WEIGHTED ESTIMATES FOR SINGULAR INTEGRAL OPERATORS ON CMO SPACES

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Dedicated to Professor Enji Sato on his sixtieth birthday

Abstract. We prove the boundedness of singular integral operators on weighted CMO spaces. We also show that our result is optimal.

1 Introduction Since Beurling [1] introduced the Beurling algebras and Herz [5] generalized these spaces, many studies have been done for these spaces (see, for example, [8] and [12]). Chen and Lau [2] and García-Cuerva [3] introduced the CMO spaces, which are the dual spaces of the Beurling-type Hardy spaces, and the authors [7] proved the boundedness of singular integral operators on CMO spaces. Weighted Herz spaces are also considered in [6], [9] [10] and [11].

In this paper we consider the boundedness of singular integral operators on weighted CMO spaces. We also show that our reslut is optimal by giving a couterexample.

2 Definitions and Theorems The following notation is used: For a set $E \subset \mathbb{R}^n$ we denote the Lebesgue measure of E by $|E|$. We denote the characteristic function of E by χ_E. We indicate a ball of radius R centered at the origin by $B(0,R) = \{x \in \mathbb{R}^n : |x| \le R\}$. For a locally integrable nonnegative function, i.e. weight function w, we write $w(E) = \int_E w(x)dx$.

First we define nonhomogeneous CMO spaces [3].

Definition 1. For $1 \le p < \infty$ and $n/(n+1) < q \le 1$,

$$CMO^p_q(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{CMO^p_q} < \infty \right\},$$

where

$$\|f\|_{CMO^p_q} = \sup_{R\ge 1} \inf_c |B(0,R)|^{1-1/p-1/q} \left\{ \int_{B(0,R)} |f(x)-c|^p dx \right\}^{1/p}.$$

We denote $CMO^p = CMO^p_1$.

The authours [7] defined weak CMO spaces.

Definition 2. For $n/(n+1) < q \le 1$,

$$WCMO^1_q(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{WCMO^1_q} < \infty \right\},$$

where

$$\|f\|_{WCMO^1_q} = \sup_{R\ge 1} |B(0,R)|^{-1/q} \inf_c \sup_{\lambda>0} \lambda |\{ x \in B(0,R) : |f(x)-c| > \lambda \}|.$$

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Next we define weighted CMO spaces and weighted weak CMO spaces. .

Definition 3. For $1 \le p < \infty, n/(n+1) < q \le 1$ and a weight function w,

$$CMO^p_q(w)(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{CMO^p_q(w)} < \infty \right\},$$

where

$$\|f\|_{CMO^p_q(w)} = \sup_{R \ge 1} \inf_c |B(0,R)|^{1-1/p-1/q} \left\{ \int_{B(0,R)} |f(x) - c|^p w(x) dx \right\}^{1/p}.$$

Definition 4. For $n/(n+1) < q \le 1$ and a weight function w,

$$WCMO^1_q(w)(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{WCMO^1_q(w)} < \infty \right\},$$

where

$$\|f\|_{WCMO^1_q(w)} = \sup_{R \ge 1} |B(0,R)|^{-1/q} \inf_c \sup_{\lambda > 0} \lambda w(\{x \in B(0,R) : |f(x) - c| > \lambda\}).$$

Next we define some classes of weight functions.

Definition 5. Let $1 < p < \infty$. For a weight function w, we say that $w \in A_p$ if there exists a constant C such that

$$\Big(\frac{1}{|Q|} \int_Q w(x) dx \Big) \Big(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \Big)^{p-1} \le C$$

for all balls $Q \subset \mathbb{R}^n$.

Definition 6. For a weight function w, we say that $w \in A_1$ if there exists a constant C such that

$$\frac{1}{|Q|} \int_Q w(x) dx \le C \operatorname*{essinf}_{x \in Q} w(x)$$

for all balls $Q \subset \mathbb{R}^n$.

Definition 7 (centered reverse doubling). Let w be a weight function and $\delta > 0$. We say $w \in RD(\delta)$ if there exists a constant C such that for any $R > 0$ and $j > 0$,

$$\frac{w(B(0, 2^j R)}{w(B(0,R))} \ge C 2^{\delta j}. \tag{1}$$

The following lemmas are well-known (see, for example, [4] and [14]).

Lemma 1. *If $w \in A_p$, then $w \in RD(\delta)$ for some $\delta > 0$.*

Lemma 2. *Let $1 < p < \infty, n/(n+1) < q \le 1$ and $w_\alpha(x) = |x|^\alpha$ $(\alpha \in \mathbb{R})$. Then $w_\alpha \in A_p$ if and only if $-n < \alpha < n(p-1)$. Furthermore $w_\alpha \in RD(n+\alpha)$.*

Next we define a standard singular integral operator T and its modified singular integral operator $\widetilde{T}$.

Definition 8. We say that T is a standard singular integral operator, if there exists a function K which satisfies the following conditions:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

exists almost everywhere for $f \in L^2(\mathbb{R}^n)$,

$$|K(x)| \le \frac{C_K}{|x|^n} \quad \text{and} \quad |\nabla K(x)| \le \frac{C_K}{|x|^{n+1}} \quad \text{where} \quad x \neq 0,$$

$$\int_{\varepsilon<|x|<N} K(x)dx = 0 \quad \text{for all} \quad 0 < \varepsilon < N < \infty.$$

Remark . We can weaken the conditions in Definition 8, but we assume these conditions for the simplicity.

Definition 9. For a standard singular integral operator T, we define the modified singular integral operator $\widetilde{T}$ as follows.

$$\widetilde{T}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \left\{K(x-y) - K(-y)\chi_{\{|y|\ge 1\}}\right\} f(y)dy.$$

Note that If $f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $\widetilde{T}f(x) = Tf(x) + C_f$ a.e., where C_f is a constant.

The authors [7] proved the following.

Theorem A . *Let $1 < p < \infty$ and $n/(n+1) < q \le 1$. Then $\widetilde{T}$ is bounded on $CMO^p_q(\mathbb{R}^n)$;*

$$\|\widetilde{T}f\|_{CMO^p_q} \le C\|f\|_{CMO^p_q}.$$

Theorem B . *Let $n/(n+1) < q \le 1$. Then $\widetilde{T}$ is bounded from $CMO^1_q(\mathbb{R}^n)$ to $WCMO^1_q(\mathbb{R}^n)$;*

$$\|\widetilde{T}f\|_{WCMO^1_q} \le C\|f\|_{CMO^1_q}.$$

Our resutls are the following.

Theorem 1. *Let $1 < p < \infty$ and $n/(n+1) < q \le 1$. If $w \in A_p$ and $w \in RD(\delta)$ where $\delta/p > n(1/p + 1/q - 1) - 1$, then $\widetilde{T}$ is bounded on $CMO^p_q(w)(\mathbb{R}^n)$.*

Theorem 2. *Let $n/(n+1) < q \le 1$. If $w \in A_1$ and $w \in RD(\delta)$ where $\delta > n/q - 1$, then $\widetilde{T}$ is bounded from $CMO^1_q(w)(\mathbb{R}^n)$ to $WCMO^1_q(w)(\mathbb{R}^n)$.*

Corollary . *Let $w_\alpha(x) = |x|^\alpha$. If $\max(-n, p(n/q - n - 1)) < \alpha < n(p-1)$, then $\widetilde{T}$ is bounded on $CMO^p_q(w_\alpha)(\mathbb{R}^n)$.*

Proof. Note that $w_\alpha \in RD(n + \alpha)$. □

Remark . Compared with A_p condition (see Lemma 2), the condition $p(n/q - n - 1) < \alpha$ is strong, but we shall show our reslut is optimal by giving a counterexample in Section 4. Note that if $n = 1$ and $q = 1$, then the condition above coincides with the condition $w_\alpha \in A_p$.

3 Proofs First we shall show some lemmas. The following two lemmas are well-known (see, for example, [4] and [14]).

Lemma 3. *Let $1 < p < \infty$. If $w \in A_p$, then standard singular integral operators T are bounded on weighted L^p space $L^p(w)(\mathbb{R}^n)$.*

Lemma 4. *If $w \in A_1$, then standard singular integral operators T are bounded from $L^1(w)(\mathbb{R}^n)$ to weighted weak L^1 space $WL^1(w)(\mathbb{R}^n)$.*

The next lemma is easily obtained from Hörder's inequality and the definition of A_p weight. We denote the mean value of f on a ball $Q \subset \mathbb{R}^n$ by $f_Q = |Q|^{-1} \int_Q f(x)dx$.

Lemma 5. *Let $1 < p < \infty$ and $n/(n+1) < q \leq 1$. If $w \in A_p$ and $f \in CMO^p_q(w)(\mathbb{R}^n)$, then for any $R > 1$,*

$$\left\{ \int_{B(0,R)} |f(x) - f_{B(0,R)}|^p w(x)dx \right\}^{1/p} \leq C\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q-1)}.$$

Throughout this paper, C is a positive constant which is independent of essential parameters and not necessarily same at each occurrence.

By using this lemma we have the following.

Lemma 6. *Let $1 < p < \infty$ and $n/(n+1) < q \leq 1$. If $w \in A_p$ and $f \in CMO^p_q(w)(\mathbb{R}^n)$, then for any $R > 1$,*

$$\int_{B(0,R)} |f(x) - f_{B(0,R)}|dx \leq C\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q)} w(B(0,R))^{-1/p}. \tag{2}$$

By using Lemma 6, we obtain the following.

Lemma 7. *Let $1 < p < \infty$ and $n/(n+1) < q \leq 1$. If $w \in A_p$ and $f \in CMO^p_q(w)(\mathbb{R}^n)$, then for any $R > 1$,*

$$|f_{B(0,R)} - f_{B(0,2R)}| \leq C\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q-1)} w(B(0,R))^{-1/p}. \tag{3}$$

By using Lemmas 6 and 7, we have the following.

Lemma 8. *Let $1 < p < \infty, n/(n+1) < q \leq 1$ and w be a weight function. If $w \in A_p, w \in RD(\delta)$ and $f \in CMO^p_q(w)(\mathbb{R}^n)$, then for any $R > 1$ and $j \in \mathbb{N}$,*

$$\begin{aligned} &\int_{B(0,2^jR)} |f(x) - f_{B(0,R)}|dx \\ &\leq \begin{cases} C2^{j(n/p+n/q-\delta/p)}\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q)} w(B(0,R))^{-1/p} \\ \qquad\qquad\qquad \text{if} \quad \delta/p < n(1/p+1/q-1), \\ Cj2^{jn}\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q)} w(B(0,R))^{-1/p} \\ \qquad\qquad\qquad \text{if} \quad \delta/p \geq n(1/p+1/q-1). \end{cases} \end{aligned}$$

Proof. Let $B = B(0,R)$ and $2^jB = B(0,2^jR)$. By (2) and (3),

$$\begin{aligned}
&\int_{B(0,2^jR)} |f(x)-f_B|dx \le \int_{B(0,2^jR)} |f(x)-f_{2^jB}|dx + C(2^jR)^n|f_B - f_{2^jB}| \\
\le& C\|f\|_{CMO^p_q(w)}(2^jR)^{n(1/p+1/q)}w(2^jB)^{-1/p} + C(2^jR)^n\sum_{k=0}^{j-1}|f_{2^kB}(x) - f_{2^{k+1}B}| \\
\le& C\|f\|_{CMO^p_q(w)}(2^jR)^{n(1/p+1/q)}w(2^jB)^{-1/p} \\
&+ C\|f\|_{CMO^p_q(w)}(2^jR)^n\sum_{k=0}^{j-1}(2^kR)^{n(1/p+1/q-1)}w(2^kB)^{-1/p}.
\end{aligned}$$

By (1) we have

$$\begin{aligned}
(4)\qquad &(2^jR)^{n(1/p+1/q)}w(2^jB)^{-1/p} \\
&\le \begin{cases} C2^{j(n/p+n/q-\delta/p)}R^{n(1/p+1/q)}w(B)^{-1/p} & \text{if} \quad \delta/p < n(1/p+1/q-1), \\ C2^{jn}R^{n(1/p+1/q)}w(B)^{-1/p} & \text{if} \quad \delta/p \ge n(1/p+1/q-1), \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(5)\qquad &(2^jR)^n\sum_{k=0}^{j-1}(2^kR)^{n(1/p+1/q-1)}w(2^kB)^{-1/p} \\
&\le CR^{n(1/p+1/q)}w(B)^{-1/p}2^{jn}\sum_{k=0}^{j-1}2^{k(n/p+n/q-n-\delta/p)} \\
&\le \begin{cases} C2^{j(n/p+n/q-\delta/p)}R^{n(1/p+1/q)}w(B)^{-1/p} & \text{if} \quad \delta/p < n(1/p+1/q-1), \\ Cj2^{jn}R^{n(1/p+1/q)}w(B)^{-1/p} & \text{if} \quad \delta/p \ge n(1/p+1/q-1). \end{cases}
\end{aligned}$$

□

Now we shall prove Theorem 1.

Proof of Theorem 1. We use the same argument as in [7]. Let $R \ge 1$ and fix a ball $B(0,R)$. We denote $2B = B(0,2R)$. Since

$$\text{p.v.}\int_{\mathbb{R}^n}\left\{K(x-y) - K(-y)\chi_{\{|y|\ge 1\}}\right\}dy = 0,$$

it follows that for $x \in B(0,R)$,

$$\begin{aligned}
\widetilde{T}f(x) =& T((f-f_{2B})\chi_{2B})(x) - \int_{\mathbb{R}^n}K(-y)\chi_{\{|y|\ge 1\}}\left(f(y)-f_{2B}\right)\chi_{2B}(y)dy \\
&+ \int_{|y|\ge 2R}\{K(x-y)-K(-y)\}\left(f(y)-f_{2B}\right)dy.
\end{aligned}$$

Let

$$C_R = -\int_{\mathbb{R}^n}K(-y)\chi_{\{|y|\ge 1\}}\left(f(y)-f_{2B}\right)\chi_{2B}(y)dy,$$

and we write

$$\widetilde{T}f(x) - C_R = T((f - f_{2B})\chi_{2B})(x) + \int_{|y|\geq 2R} \{K(x-y) - K(-y)\}(f(y) - f_{2B})\,dy =: I + II.$$

First we estimate I. By Lemmas 3 and 5 we have

$$\begin{aligned}&\Big\{\int_{B(0,R)} |T((f-f_{2B})\chi_{2B})(x)|^p w(x)dx\Big\}^{1/p}\\ \leq &C\Big\{\int |(f(x) - f_{2B})\chi_{2B}|^p w(x)dx\Big\}^{1/p} \leq C\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q-1)}.\end{aligned}$$

Next we estimate II. By the regularity condition about K and Lemma 8, we have for $x \in B(0,R)$,

$$\begin{aligned}&\left|\int_{|y|\geq 2R}\{K(x-y) - K(-y)\}(f(y) - f_{2B})dy\right|\\ &\leq \sum_{j=1}^{\infty}\int_{B(0,2^{j+1}R)\setminus B(0,2^jR)} |K(x-y) - K(-y)||f(y) - f_{2B}|dy\\ &\leq C\sum_{j=1}^{\infty}\int_{B(0,2^{j+1}R)\setminus B(0,2^jR)} \frac{|x|}{|y|^{n+1}}|f(y) - f_{2B}|dy\\ &\leq C\sum_{j=1}^{\infty}\frac{R}{(2^jR)^{n+1}}\int_{B(0,2^{j+1}R)} |f(y) - f_{2B}|\,dy\\ &\leq C\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q-1)} w(B)^{-1/p}\sum_{j=1}^{\infty} 2^{-j(n+1)}\max(j2^j, 2^{j(n/p+n/q-\delta/p)})\\ &\leq C\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q-1)} w(B)^{-1/p}.\end{aligned}$$

Therefore

$$\Big\{\int_{B(0,R)} |II|^p w(x)dx\Big\}^{1/p} \leq C\|f\|_{CMO^p_q(w)} R^{n(1/p+1/q-1)}.$$

□

Theorem 2 is proved similarly. We use Lemm 4 and the following lemma instead of Lemma 5 (see also [7]), therefore we omit the proof.

Lemma 9. *Let $n/(n+1) < q \leq 1$. If $w \in A_1$ and $f \in CMO^1_q(w)(\mathbb{R}^n)$, then for any $R > 1$,*

$$\int_{B(0,R)} |f(x) - f_{B(0,R)}|dx \leq C\|f\|_{CMO^1_q(w)} |B(0,R)|^{1+1/q} w(B(0,R))^{-1}.$$

Proof. Note that

$$\operatorname*{esssup}_{x\in B(0,R)} w^{-1}(x) \leq C\frac{|B(0,R)|}{w(B(0,R))}.$$

□

4 Counterexample We show that the condition $p(n/q-n-1)<\alpha$ in Corollary is optimal by giving a counterexample.

Let $1<p<\infty, n/(n+1)<q\le 1$ and $w_\alpha(x)=|x|^\alpha$ where $\alpha=p(n/q-n-1)$. We assume $\alpha>-n$. We shall give a standard singular integral operator T satisfying the conditions in Definition 8, but $\widetilde{T}f$ is not well-defined for some $f\in CMO^p_q(w_\alpha)(\mathbb{R}^n)$.

Let $\mathbb{R}^n_-=\{x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n : x_k<0 \quad \text{for all } k\}$. We take a function $\Omega(x')$ defined on the unit sphere S^{n-1} which satisfies the following conditions:

$$\Omega(x')=1 \quad \text{if} \quad x'\in S^{n-1}\cap\mathbb{R}^n_-,$$
$$\Omega\in C^\infty(S^{n-1}) \quad \text{and} \quad \int_{S^{n-1}}\Omega(x')d\sigma(x')=0,$$

where $d\sigma$ is the induced Euclidean measure on S^{n-1}. We define $\Omega(x)=\Omega(x/|x|)$ when $x\neq 0$. Let

$$Tf(x)=\text{p.v.}\int_{\mathbb{R}^n}K(x-y)f(y)dy \quad \text{where} \quad K(x)=\frac{\Omega(x)}{|x|^n}.$$

Then T satisfies the conditions in Definition 8 (see, for example, [4] and [14]).

Let

$$Q_j=\{x\in\mathbb{R}^n : 2^j<x_k<2^{j+1} \quad \text{for all } k\} \quad (j\in\mathbb{N}\cup\{0\})$$

and

$$f(x)=\sum_{j=10}^{\infty}2^j\chi_{Q_j}(x).$$

We show the following.

Counterexample .

$$f\in CMO^p_q(w_\alpha)(\mathbb{R}^n). \tag{6}$$
$$\widetilde{T}f(x)=\infty \quad \textit{where} \quad x\in Q_0. \tag{7}$$

Proof. The proof of (6) is straightforward. Take $R\ge 2^{10}$ and pick a j_0 such that $2^{j_0}\le R<2^{j_0+1}$. Since we assume $n+\alpha>0$, we have

$$\Big\{\int_{B(0,R)}|f(x)|^p w_\alpha(x)dx\Big\}^{1/p}\le C\Big\{\sum_{j=10}^{j_0}2^{jp}2^{j(n+\alpha)}\Big\}^{1/p}\le CR^{n(1/p+1/q-1)}.$$

Next we prove (7). Let $j\ge 10$. If $x\in Q_0$ and $y\in Q_j$, then $x_k-y_k<0$ and $-y_k<0$ for all k. Therefore

$$\begin{aligned}K(x-y)-K(-y)=\frac{1}{|x-y|^n}-\frac{1}{|y|^n}&=\frac{|y|^{2n}-|x-y|^{2n}}{|x-y|^n\ |y|^n\ (|y|^n+|x-y|^n)}\\ &\ge\frac{(|y|^2-|x-y|^2)|y|^{2(n-1)}}{|x-y|^n\ |y|^n\ (|y|^n+|x-y|^n)}.\end{aligned}$$

Since

$$|y|^2-|x-y|^2=\sum_{k=1}^n x_k(2y_k-x_k)\ge\sum_{k=1}^n(2y_k-2)\ge\sum_{k=1}^n y_k\ge|y|,$$

and $|x-y| \le 2|y|$, we obtain

$$K(x-y) - K(-y) \ge \frac{C}{|y|^{n+1}} \quad \text{for some positive constant } C.$$

Therefore we have

$$\widetilde{T} f(x) \ge C \sum_{j=10}^{\infty} 2^j \int_{Q_j} \frac{dy}{|y|^{n+1}} \ge C \sum_{j=10}^{\infty} 2^j 2^{-j} = \infty.$$

□

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References

[1] A. Beurling, *Construction and analysis of some convolution algebra*, Ann. Inst. Fourier, **14** (1964), 1–32.

[2] Y. Chen and K. Lau, *Some new classes of Hardy spaces*, J. Func. Anal., **84** (1989), 255–278.

[3] J. García-Cuerva, *Hardy spaces and Beurling algebras*, J. London Math. Soc. (2), **39** (1989), 499–513.

[4] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[5] C. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech., **18** (1968), 283–324.

[6] Y. Komori, *Weak type estimates for Calderon-Zygmund operators on Herz spaces at critical indexes*, Math. Nachr., **259** (2003), 42–50.

[7] Y. Komori and K. Matsuoka, *Some weak-type estimates for singular integral operators on CMO spaces*, to appear in Hokkaido Math. J. **39** (2010).

[8] S. Z. Li and D. C. Yang, *Boundedness of some sublinear operators on Herz spaces*, Illinois J. Math., **40** (1996), 484–501.

[9] S. Z. Lu and F. Soria, *On the Herz spaces with power weights*, *Fourier Analysis and Partial Differential Equations*, CRC Press, (1995), 227–236.

[10] S. Z. Lu and D. C. Yang, *The decomposition of weighted Herz space on R^n and its applications*, Sci. in China **38** (1995), 147–158.

[11] S. Z. Lu and D. C. Yang, *Hardy-Littlewood-Sobolev theorems of fractional integration on Herz-type spaces and its applications*, Canad. J. Math. **48** (1996), 363–380.

[12] S. Z. Lu and D. C. Yang, *The continuity of commutators on Herz-type spaces*, Michigan Math. J. **44** (1997), 255–281.

[13] S. Z. Lu and D. C Yang, *The continuity of commutators on Herz-type spaces*, Michigan Math. J. **44** (1997), 255–281.

[14] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, 1986.

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