in$ base $\mathcal{F}$ and $(f \times f)(\mathcal{F}) \in FJ_X^*$ since $(f \times f)[\chi_K] = \chi_{f \times f[K]} \in$ base $(f \times f)(\mathcal{F})$ and $(f \times f)[K]$ is a (weakly) compact subset of $(X, FJ_X) \times (X, FJ_X)$ ($f \times f$ is fuzzy uniformly continuous and thus fuzzy continuous!), i.e. $f : (Y, FJ_Y) \to (X, FJ_X*)$ is fuzzy uniformly continuous.

3.14 Remark. If $(X, FJ_X) \in |\mathbf{FSUConv}|$ then (X, FJ_X^*) as constructed in the above proof belongs to $|\mathbf{FSUConv}|$: Let $\mathcal{F} \in FJ_X^*$. Hence, there is some non-empty (weakly) compact subset K of $(X, FJ_X) \times (X, FJ_X)$ with $\chi_K \in$ base $\mathcal{F}$. Furthermore, $\mathcal{F}^{-1} \in FJ_X$ and $(\chi_K)^{-1} = \chi_{K^{-1}} \in$ base $\mathcal{F}^{-1}$ (cf. [11; proposition 1.19], where $K^{-1} = \{(x, y) : (y, x) \in K\}$ is also (weakly) compact (note that $s : (X, FJ_X) \times (X, FJ_X) \to (X, FJ_X) \times (X, FJ_X)$ defined by $s(x, y) = (y, x)$ for each $(x, y) \in X \times X$ is fuzzy uniformly continuous and thus continuous). Consequently $\mathcal{F}^{-1} \in FJ_X^*$.

3.15 Corollary. *The construct* **LC-FSUConv** *(resp.* **LWC-FSUConv***) of locally compact (resp. locally weakly compact) fuzzy semiuniform convergence spaces (and fuzzy uniformly continuous maps) is bicoreflective in* **FSUConv**.

3.16 Corollary. *Let $(f_i : (X_i, FJ_{X_i}) \to (X, FJ_X))_{i \in I}$ be a final sink in* **FPUConv** *or in* **FSUConv** *, i.e. $FJ_X = \{\mathcal{F} \in F_L(X \times X)$: there is some $i \in I$ and some $\mathcal{F}_i \in FJ_{X_i}$ with $(f \times f)(\mathcal{F}_i) \subset \mathcal{F}\} \cup \{\dot{x} \times \dot{x} : x \in X\}$, such that all (X_i, FJ_{X_i}) are locally (weakly) compact. Then (X, FJ_X) is locally (weakly) compact.*

3.17 Corollary. *Let $(f_i : (X_i, q_i) \to (X, q))_{i \in I}$ be a final sink in* **FGConv**, *i.e. $q = \{(\mathcal{F}, x) \in F_L(X) \times X$: there is some $i \in I$ and some $(\mathcal{F}_i, x_i) \in q_i$ with $f_i(\mathcal{F}_i) \subset \mathcal{F}$ and $f_i(x_i) = x\} \cup \{(\dot{x}, x) : x \in X\}$, such that all (X_i, q_i) are locally compact. Then (X, q) is locally compact.*

Proof. By 3.9.2, **FPConv** is concretely isomorphic to **FGConv** and bicoreflective in **FPU-Conv**. Thus, final structures in **FPConv** ($\cong$ **FGConv**) are formed as in **FPUConv**. Furthermore,

1° For each $(X, q) \in |\mathbf{FGConv}|$ the corresponding fuzzy preconvergence structure FJ_q is given by $FJ_q = \{\mathcal{H} \in F_L(X \times X)$: there is some $(\mathcal{F}, x) \in q$ with $\dot{x} \times \mathcal{F} \subset \mathcal{H}\}$.

2° A sink $(f_i : (X_i, q_i) \to (X, q))_{i \in I}$ in **FGConv** is final iff the sink $(f_i : (X_i, FJ_{q_i}) \to (X, FJ_q))_{i \in I}$ is final in **FPConv**.

3° $(X, q) \in |\mathbf{FGConv}|$ is locally compact iff (X, FJ_q) is locally weakly compact (use 3.12 and note that $q_{FJ_q} = q$).

Thus, 3.17 follows from 3.16.

3.18 Remark. It follows from 3.17 that *quotients of locally compact fuzzy generalized convergence spaces are locally compact*. This is highly remarkable since such a result cannot be obtained for the construct **FTop** of fuzzy topological spaces, because it is false already in case $L = \{0,1\}$, e.g. $\mathbb{R}^{\mathbb{N}}$ is (as a compactly generated topological space) a quotient space of a locally compact topological space but it is not locally compact. By the way, a fuzzy topological space (X,t) is locally compact iff its corresponding fuzzy generalized convergence space (X,q_t) is locally compact where $(\mathcal{F},x) \in q_t$ iff $\mathcal{F} \supset \mathcal{U}_t(x)$ (here $\mathcal{U}_t(x)$ denotes the fuzzy neighborhood filter of x with respect to t, i.e. $\mathcal{U}_t(x)(f) = (int_t f)(x)$ with $int_t f = \bigvee_{g \in t, g \leq f} g$ for each $f \in L^X$). Since **FTop** can be embedded into **FGConv** as a (bireflective) subconstruct (cf. [12]), *quotients in* **FGConv** *of locally compact fuzzy topological spaces are locally compact by 3.17. In other words: Quotients in* **FGConv** *are better behaved than in* **FTop**. The same is true for quotients in **FPUConv**, $\mathbf{FKConv}_s$, or **FSUConv** (note that the construct $\mathbf{FTop}_s$ of symmetric fuzzy topological spaces can be embedded in $\mathbf{FKConv}_s$ and thus in **FSUConv**, where a fuzzy topological space (X,t) is called symmetric iff its corresponding fuzzy Kent convergence space (X,q_t) is symmetric).

3.19 Proposition. *Let $((X_i, FJ_{X_i}))_{i \in I}$ be a family of locally (weakly) compact fuzzy preuniform convergence spaces such that (X_i, FJ_{X_i}) is (weakly) compact for all but finitely many $i \in I$. Then the product space $\prod_{i \in I}(X_i, FJ_{X_i})$ is locally (weakly) compact.*

Proof. Let $X = \prod_{i \in I} X_i$, $(X, FJ_X) = \prod_{i \in I}(X_i, FJ_{X_i})$, and $\mathcal{F} \in FJ_X$. Hence, for each $i \in I$, $(p_i \times p_i)(\mathcal{F}) \in FJ_{X_i}$, where $p_i : X \to X_i$ denotes the i-th projection. By assumption, there are finitely many elements $i_1, \ldots, i_n$ of I such that (X_i, FJ_{X_i}) is (weakly) compact for each $i \in I \backslash \{i_1, \ldots, i_n\}$, and for each $i \in \{i_1, \ldots, i_n\}$, there is a (weakly) compact subset K_i of $(X_i, FJ_{X_i}) \times (X_i, FJ_{X_i})$ such that $\chi_{K_i} \in$ base $(p_i \times p_i)(\mathcal{F})$. Put $K_i = X_i \times X_i$ for each $i \in I \backslash \{i_1, \ldots, i_n\}$. Let $p'_i : \prod_{i \in I} X_i \times X_i \to X_i \times X_i$ be the i-th projection and $j : \prod_{i \in I} X_i \times X_i \to \prod_{i \in I} X_i \times \prod_{i \in I} X_i$ the canonical isomorphism. Then $\chi_{K_{i_1}} \circ p'_{i_1} \wedge \cdots \wedge \chi_{K_{i_n}} \circ p'_{i_n} = \chi_{\prod K_i}$ belongs to a base of $\prod_{i \in I} p_i \times p_i(\mathcal{F})$ and $j[\chi_{\prod K_i}] = \chi_{j[\prod K_i]} \in$ base $j(\prod_{i \in I} p_i \times p_i(\mathcal{F}))$. Furthermore, $\prod_{i \in I} K_i$ is (weakly) compact by 2.12. Thus, $j[\prod K_i]$ is (weakly) compact by 2.15 since j is fuzzy uniformly continuous and $\chi_{j[\prod K_i]} \in$ base $\mathcal{F}$ because $j(\prod p_i \times p_i(\mathcal{F})) = j(\prod p'_i \times p'_i(j^{-1}(\mathcal{F}))) \subset j(j^{-1}(\mathcal{F})) = \mathcal{F}$.

3.20 Proposition. *Let (X, FJ_X) be a locally (weakly) compact fuzzy preuniform convergence space and $A \subset X$ (pre) closed. Then the subspace (A, FJ_A) of (X, FJ_X) is locally (weakly) compact.*

Proof. Let $\mathcal{F} \in FJ_A$ and let $i_A : A \to X$ be the inclusion map. By assumption, there is a non-empty (weakly) compact subset K of $(X, FJ_X) \times (X, FJ_X)$ such that $\chi_K \in$ base $(i_A \times i_A)(\mathcal{F})$. Thus, since $\bar{1} \in$ base $\mathcal{F}$, $\chi_{A \times A} = i_A \times i_A[\bar{1}] \in$ base $i_A \times i_A(\mathcal{F})$. Consequently, $\chi_{K \cap (A \times A)} = \chi_K \wedge \chi_{A \times A} \in$ base $(i_A \times i_A)(\mathcal{F})$ and $\sup \chi_{K \cap (A \times A)} = \sup(\chi_K \wedge \chi_{A \times A}) = \sup \chi_K \wedge \sup \chi_{A \times A} = 1$ if $A \neq \emptyset$ which may be assumed without loss of generality. Hence, $K \cap (A \times A) \neq \emptyset$. Furthermore, $K \cap (A \times A)$ is (weakly) compact as a (pre) closed subspace of the subspace (K, FJ_K) of $(X, FJ_X) \times (X, FJ_X)$ and thus (weakly) compact in $(A, FJ_A) \times (A, FJ_A)$. Finally, the characteristic function of $K \cap (A \times A)$ w.r.t. $A \times A$,

denoted by $\chi^{A\times A}_{K\cap(A\times A)}$, belongs to
base $\mathcal{F}$ since $\mathcal{F}(\chi^{A\times A}_{K\cap(A\times A)}) = \mathcal{F}(\chi_K \circ (i_A \times i_A)) = (i_A \times i_A)(\mathcal{F})(\chi_K) = \sup \chi_K = 1 = \sup \chi^{A\times A}_{K\cap(A\times A)}$ because $K \cap (A \times A) \neq \emptyset$.

3.21 Theorem. *The construct* **LC-FPUConv** *(resp.* **LWC-FPUConv***) as well as the construct* **LC-FSUConv** *(resp.* **LWC-FSUConv***) is a cartesian closed topological construct.*

Proof. By 3.14 **LC-FPUConv** (resp. **LWC-FPUConv**) is bicoreflective in **FPUConv** and by 3.19 closed under formation of finite products. Since **FPUConv** is cartesian closed (cf. [11]), **LC-FPUConv** (resp. **LWC-FPUConv**) is cartesian closed too and, since **FPUConv** is topological, **LC-FPUConv** (resp. **LWC-FPUConv**) is also topological (cf. [9; 3.1.7]). Similarly, **LC-FSUConv** (resp. **LWC-FSUConv**) is bicoreflective in **FSUConv** (cf. 3.15) and closed under formation of finite products since products in **FSUConv** are formed as in **FPUConv**. Thus, **LC-FSUConv** (resp. **LWC-FSUConv**) is a cartesian closed topological construct because **FSUConv** is a cartesian closed topological construct (cf. [11]).

3.22 Corollary. *The construct* **LC-FGConv** *of locally compact fuzzy generalized convergence spaces is a cartesian closed topological construct.*

Proof. Since the concrete isomorphism between **FPConv** and **FGConv** (cf. [11]) leads to a concrete isomorphism between the constructs **LWC-FPConv** of locally weakly compact fuzzy preconvergence spaces and **LC-FGConv** it suffices to prove that **LWC-FPConv** is cartesian closed: Since **LWC-FPConv** is the intersection of the bicoreflective subconstructs **LWC-FPUConv** and **FGConv** of **FPUConv** it is bicoreflective in **FPUConv**. Furthermore, it is closed under formation of finite products in **FPUConv** (cf. 3.19 and note that **FPUConv** is closed under formation of finite pro-ducts by [11]). Consequently, **LWC-FPConv** is cartesian closed because **FPUConv** is cartesian closed.

3.23 Remark. 1. The natural function spaces in **LC-FPUConv** (or **LWC-FPUConv**) and **LC-FSUConv** (or **LWC-FSUConv**) are obtained by bicoreflective modification of the natural function spaces in **FPUConv** and **FSUConv** respectively (cf.[9; 3.1.7]), e.g. if $\mathbf{X} = (X, FJ_X)$ and $\mathbf{Y} = (Y, FJ_Y)$ are locally compact fuzzy preuniform convergence spaces, then the natural function space $\mathbf{Y}^{\mathbf{X}}$ in **LC-FPUConv** is the set $[\mathbf{X},\mathbf{Y}]$ of all fuzzy uniformly continuous maps from $\mathbf{X}$ to $\mathbf{Y}$ endowed with the **LC-FPUConv**-structure $(FJ_{X,Y})_{LC} = \{\Phi \in FJ_{X,Y}$: there is some non-empty compact subset K of
$([\mathbf{X},\mathbf{Y}], FJ_{X,Y}) \times ([\mathbf{X},\mathbf{Y}], FJ_{X,Y})$ with $\chi_K \in$ base $\Phi\}$, where $FJ_{X,Y} = \{\Phi \in F_L([\mathbf{X},\mathbf{Y}] \times [\mathbf{X},\mathbf{Y}]) : e_{X,Y} \times e_{X,Y}(\mathcal{H} \times \Phi) \in FJ_Y$ for each $\mathcal{H} \in FJ_X\}$ and $e_{X,Y} : X \times [\mathbf{X},\mathbf{Y}] \to Y$ denotes the evaluation map, i.e. $e_{X,Y}(x, f) = f(x)$ for each $(x, f) \in X \times [\mathbf{X},\mathbf{Y}]$, and $(X \times [\mathbf{X},\mathbf{Y}]) \times (X \times [\mathbf{X},\mathbf{Y}])$ is identified with $(X \times X) \times ([\mathbf{X},\mathbf{Y}] \times [\mathbf{X},\mathbf{Y}])$.

2. For each $(X, q) \in |\mathbf{FGConv}|$, $1_X : (X, q^*) \to (X, q)$ with $q^* = \{(\mathcal{F}, x) \in q$: there is some non-empty compact subset K of (X, q) with $\chi_K \in$ base $\mathcal{F}\}$ is the bicoreflection of (X, q) w.r.t. **LC-FGConv** since (X, q) and (X, q^*) have the same compact subsets (cf. also the proof of 3.13). Furthermore, **LC-FGConv** is closed under formation of finite products formed in **FGConv** (cf. the proof of 3.22 and note that **LC-FGConv** and **FGConv** are concretely isomorphic to **LWC-FPConv** and **FPConv** respectively and **FPConv** is closed under formation of finite products in **FPUConv** by

[11].). Thus, the natural function spaces in **LC-FGConv** are constructed by bicoreflective modification of the natural function spaces in **FGConv**, i.e. if $\mathbf{X} = (X, q)$ and $\mathbf{X}' = (X', q')$ are locally compact fuzzy generalized convergence spaces then the power-object $\mathbf{X}'^{\mathbf{X}}$ (= natural function space) in **LC-FGConv** is the set $[\mathbf{X}, \mathbf{X}']$ of all fuzzy continuous maps from $\mathbf{X}$ into $\mathbf{X}'$ endowed with the **LC-FGConv** structure $(\hat{q})^*$ defined by: $(\Phi, f) \in (\hat{q})^* \Leftrightarrow$ $(\Phi, f) \in \hat{q}$ and there is a non-empty compact subset K of $([\mathbf{X}, \mathbf{X}'], \hat{q})$ with $\chi_K \in$ base Φ
where $\hat{q}$ is the *structure of fuzzy continuous convergence* which means $(\Phi, f) \in \hat{q} \Leftrightarrow (e_{X,X'}(\mathcal{F} \times \Phi), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$.

3. Since **LC-FGConv** is cartesian closed, *the construct* $\mathbf{LCT_2}$**-FGConv** *of locally compact T_2 fuzzy generalized convergence spaces (and fuzzy continuous maps) is cartesian closed* too where a fuzzy generalized convergence space is T_2 iff the fuzzy filter convergence is unique (note: 1° The fuzzy natural function space structure in **FGConv** is the structure of fuzzy continuous convergence which is T_2 whenever its codomain is T_2. 2° This structure is coarser than the fuzzy natural function space structure in **LC-FGConv** and thus it is T_2 whenever the latter one is T_2, i.e. $\mathbf{LCT_2}$**-FGConv** is closed under formation of function spaces in **LC-FGConv**. 3° $\mathbf{LCT_2}$**-FGConv** is closed under formation of finite products in **LC-FGConv** since these ones are formed as in **FGConv**.). In particular, the fuzzy natural function spaces in $\mathbf{LCT_2}$**-FGConv** are formed as in **LC-FGConv**. *The cartesian closedness of* $\mathbf{LCT_2}$**-FGConv** *is highly remarkable* because the construct $\mathbf{LCT_2}$**-FTop** of locally compact T_2 fuzzy topological spaces is not even cartesian closed in case $L = \{0, 1\}$, i.e. in the non-fuzzy case, which is well-known (remember the fact that for locally compact Hausdorff spaces $\mathbf{X}$ and $\mathbf{Y}$ the compact open topology on the set $C(\mathbf{X}, \mathbf{Y})$ of all continuous maps form $\mathbf{X}$ into $\mathbf{Y}$ is proper [=splitting] and admissible [=conjoining] but not locally compact in general).

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