ON DERIVATIONS OF INCLINE ALGEBRAS

N. O. Alshehri*

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Abstract. In this paper we introduce the notion of derivation of incline algebras, and we prove some result on derivation of an incline and integral incline. Moreover, we show that if a derivation is nonzero on an integral incline K , then it is nonzero on any nonzero ideal of K.

1 Introduction

Z. Q. Cao, K. H. Kim, and F. W. Roush [5] introduced the notion of incline algebras in their book, Incline algebra and applications, and was studied by some authors (see [1, 2, 6, 7]). Inclines are a generalization of both Boolean and fuzzy algebras, and a special type of a semiring, and they give a way to combine algebras with ordered structures to express the degree of intensity of binary relations.

The notion of derivation, introduced from the analytic theory, is helpful to the research of structure and property in algebraic system. Several authors (see [3, 4, 9, 13]). studied derivations in rings and near-rings. Jun and Xin [8] applied the notion of derivation in ring and near-ring theory to BCI-algebras. In this paper, we introduced the concept of derivations for an incline and investigate some of its properties. We show that if d is a derivation of an integral incline K , and $a \in K$ such that $a * dx = 0$ or $dx * a = 0$ for all $x \in X$, then either $a = 0$ or d is zero, and we prove that if $d^2 = 0$,then d is zero. Also we show that if $d_1, d_2$are two derivations of an incline, and $d_1 d_2 = 0$, then $d_1 d_2$ is a derivation of K. Finally we prove that if M is a nonzero ideal of an integral incline K, and d is a nonzero derivation of K. Then d is nonzero on M.

2 Preliminaries

An incline (algebra) is a set K with two binary operations denoted by $+$ and $*$ satisfying the following axioms for all $x, y, z \in K$:

$(K1)$ $x + y = y + x$,
$(K2)$ $x + (y + z) = (x + y) + z$,
$(K3)$ $x * (y * z) = (x * y) * z$,
$(K4)$ $x * (y * z) = (x * y) + (x * z)$,
$(K5)$ $(y + z) * x = (y * x) + (z * x)$,
$(K6)$ $x + x = x$,
$(K7)$ $x + (x * y) = x$,
$(K8)$ $y + (x * y) = y$.

For convenience, we pronounce $+$ (resp.$*$) as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in K$.

Note that $x \leq y \iff x + y = y$ for all $x, y \in K$. It is easy to see that $\leq$ is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$. It follows that

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(1) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in K$;
(2) $y \leq z$ implies $x * y \leq x$ and $y * x \leq z * x$ for any $x, y, z \in K$;
(3) if $x \leq y$, $a \leq b$, then $x + a \leq y + b, x * a \leq y * b$.

Asubincline of an incline K is a non-empty subset M of K which is closed under addition and multiplication.

Asubincline M is said to be an ideal of an incline K if $x \in M$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra K is a zero element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$, for any $x \in K$. An element 1 ($\neq$ zero element) in an incline algebra K is called a multiplicative identity if for any $x \in K, x * 1 = 1 * x = x$. A non-zero element a in an incline algebra K with zero element is said to be a left (resp. right) zero divisor if there exists a non-zero $b \in K$ such that $a * b = 0$ (resp. $b * a = 0$). A zero divisor is an element of K which is both a left zero divisor and a right zero divisor. An incline K with multiplicative identity 1 and zero element 0 is called an integral incline if it has no zero divisors. For more details, refer to ([10, 11, 12, 14]).

3 Derivations of incline algebras

Definition 3.1. Let K be an incline and $d : K \to K$ be a function. We call d a derivation of K , if it satisfies the following condition for all $x, y \in K$.

$$d\ (x * y) = (d\ x * y\ ) + (x * d\ y)$$

We often abbreviate $d\ (x)$ to $d\ x$.

Example 3.2. If $K = \{a, b, c, d\}$, and we define the sum " + " and product " $*$ " on K as:

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	b
d	d	b	b	d

Table 1

$*$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	a
d	a	d	a	d

Table 2

Then $(K, +, *)$ is an incline algebra. Define a map $d : K \to K$ by:

$$dx = \begin{cases} a & if x = a, bd \\ c & if x = c \end{cases}$$

Then we can see that d is a derivation of K .

Example 3.3. If $K = \{0, a, b, 1\}$, and we define the sum " + " and product " $*$ " on K as:

$+$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Table 1

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 2

Then $(K, +, *)$ is an incline algebra. Define a map $d : K \to K$ by:

$$dx = \begin{cases} a & if x = 1 \\ c & otherwise \end{cases}$$

Then we can see that d is a derivation of K .

Proposition 3.4. Let K be an incline and d be a derivation of K . Then the following hold for all $x, y \in K$:
(i) $d\ (x * y) \leq d\ x + d\ y$;
(ii) If $x \leq y$,then $d\ (x * y) \leq y$;
(iii) If K is a distributive lattice,then $d\ x \leq x$.

Proof .

(i)Lat $x, y \in K$,from (1)we have that $d\ x * y \leq d\ x$,$x*\ d\ y \leq d\ y$ .Then by (3) :

$$d\,(x * y) = (d\ x * y) + (x * d\ y) \leq dx + d\ y.$$

(ii)Let $x \leq y$,from(3)and(1)we get $x * d\ y \leq y\ *d\ y \leq y$.Also $d\ x * y \leq y$,thus

$$d\,(x * y) = (d\ x * y\ ) + (x * d\ y\ ) \leq y + y = y.$$

(iii) If K is a distributive lattice, then

$$dx = d\ \ (x * x) = (d\ x * x) + (x * d\ x)$$

and so,

$$dx + x = (d\ x * x) + ((x * d\ x) + x)\,.$$

By $(K\ 7)$we have that $d\ x + x = (d\ x * x) + x$,and $(K\ 8)$implies $d\ x + x = x$, thus,d $x \leq x$.

Proposition 3.5**.** Let K be an incline with a zero element and d be a derivation of K. Then $d\ 0 = 0$.

Proof.
Let $x \in K$, then

$$d\ 0 = d\ \ (x * 0) = (d\ x * 0) + (x * d\ 0) = x * d0.$$

Putting $x = 0$,we get $d\ 0 = 0$.

Proposition 3.6. Let K be an incline with multiplicative identity and d be a derivation of K. Then the following hold for all $x \in K$:

(i) $x* \ d\ 1 \leq d\ x$;
(ii) If $d\ 1 = 1$,then $x \leq d\ x$.
(iii) If K is a distributive lattice, then $d\ 1 = 1$ and only if d is an identiy derivation.

Proof.
(i) Let $x \in K, d\ x = d\ (x * 1) = d\ x + (x * d\ 1)$. Therefore $x * d\ 1 \leq d\ x$.
(ii) follows directly from (i).
(iii) We only need to prove the necessity. Assume $d\ 1 = 1$. Using(ii) ,we get that $x \leq d\ x$ for all $x \in K$. But Kisa distributive lattice, so by proposition 3.4(iii)we have $d\ x \leq x$.Thus
$d\ x = x$.

Proposition 3.7. Let d be a derivation of an integral incline K , and a be an element of K. Then

(i) If $a * d\ x = 0$ for all $x \in K$, then either $a = 0$ or d is zero.
(ii) If $d\ x * a = 0$ for all $x \in K$ then either $a = 0$ or d is zero.

Proof.
(i) Let $a * d\ x = 0$ for all $x \in K$. Let $y \in K$, replace x by $x * y$ then

$$0 = a * d\ (x * y) = a * (d\ x * y) + a * (x * d\ y) = a * (x * d\ y).$$

Putting $x = 1$ we get that $a * d\ y = 0$, But K has no zero divisors, so $a = 0$ or $d\ y = 0$ for all $y = K$. Thus we have that $a = 0$ or d is zero.
(ii) Similar to (i).

Definition 3.8. Let d be a derivation of an incline K. If $x \leq y$ implies $d\ x \leq d\ x$ for all $x, y \in K$, d is called an isotone derivation.

Proposition 3.9**.** Let K be an incline and d be a derivation of K. If $d\ (x + y) = d\ x + d\ y$, for all $x, y \in K$, then the following hold for all $x, y \in K$:

(i) $d\ (x * y) \leq d\ x$,
(ii) $d\ (x * y) \leq d\ y$,
(iii) d is an isotone derivation.

Proof.
(i) Let $x, y \in K$,then by $(K\ 7)$we have:

$$dx = d\ (x + x * y) = d\ x + d\ (x * y).$$

Hence $d\ (x * y) \leq d\ x$.
(ii) Similar to (i).

(iii) Let $x \leq y$,then $x + y = y$,and so

$$dy = d(x + y) = dx + dy$$

Hence $d\,x \leq d\,y$. Completing the proof.

Proposition 3.10. Let K be an integral incline and d be a derivation of K, Denote $d^{-1}(0) = \{x \in K \ |d\,x = 0\}$. If $d\,(x + y) = d\,x + d\,y$, for all $x, y \in K$,then $d^{-1}(0)$ is an ideal of K.

Proof.
Let $x, y \in d^{-1}(0)$,thus $d\,x = d\,y = 0$. From the hypothesies we get $x + y \in d^{-1}(0)$, also $d\,(x * y) = 0$, and so $x * y \in d^{-1}(0)$. Then $d^{-1}(0)$ is a subincline of K. Now let $x \in K$ and $y \in d^{-1}(0)$ such that $x \leq y$, thus $d\,y = 0$, and $(K\ 8)$ give us that

$$0 = d\,y = d\ \ (y + (x * y)) = d\,y + d\ \ (x * y)\,,$$

hence $d\,(x * y) = 0$.Then

$$0 = d\ \ (x * y) = d\,x * y + x * d\,y = d\,x * y\ .$$

Since K has no zero divisors, then either $d\,x = 0$ or $y = 0$. If $d\,x = 0$, then $x \in d^{-1}(0)$ .If $y = 0$,then $x = 0$, by proposition 3.3 $x \in d^{-1}(0)$, completing the proof.

Theorem 3.11. Let d be a derivation of an integral incline K. Define $d^{2}(x) = d\,(d\,x)$ for all $x \in K$. If $d^{2} = 0$, then d is zero.

Proof.
Let $x, y \in K$, then

$$\begin{aligned} 0 &= d^2(x * y) = d\ \ (d\,x * y + x * dy) \\ &= d^{2}x * y + d\,x * dy + d\,x * d\,y + x * d^2 y \\ &= d\,x * d\,y + d\,x * d\,y\ . \end{aligned}$$

From$(K\ 6)$we get that $d\,x * d\,y = 0$,Since K has no zero divisors we have that, $d\,x = 0$ for all $x \in K$ or $d\,y = 0$ for all $y \in K$.In two cases we have $d = 0$.

Theorem 3.12. Let K be an incline and $d_{1}d_{2}$ derivations of K. Define $d_{1}d_{2}(x) = d_{1}(d_{2}x\ )$ for all $x \in K$. If $d_{1}d_{2} = 0$, then $d_{2}d_{1}$ is a derivation of K.

Proof.
Let $x, y \in K$, then

$$\begin{aligned} 0 &= d_{1}d_{2}(x * y) = d_{1}(d_{2}x\ * y + x * d_{2}y) \\ &= d_{1}d_{2}x * y + d_{2}x * d_{1}y\ + d_{1}x * d_{2}y + x * d_{1}d_{2}y \\ &= d_{2}x * d_{1}y + d_{1}x * d_{2}y. \end{aligned}$$

Then

$$\begin{aligned} d_2 d_1 (x * y) &= d_2 (d_1 x * y + x * d_1 y) \\ &= d_2 d_1 x * y + d_1 x * d_2 y + d_2 x * d_1 y + x * d_2 d_1 y \\ &= d_2 d_1 x * y + x * d_2 d_1 y. \end{aligned}$$

This implies that $d_2 d_1$ is a derivaton.

Theorem 3.13. Let M be a nonzero ideal of an integral incline K. If d is a nonzero derivation of K, then d is nonzero on M.

Proof.

Assume that $d = 0$ on M and $x \in M$,then $d\,x = 0$. Let $y \in K$,since $x * y \leq x$ and M is an ideal of K,thus we have $x * y \in M$ .Therefore $d\,(x * y) = 0$,then we get that

$$0 = d\,(x * y) = d\,x * y + x * d\,y = x * d\,y\ .$$

But Khas no zero divisors, so $x = 0$ for all $x \in M$ or $d\,y = 0$ for all $y \in K$,Since $M \neq 0$, we get that $d\,y = 0$ for all $y \in K$ .this contradicts $d \neq 0$ on K .

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*Department of Mathematics, King Abdulaziz University, Faculty of Education, Science Sections, Jeddah, Saudi Arabia.

E-mail: n_alshehry@yahoo.com