A NOTE ON L^p ESTIMATES FOR SINGULAR INTEGRALS

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Received January 26, 2010

Abstract. In this note we introduce a function space, which is used to define kernels of singular integrals. The space is useful in proving L^p boundedness of certain singular integrals via extrapolation arguments under a sharp condition on their kernels.

1. Introduction

Let Δ_s, $s \geq 1$, denote the family of measurable functions h on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ such that

$$\|h\|_{\Delta_s} = \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s \, dt/t \right)^{1/s} < \infty,$$

where $\mathbb{Z}$ denotes the set of integers. We note that $\Delta_s \subset \Delta_t$ if $s > t$. Let $P(y) = (P_1(y), P_2(y), \dots, P_d(y))$ be a polynomial mapping, where each P_j is a real-valued polynomial on $\mathbb{R}^n$. We assume $n \geq 2$. Define the singular Radon transform $T(f)$ by

$$(1.1) \qquad \begin{aligned} T(f)(x) &= \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y))K(y)\, dy \\ &= \lim_{\epsilon\to 0} \int_{|y|>\epsilon} f(x - P(y))K(y)\, dy, \end{aligned}$$

for an appropriate function f on $\mathbb{R}^d$, where $K(y) = h(|y|)\Omega(y')|y|^{-n}$, $y' = |y|^{-1}y$, $h \in \Delta_1$ and Ω is a function in $L^1(S^{n-1})$ satisfying

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0.$$

Here $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. We denote by $L^q(S^{n-1})$ the space of functions F on S^{n-1} such that $\|F\|_q = \left(\int_{S^{n-1}} |F|^q \, d\sigma\right)^{1/q} < \infty$.

Also, we consider the maximal operator

$$(1.2) \qquad T^*(f)(x) = \sup_{N,\epsilon>0} \left| \int_{\epsilon<|y|<N} f(x - P(y))K(y)\, dy \right|,$$

where P and K are as in (1.1).

In what follows we assume that the polynomial mapping P in (1.1) satisfies $P(-y) = -P(y)$, $P \neq 0$. The following result is known (see [2]).

Theorem A. *If $\Omega \in L^q(S^{n-1})$, $q \in (1,2]$ and $h \in \Delta_s$, $s \in (1,2]$, then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_q \|h\|_{\Delta_s} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1,\infty)$, where the constant C_p is independent of q, s, Ω, h, and the polynomials P_j if each of them is of fixed degree.

2000 *Mathematics Subject Classification.* Primary 42B20, 42B35.
Key Words and Phrases. Singular integrals, extrapolation.

We recall that $\mathcal{L}_a$, $a > 0$, is the space of functions h on $\mathbb{R}_+$ satisfying $L_a(h) < \infty$, where

$$L_a(h) = \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| \left(\log(2 + |h(r)|)\right)^a \, dr/r,$$

and $\mathcal{N}_a$ is defined to be the space of functions h on $\mathbb{R}_+$ such that $N_a(h) < \infty$, where

$$N_a(h) = \sum_{m\geq 1} m^a 2^m d_m(h),$$

with $d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k,m)|$,

$$E(k,m) = \{r \in (2^k, 2^{k+1}] : 2^{m-1} < |h(r)| \leq 2^m\}, \quad m \geq 2,$$
$$E(k,1) = \{r \in (2^k, 2^{k+1}] : 0 < |h(r)| \leq 2\}.$$

It was observed in [2] that $N_a(h) < \infty$ implies $L_a(h) < \infty$ and that $N_a(h) < \infty$ if $L_{a+b}(h) < \infty$ for some $b > 1$.

Let $L \log L(S^{n-1})$ be the Zygmund class of functions F on S^{n-1} satisfying

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|) \, d\sigma(\theta) < \infty.$$

Theorem A implies the following result via an extrapolation argument (see [2] and also [1, 3, 4, 5]).

Theorem B. *If $\Omega \in L\log L(S^{n-1})$ and $h \in \mathcal{N}_1$, then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)} \quad \textit{for all } p \in (1,\infty),$$

where the constant C_p is independent of the polynomials P_j if they are of fixed degree.

For $a > 0$, let $\mathcal{M}_a$ be the collection of functions h on $\mathbb{R}_+$ such that there exist a sequence $\{h_k\}_{k=1}^{\infty}$ of functions on $\mathbb{R}_+$ and a sequence $\{a_k\}_{k=1}^{\infty}$ of non-negative real numbers satisfying $h = \sum_{k=1}^{\infty} a_k h_k$, $\sup_{k\geq 1} \|h_k\|_{\Delta_{1+1/k}} \leq 1$ and $\sum_{k=1}^{\infty} k^a a_k < \infty$. For $h \in \mathcal{M}_a$, define

$$\|h\|_{\mathcal{M}_a} = \inf_{\{a_k\}} \sum_{k=1}^{\infty} k^a a_k,$$

where the infimum is taken over all sequences $\{a_k\}$ of non-negative real numbers such that $\sum_{k=1}^{\infty} k^a a_k < \infty$ and $h = \sum_{k=1}^{\infty} a_k h_k$ for some $\{h_k\}$ satisfying $\sup_{k\geq 1} \|h_k\|_{\Delta_{1+1/k}} \leq 1$.

We note the following.

Proposition. *For $a > 0$, let $\mathcal{M}_a$, $\mathcal{N}_a$, $\mathcal{L}_a$ be as above. Then,*

(1) *$\|\cdot\|_{\mathcal{M}_a}$ is a norm on the space $\mathcal{M}_a$;*
(2) *if $h \in \mathcal{N}_a$, then $h \in \mathcal{M}_a$;*
(3) *if $h \in \mathcal{M}_a$, then $h \in \mathcal{L}_a$.*

The space $\mathcal{M}_1$ is useful in an extrapolation argument. Indeed, Theorem A and extrapolation imply the following L^p boundedness of the singular integral operator T.

Theorem 1. *Let T be as in (1.1) with a kernel $K(y) = h(|y|)\Omega(y')|y|^{-n}$. If $h \in \mathcal{M}_1$ and $\Omega \in L\log L(S^{n-1})$, then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)} \quad \textit{for all } p \in (1,\infty),$$

where the constant C_p is independent of the polynomials P_j if each of them is of fixed degree.

Theorem B follows from Theorem 1 by Proposition (2).

Similarly, Theorem 3 of [2] implies the following.

Theorem 2. *Let T^* be as in* (1.2). *Suppose that $h \in \mathcal{M}_1$ and $\Omega \in L\log L(S^{n-1})$. Then*

$$\|T^*(f)\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)} \qquad \text{for all } p \in (1,\infty),$$

where the constant C_p is independent of the polynomials P_j, as in Theorem 1.

Let $\{A_t\}_{t>0}$ be a dilation group on $\mathbb{R}^n$ defined by $A_t = t^P = \exp((\log t)P)$, where P is an $n\times n$ real matrix whose eigenvalues have positive real parts. Let r be a norm function on $\mathbb{R}^n$ associated with $\{A_t\}_{t>0}$ such that

(1) r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$;
(2) $r(A_t x) = tr(x)$ for all $t>0$ and $x \in \mathbb{R}^n$;
(3) $r(x+y) \le C(r(x)+r(y))$ for some $C>0$;
(4) $dx = t^{\gamma-1}\, d\sigma\, dt$, that is,

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_\Sigma f(A_t\theta)t^{\gamma-1}\,d\sigma(\theta)\,dt,$$

with $d\sigma = \omega\, d\sigma_0$, for an appropriate function f, where $\gamma = \text{trace } P$, ω is a strictly positive C^∞ function on $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$ and $d\sigma_0$ is the Lebesgue surface measure on Σ;
(5) $\Sigma = \{\theta \in \mathbb{R}^n : \langle B\theta, \theta\rangle = 1\}$ for a positive symmetric matrix B, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$;
(6)

$$\begin{aligned} c_1|x|^{\alpha_1} &\le r(x) \le c_2|x|^{\alpha_2} \quad \text{if } r(x) \ge 1,\\ c_3|x|^{\beta_1} &\le r(x) \le c_4|x|^{\beta_2} \quad \text{if } r(x) \le 1, \end{aligned}$$

for some positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1$ and β_2.

Let Ω be locally integrable in $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \ne 0$, $t > 0$. We assume that

$$\int_\Sigma \Omega(\theta)\,d\sigma(\theta) = 0.$$

We consider a singular integral operator on $\mathbb{R}^n$ of the form

$$S(f)(x) = \text{p.v.}\int_{\mathbb{R}^n} f(x-y)K(y)\,dy. \tag{1.3}$$

where $K(y) = h(r(y))\Omega(y')r(y)^{-\gamma}$, $y' = A_{r(y)^{-1}}y$, $h \in \Delta_1$.

Theorem 1.3 of [3] and an extrapolation argument similar to that for Theorem 1 imply the following.

Theorem 3. *Let S be as in* (1.3) *with the functions h and Ω satisfying $h \in \mathcal{M}_1$,*

$$\int_\Sigma |\Omega(\theta)|\log(2+|\Omega(\theta)|)\,d\sigma(\theta) < \infty.$$

Then

$$\|S(f)\|_p \le C_p\|f\|_p \qquad \text{for all } p \in (1,\infty).$$

Theorem 1.4 of [3] follows from this (see also Remark in Section 3 of [3]). We shall prove Proposition in Section 2 and Theorem 1 in Section 3 by applying results of [2]. The letter C will be used to denote non-negative constants which may be different in different occurrences.

2. Proof of Proposition

Proof of (1). We have to prove the following.

(i) $\|h\|_{\mathcal{M}_a} = 0$ if and only if $h = 0$;
(ii) if $h \in \mathcal{M}_a$ and λ is a complex number, then $\lambda h \in \mathcal{M}_a$ and $\|\lambda h\|_{\mathcal{M}_a} = |\lambda|\|h\|_{\mathcal{M}_a}$;
(iii) if $h, \ell \in \mathcal{M}_a$, then $h + \ell \in \mathcal{M}_a$ and $\|h + \ell\|_{\mathcal{M}_a} \le \|h\|_{\mathcal{M}_a} + \|\ell\|_{\mathcal{M}_a}$.

It is not difficult to prove these results. To prove (i), note that $\|h\|_{\Delta_1} \le \sum_k a_k \le \sum_k k^a a_k$ if $h = \sum_k a_k h_k$ as in the definition, since $\|h_k\|_{\Delta_1} \le \|h_k\|_{\Delta_{1+1/k}} \le 1$. This shows that $\|h\|_{\Delta_1} = 0$ and hence $h = 0$ if $\|h\|_{\mathcal{M}_a} = 0$. The converse is obvious. We omit proofs of (ii) and (iii).

Proof of (2). This follows from results in Section 3 of [2]. Let $h \in \mathcal{N}_a$ and $E_1 = \{r \in \mathbb{R}_+ : 0 < |h(r)| \le 2\}$, $E_m = \{r \in \mathbb{R}_+ : 2^{m-1} < |h(r)| \le 2^m\}$ for $m \ge 2$. Then

$$\|h\chi_{E_m}\|_{\Delta_{1+1/m}} \le 2^m (d_m(h))^{m/(m+1)}, \tag{2.1}$$

where χ_S denotes the characteristic function of a set S. Define $h_m = 2^{-m}(d_m(h))^{-m/(m+1)} h\chi_{E_m}$ if $d_m(h) \neq 0$ and $h_m = 0$ if $d_m(h) = 0$. Put $a_m = 2^m(d_m(h))^{m/(m+1)}$. Then, by (2.1) $\|h_m\|_{\Delta_{1+1/m}} \le 1$, and $h = \sum_{m=1}^\infty a_m h_m$. To show $h \in \mathcal{M}_a$, it suffices to prove that $\sum_{m=1}^\infty m^a a_m < \infty$. To see this we use Young's inequality

$$\alpha\beta \le p^{-1}\alpha^p + q^{-1}\beta^q, \quad \alpha, \beta \ge 0, \quad 1 < p, q < \infty, \quad 1/p + 1/q = 1. \tag{2.2}$$

Using (2.2) with $\alpha = 1/3$, $\beta = (d_m(h))^{m/(m+1)}$, $p = m+1$ and $q = (m+1)/m$, we have

$$\begin{aligned} \sum_{m=1}^\infty m^a a_m &= \sum_{m=1}^\infty m^a 2^m (d_m(h))^{m/(m+1)} \\ &\le 3\sum_{m=1}^\infty m^a 2^m (m+1)^{-1} 3^{-m-1} + 3\sum_{m=1}^\infty m^a 2^m (m/(m+1)) d_m(h) \\ &\le C\left(1 + N_a(h)\right). \end{aligned}$$

This completes the proof of part (2).

Proof of (3). The following elementary lemmas are useful.

Lemma 1. *Suppose $x \ge 2$, $1 < p < \infty, a > 0$. Then*

$$x(\log x)^a \le C_a (p-1)^{-a} x^p,$$

where the constant C_a depends only on a.

Lemma 2. *If $f(x) = x(\log x)^a$, $a > 0$, $x > e^{1-a}$, then $f''(x) > 0$.*

Let $h \in \mathcal{M}_a$ and $h = \sum_k a_k h_k$ as in the definition. To show that $h \in \mathcal{L}_a$, we may assume that $\sum_k a_k = 1$. Since Lemma 2 implies that the function $(e+x)(\log(e+x))^a$ is convex for $x \ge 0$,

$$\begin{aligned} |h|\left(\log\left(2 + |h|\right)\right)^a &\le \left(e + |h|\right)\left(\log\left(e + |h|\right)\right)^a \\ &\le \sum a_k \left(e + |h_k|\right)\left(\log\left(e + |h_k|\right)\right)^a. \end{aligned} \tag{2.3}$$

By Lemma 1 with $p = 1 + 1/k$, we have

$$\begin{aligned} \left(e + |h_k|\right)\left(\log\left(e + |h_k|\right)\right)^a &\le C_a k^a (e + |h_k|)^{1+1/k} \\ &\le C_a k^a 2^{1/k}(e^{1+1/k} + |h_k|^{1+1/k}) \le Ck^a(e^2 + |h_k|^{1+1/k}). \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we see that

$$\sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} |h| \left(\log\left(2+|h|\right)\right)^a \, dr/r$$
$$\leq C \sum k^a a_k \left(e^2 + \|h_k\|_{\Delta 1+1/k}^{1+1/k}\right) \leq C \sum k^a a_k < \infty.$$

This completes the proof.

3. Proof of Theorem 1

We prove Theorem 1 by applying Theorem A with extrapolation. By well-known arguments we have the following (see [6, Chap. XII, pp. 119–120] for relevant results).

Lemma 3. *Suppose $F \in L^1(S^{n-1})$ and $a > 0$. Then, the following two statements are equivalent*:

(1) $\int_{S^{n-1}} |F| \left(\log(2+|F|)\right)^a \, d\sigma < \infty$ *and* $\int_{S^{n-1}} F \, d\sigma = 0$;
(2) *there exist a sequence $\{F_m\}_{m=1}^\infty$ of functions on S^{n-1} and a sequence $\{b_m\}_{m=1}^\infty$ of non-negative real numbers such that $F = \sum_{m=1}^\infty b_m F_m$, $\sup_{m\geq 1} \|F_m\|_{1+1/m} \leq 1$, $\int_{S^{n-1}} F_m \, d\sigma = 0$, $\sum_{m=1}^\infty m^a b_m < \infty$.*

Proof. First, we prove that (1) implies (2). Put

$$U_m = \{\theta \in S^{n-1} : 2^{m-1} < |F(\theta)| \leq 2^m\} \quad \text{for } m \geq 2,$$
$$U_1 = \{\theta \in S^{n-1} : |F(\theta)| \leq 2\}.$$

Then, we decompose F as $F = \sum_{m=1}^\infty \tilde{F}_m$, where $\tilde{F}_m = F\chi_{U_m} - \sigma(S^{n-1})^{-1} \int_{U_m} F \, d\sigma$. Note that $\int \tilde{F}_m \, d\sigma = 0$. Set $e_m = \sigma(U_m)$, $m \geq 1$. Then

$$\|\tilde{F}_m\|_{1+1/m} \leq 22^m e_m^{m/(m+1)} \quad \text{for } m \geq 1. \tag{3.1}$$

Define $F_m = 2^{-m-1} e_m^{-m/(m+1)} \tilde{F}_m$ if $e_m \neq 0$, $F_m = 0$ if $e_m = 0$, and $b_m = 2^{m+1} e_m^{m/(m+1)}$ for $m \geq 1$. Then, $F = \sum_{m=1}^\infty b_m F_m$, $\int F_m \, d\sigma = 0$, and (3.1) implies that $\sup_{m\geq 1} \|F_m\|_{1+1/m} \leq 1$. Also, by (2.2) we have

$$\begin{aligned} \sum_{m=1}^\infty m^a b_m &= \sum_{m=1}^\infty m^a 2^{m+1} e_m^{m/(m+1)} \\ &\leq 2 \sum_{m=1}^\infty (m/(m+1)) m^a 2^{(m+1)(1+1/m)} e_m + 2 \sum_{m=1}^\infty m^a 2^{-m-1}/(m+1) \\ &\leq C \sum_{m=1}^\infty m^a 2^m e_m + C \leq C \int_{S^{n-1}} |F| \left(\log(2+|F|)\right)^a \, d\sigma + C. \end{aligned} \tag{3.2}$$

Conversely, by the proof of Proposition (3) we can see that (2) implies (1). □

Fix $p \in (1,\infty)$ and a function f with $\|f\|_p \leq 1$. Set $S(h,\Omega) = \|T(f)\|_p$, where $T(f)$ is as in (1.1). Let $h \in \mathcal{M}_1$ and $\Omega \in L\log L(S^{n-1})$. Write $h = \sum_{k=1}^\infty a_k h_k$ as in the definition of $\mathcal{M}_1$. We may assume $\sum_{k=1}^\infty k a_k \leq 2\|h\|_{\mathcal{M}_1}$. Also, we have $\Omega = \sum_{m=1}^\infty b_m \Omega_m$ by applying Lemma 3 with $a = 1$, where $\sup_{m\geq 1} \|\Omega_m\|_{1+1/m} \leq 1$, $\int_{S^{n-1}} \Omega_m \, d\sigma = 0$, $b_m \geq 0$, $\sum_{m=1}^\infty m b_m < \infty$. We may assume that $\sum_{m=1}^\infty m b_m \leq C \int_{S^{n-1}} |\Omega| \log(2+|\Omega|) \, d\sigma + C$ by

(3.2). Now, the subadditivity of S and Theorem A imply

$$\begin{aligned} S(h,\Omega) &\le \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_k b_m S(h_k,\Omega_m) \\ &\le C\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} k a_k m b_m \|h_k\|_{\Delta_{1+1/k}} \|\Omega_m\|_{1+1/m} \\ &\le C\|h\|_{\mathcal{M}_1}\left(1+\int_{S^{n-1}} |\Omega|\log(2+|\Omega|)\,d\sigma\right). \end{aligned}$$

The conclusion of Theorem 1 follows from this.

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