THE SET OF REGULAR ELEMENTS IN ORDERED SEMIGROUPS

Niovi Kehayopulu and Michael Tsingelis

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Abstract. For a semigroup or an ordered semigroup S, we denote by $Reg(S)$, $LReg(S)$, $Gr(S)$ the set of regular, left regular, completely regular elements of S respectively, and for a subsemigroup T of S, we denote by $reg(T)$ the set of elements of T which are regular in S. For a subset H of an ordered semigroup S, $(H]$ denotes the set of elements $t \in S$ such that $t \leq h$ for some $h \in H$. We characterize the ordered semigroups S in which the set of regular elements is a subset of the set of left regular elements as the ordered semigroups such that $reg(Sa) = Reg(Sa]$ for every $a \in S$. We prove that this type of ordered semigroups is actually the class of semigroups for which $reg(Se) = Reg(Se]$ for every $e \in S$ such that $e \leq e^2$. As a consequence, for a semigroup S (without order), condition $reg(Se) = Reg(Se)$ for every idempotent element of S is equivalent to the condition $reg(Sa) = Reg(Sa)$ for every $a \in S$. For an ordered semigroup S it remains an open problem if condition $Reg(S) \subseteq LReg(S)$ implies $Reg(S) = Gr(S)$.

1. Introduction and prerequisites.

An element a of a semigroup S is called *regular* if there exists an element $x \in S$ such that $a = axa$; *left* (resp. *right*) *regular* if there exists $x \in S$ such that $a = xa^2$ (resp. $a = a^2x$) [1]; *completely regular* if there exists $x \in S$ such that $a = a^2xa^2$ [8]. Keeping the notations given in [7], $Reg(S)$ denotes the set of regular elements of S, $LReg(S)$ (resp. $RReg(S)$) the set of left (resp. right) regular elements of S and, for every subsemigroup T of S, $reg(T)$ denotes the intersection $T \cap Reg(S)$, that is the set of elements of T which are regular in S. As usual, $E(S)$ is the set of idempotent elements of S. As far as the set of completely regular elements is concerned, although $CReg(S)$ suits better in our case (i.e. for ordered semigroups), we will keep the notation $Gr(S)$ already existed in the bibliography to be easier for the reader to follow the aim of the paper. According to the main result in [7], for a semigroup S, the following five conditions are equivalent and the right analogue of the same results also hold:

(i) $reg(Se) = Gr(Se) \ \ \forall\, e \in E(S)$
(ii) $reg(Se) = Reg(Se) \ \ \forall\, e \in E(S)$
(iii) $reg(Se) \subseteq LReg(Se) \ \ \forall\, e \in E(S)$
(iv) $Reg(S) \subseteq LReg(S)$
(v) $Reg(S) = Gr(S)$.

The following question arises: Are there similar characterizations in case of ordered semigroups? In positive case, we obtain generalizations of the corresponding results on semigroups as every semigroup endowed with the equality relation is an ordered semigroup. This is by no means gratuitous generalization: the structure of order on an algebraic system is a very natural additional structure to require, and the quasi-topological nature of the structure creates substantial changes in the methodology.

If $(S, ., \leq)$ is an ordered semigroup, then a nonempty subset A of S is called a *left* (resp. *right*) *ideal* of S if 1) $SA \subseteq A$ (resp. $AS \subseteq A$) and 2) If $a \in A$ and $b \in S$ such that $b \leq a$,

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then $b \in A$ [2]. For an ordered semigroup $(S, ., \leq)$ and a subset H of S, we denote by $(H]$ the subset of S defined by:

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

Recall that, if S is a semigroup and $a \in S$, then the left ideal of S generated by Sa is the set Sa; the right ideal of S generated by aS is the set aS. If S is an ordered semigroup and a an element of S, then the left ideal of S generated by $(Sa]$ is the set $(Sa]$, and the right ideal of S generated by $(aS]$ is the set $(aS]$.

An element a of an ordered semigroup $(S, ., \leq)$ is called *regular* if there exists $x \in S$ such that $a \leq axa$ [5]; *left* (resp. *right*) *regular* if there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$) [3]; *completely regular* if it is regular, left regular and right regular, in other words, if there exists $x \in S$ such that $a \leq a^2xa^2$ [4]. As in semigroups, for an ordered semigroup S we denote by $Reg(S)$, $LReg(S)$, $RReg(S)$, $Gr(S)$ the set of regular, left regular, right regular, and completely regular elements of S, respectively. For a subsemigroup T of S, we denote $reg(T) := T \cap reg(S)$ that is, the elements of T which are regular in S. We have $Reg(T) \subseteq reg(T)$ and the inclusion can be strict as every ordered semigroup can be embedded in a regular ordered (in particular, *poe*-) semigroup [6]. The present paper characterizes the ordered semigroups S in which the set of regular elements is a subset of the set of left regular elements. We show that this type of ordered semigroups satisfies the condition $reg(Se] = Reg(Se]$ for every $e \in S$ such that $e \leq e^2$ if and only if $reg(Sa] = Reg(Sa]$ for every $a \in S$. As a consequence, for a semigroup S (without order) we have the additional information that

$$reg(Se) = Reg(Se) \ \forall\, e \in E(S) \Longleftrightarrow reg(Sa) = Reg(Sa) \ \forall\, a \in S.$$

We add the condition $reg(Sa) = Reg(Sa) \ \ \forall\, a \in S$ among the five equivalent conditions of the main Theorem in [7]. So the characterization of semigroups with $reg(Se) = Reg(Se) \ \forall\, e \in E(S)$ (the so called "related by idempotents" [7]) is actually the characterization of semigroups with $reg(Sa) = Reg(Sa) \ \forall\, a \in S$. The right analogue of our results also hold. Part of the results in [7] can be also obtained as application of the results of the present paper. On the other hand, while for semigroups (without order) we have $Reg(S) \subseteq LReg(S)$ if and only if $Reg(S) = Gr(S)$, for an ordered semigroup S the situation is as follows: $Reg(S) = Gr(S)$ implies $Reg(S) \subseteq LReg(S)$, but it remains as an open problem if condition $Reg(S) \subseteq LReg(S)$ implies $Reg(S) = Gr(S)$.

2. Main results

Notation 1. For an ordered semigroup $(S, ., \leq)$, let $E(S) := \{x \in S \mid x \leq x^2\}$.

Definition 2. If $(S, ., \leq)$ is an ordered semigroup, we denote by $Reg(S)$ the set of regular elements of S, by $LReg(S)$ the set of left regular elements, and by $RReg(S)$ the set of right regular elements of S. That is,

$$Reg(S) := \{a \in S \mid a \leq axa \text{ for some } x \in S\}$$

$$LReg(S) := \{a \in S \mid a \leq xa^2 \text{ for some } x \in S\}$$

$$RReg(S) := \{a \in S \mid a \leq a^2x \text{ for some } x \in S\}.$$

We write $Reg(Sa]$ (resp. $reg(Sa]$) instead of $Reg((Sa])$ (resp. $reg((Sa])$) $(a \in S)$.

Definition 3. If $(S, ., \leq)$ is an ordered semigroup and T a subsemigroup of S, we define

$$reg(T) := T \cap Reg(S).$$

Lemma 4. *If $(S,.,\le)$ is an ordered semigroup and $a\in S$, then the set $(Sa]$ is a subsemigroup of S.*

Proof. First of all, $\emptyset\ne(Sa]\subseteq S$ (since $a^2\in Sa\subseteq(Sa]$). If now $b,c\in(Sa]$, then $b\le sa$ for some $s\in S$ and $c\le ta$ for some $t\in S$. Then $bc\le(sat)a\in Sa$, so $bc\in(Sa]$. □

Lemma 5. *Let $(S,.,\le)$ be an ordered semigroup and $a,x\in S$ such that $a\le axa$. Then $ax,xa\in E(S)$.*

Proof. Since $a\le axa$, we have $ax\le(axa)x=(ax)^2$ and $xa\le x(axa)=(xa)^2$, so ax and xa belong to $E(S)$. □

Lemma 6. *Let $(S,.,\le)$ be an ordered semigroup and $a,b\in S$. Then the set aSb is a subsemigroup of S.*

Proof. Clearly $\emptyset\ne aSb\subseteq S$. If now $c,d\in aSb$, then $c=asb$ for some $s\in S$ and $d=atb$ for some $t\in S$. Then $cd=a(sbat)b\in aSb$. □

Proposition 7. *Let $(S,.,\le)$ is an ordered semigroup and T a subsemigroup of S. Then $Reg(T)\subseteq reg(T)$.*

Proof. Since T is a subsemigroup of S, $Reg(T)\subseteq T$ and $Reg(T)\subseteq Reg(S)$, we have $Reg(T)\subseteq T\cap Reg(S):=reg(T)$. □

Theorem 8. *Let $(S,.,\le)$ be an ordered semigroup. The following are equivalent:*
1) $reg(Sa]=Reg(Sa]\quad\forall\, a\in S$
2) $reg(Se]=Reg(Se]\quad\forall\, e\in E(S)$
3) $reg(Se]\subseteq LReg(Se]\quad\forall\, e\in E(S)$
4) $Reg(S)\subseteq LReg(S)$.

Proof. 1) $\Longrightarrow$ 2). This is obvious since $E(S)\subseteq S$.
2) $\Longrightarrow$ 3). Let $e\in E(S)$ and $b\in reg(Se]$. Since $e\in E(S)$, by 2), we have $reg(Se]=Reg(Se]$. Then $b\in Reg(Se]$, so $b\in(Se]$ and $b\le bxb$ for some $x\in(Se]$. Since $b\le bxb\in Sxb$, we have $b\in(Sxb]$. Since $b\in S$ and $b\le bxb$, $x\in S$, we have $b\in Reg(S)$. Thus we have $b\in(Sxb]\cap Reg(S)$. Since $xb\in S$, by Lemma 4, $(Sxb]$ is a subsemigroup of S. Then, by Definition 3, $reg(Sxb]=(Sxb]\cap Reg(S)$. Hence we obtain $b\in reg(Sxb]$. Since $b,x\in S$ and $b\le bxb$, by Lemma 5, we have $xb\in E(S)$. Then, by 2), $reg(Sxb]=Reg(Sxb]$, so $b\in Reg(Sxb]$. Then $b\le byb$ for some $y\in(Sxb]$ and $y\le sxb$ for some $s\in S$. Then we have $b\le b(sxb)b=bsxb^2$. Since $x\in(Se]$, $x\le te$ for some $t\in S$. Then $b\le(bste)b^2$. Since $b\in(Se]$, $b\le(bste)b^2$ and $bste\in(Se]$, we get $b\in LReg(Se]$.

3) $\Longrightarrow$ 4). Let $b\in Reg(S)$. Then $b\in S$ and $b\le bxb$ for some $x\in S$. Since $b\le bxb\in Sxb$, we have $b\in(Sxb]$. Thus $b\in(Sxb]\cap Reg(S)$. By Lemma 6, Sxb is a subsemigroup of S. Then, by Definition 3, $reg(Sxb]=(Sxb]\cap Reg(S)$, so $b\in reg(Sxb]$. Since $b,x\in S$ and $b\le bxb$, by Lemma 5, we have $xb\in E(S)$. Then, by 3), $reg(Sxb]\subseteq LReg(Sxb]$. Thus $b\in LReg(Sxb]$, that is, $b\le zb^2$ for some $z\in(Sxb]$. Since $b,z\in S$ and $b\le zb^2$, we have $b\in LReg(S)$.

4) $\Longrightarrow$ 1). Let $a\in S$. Since $(Sa]$ is a subsemigroup of S (cf. Lemma 4), by Proposition 7, we have $Reg(Sa]\subseteq reg(Sa]$. Let now $b\in reg(Sa]$. Again since $(Sa]$ is a subsemigroup of S, by Definition 3, we have $reg(Sa]=(Sa]\cap Reg(S)$. Since $b\in reg(Sa]$, we have $b\in(Sa]$ and $b\in Reg(S)$. Since $b\in Reg(S)$, we have $b\le bxb$ for some $x\in S$. By 4), $Reg(S)\subseteq LReg(S)$, so $b\in LReg(S)$. Then $b\le yb^2$ for some $y\in S$, and $b\le b(xyb)b$. Since $b\in(Sa]$, we have $b\le wa$ for some $w\in S$. Thus we have $xyb\le xywa\in Sa$, so $xyb\in(Sa]$. Since $b\in(Sa]$, $b\le b(xyb)b$ and $xyb\in(Sa]$, we have $b\in Reg(Sa]$. Thus $reg(Sa]\subseteq Reg(Sa]$ and the proof is complete. □

Proposition 9. *If $(S,.,\leq)$ is an ordered semigroup then, for each $e\in E(S)$, we have*

$$reg(eSe)=Reg(eSe).$$

Proof. Let $e\in E(S)$. Since eSe is a subsemigroup of S (cf. Lemma 6), by Proposition 7, we have $Reg(eSe)\subseteq reg(eSe)$. Let now $a\in reg(eSe)$. Again since eSe is a subsemigroup of S, by Definition 3, we have $reg(eSe)=eSe\cap Reg(S)$, thus $a\in eSe$ and $a\in Reg(S)$. Since $a\in eSe$, we have $a=ese$ for some $s\in S$. Since $a\in Reg(S)$, we have $a\leq axa$ for some $x\in S$. Therefore we have

$$\begin{aligned} a\leq axa &= (ese)x(ese)\leq ese^2xe^2se \text{ (since } e\leq e^2) \\ &= (ese)(exe)(ese)=a(exe)a. \end{aligned}$$

Since eSe is a subsemigroup of S, $a\in eSe$, $a\leq a(exe)a$ and $exe\in eSe$, we obtain $a\in Reg(eSe)$. Thus $reg(eSe)\subseteq Reg(eSe)$ and the proof is complete. □

Notation 10. For an ordered semigroup $(S,.,\leq)$, we denote by $Gr(S)$ the set of completely regular elements of S. That is,

$$Gr(S):=\{a\in S\mid a\leq a^2xa^2 \text{ for some } x\in S\}.$$

We write $Gr(Sa]$ instead of $Gr((Sa])$ $(a\in S)$.

Remark 11. For every subsemigroup T of S, we have $Gr(T)\subseteq T$ and $Gr(T)\subseteq Gr(S)$, so $Gr(T)\subseteq T\cap Gr(S)$.

Proposition 12. *Let $(S,.,\leq)$ be an ordered semigroup and $a\in S$. Then*

$$Gr(Sa]=(Sa]\cap Gr(S).$$

Proof. Since $(Sa]$ is a subsemigroup of S (cf. Lemma 4), the set $Gr(Sa]$ is well defined and, by Remark 11, $Gr(Sa]\subseteq(Sa]\cap Gr(S)$. Let now $b\in(Sa]\cap Gr(S)$. Since $b\in(Sa]$, $b\leq ta$ for some $t\in S$. Since $b\in Gr(S)$, $b\leq b^2sb^2$ for some $s\in S$. Then we have

$$b\leq b^2sb^2\leq b^2s(b^2sb^2)b=b^2sb^2sbb^2\leq b^2(sb^2sta)b^2.$$

Since $(Sa]$ is a semigroup, $b\in(Sa]$, $b\leq b^2(sb^2sta)b^2$, and $sb^2sta\in Sa\subseteq(Sa]$, we have $b\in Gr(Sa]$. Thus $(Sa]\cap Gr(S)\subseteq Gr(Sa]$, and so $(Sa]\cap Gr(S)=Gr(Sa]$. □

Proposition 13. *Let $(S,.,\leq)$ be an ordered semigroup such that $Reg(S)=Gr(S)$. Then, for every $a\in S$, we have $reg(Sa]=Gr(Sa]$.*

Proof. Let $a\in S$. Since $(Sa]$ is a semigroup, the sets $reg(Sa]$ and $Gr(Sa]$ are well defined and by hypothesis and Proposition 12, we obtain

$$reg(Sa]:=(Sa]\cap Reg(S)=(Sa]\cap Gr(S)=Gr(Sa]. \quad \square$$

Proposition 14. *If $(S,.,\leq)$ is an ordered semigroup and T a subsemigroup of S, then*

$$Gr(T)\subseteq Reg(T)\subseteq reg(T).$$

Proof. Let $a\in Gr(T)$. Then $a\in T$ and $a\leq a^2xa^2$ for some $x\in T$. Then $a\leq a(axa)a$. Since $a,x\in T$ and T is a subsemigroup of S, we have $axa\in T$. Since $a\in T$, $a\leq a(axa)a$ and $axa\in T$, we have $a\in Reg(T)$. That is, $Gr(T)\subseteq Reg(T)$. Moreover, by Proposition 7, $Reg(T)\subseteq reg(T)$. □

Proposition 15. *Let* $(S, ., \leq)$ *be an ordered semigroup such that* $reg(Sa] = Gr(Sa]$ *for every* $a \in S$. *Then, for every* $a \in S$, *we have* $reg(Sa] = Reg(Sa]$.

Proof. Let $a \in S$. Since $(Sa]$ is a subsemigroup of S the sets $reg(Sa]$ and $Reg(Sa]$ are well defined and, by hypothesis and Proposition 14, we get

$$Gr(Sa] \subseteq Reg(Sa] \subseteq reg(Sa] = Gr(Sa],$$

so $reg(Sa] = Reg(Sa]$. □

Remark 16. By Proposition 14, we have $Reg(S) = Gr(S)$ if and only if $Reg(S) \subseteq Gr(S)$.

Remark 17. The right analogue of the results of this paper also hold. As far as the Theorem 8 is concerned, its right analogue reads as follows: For an ordered semigroup S, the following are equivalent: 1) $reg(aS] = Reg(aS]\ \forall a \in S$. 2) $reg(eS] = Reg(eS]\ \forall e \in E(S)$. 3) $reg(eS] \subseteq RReg(eS]\ \forall e \in E(S)$. 4) $Reg(S) \subseteq RReg(S)$.

Problem 18. According to Propositions 13 and 15,

$$\begin{aligned} Reg(S) = Gr(S) \quad &\Longrightarrow \quad reg(Sa] = Gr(Sa]\ \ \forall a \in S \\ &\Longrightarrow \quad reg(Sa] = Reg(Sa]\ \ \forall a \in S. \end{aligned}$$

Let us add two more conditions the condition 5) and 6) below in conditions 1)–4) of Theorem 8.

5) $Reg(S) = Gr(S)$
6) $reg(Sa] = Gr(Sa]\ \ \forall a \in S$.

We have already proved that

$$5) \Longrightarrow 6) \Longrightarrow 1) \Longleftrightarrow 2) \Longleftrightarrow 3) \Longleftrightarrow 4).$$

It is interesting to know under what conditions the implication 4) $\Longrightarrow$ 5) holds. We set it as an open problem.

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University of Athens
Department of Mathematics
157 84 Panepistimiopolis
Athens, Greece