A HERMITE'S INTERPOLATION FORMULA WITH GENERALIZED QUOTIENT AND REMAINDER THEOREMS

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Abstract. The purpose of this paper is to study the characterization of a Hermite's interpolation formula to produce the generalized quotient and remainder theorem of polynomials and its formulae.

1. Introduction

If $f(x)$ is a polynomial of degree $m \leq n$ over a field F and $b_1, b_2, \ldots, b_n$, are any n distinct elements in F, the Newton's interpolation formula asserts that, there is a unique representation

$$f(x) = c_0 + c_1(x - b_1) + c_2(x - b_1)(x - b_2) + \ldots + c_n(x - b_1)(x - b_2) \ldots (x - b_n)$$

with coefficients c_i in F. The proofs of these results can be found in many standard books (see, for instance, [1], p. 111).

In this paper, we shall be interested in the generalization of the above Newton's interpolation formula by $b_1, b_2, \ldots, b_n$ are not necessary distinct points in F. Suppose that only s of them, $b_1, b_2, \ldots, b_s$ are distinct. Let $m_1, m_2, \ldots, m_s$ be nonnegative integers, and $m_1 + m_2 + \ldots + m_s = n$, the "Newton's interpolation formula at these coincident points" is called "Hermite's interpolation formula" become

$$\begin{aligned} f(x) \quad = \quad & r_0(x) + r_1(x)(x - b_1)^{m_1} + r_2(x)(x - b_1)^{m_1}(x - b_2)^{m_2} + \ldots \\ & + r_s(x)(x - b_1)^{m_1}(x - b_2)^{m_2} \ldots (x - b_s)^{m_s}, \end{aligned}$$

where $r_j(x)$ is polynomials in $F[x]$, $j = 0, 1, \ldots, s$. This formula can apply to produce formulae of generalized quotient and remainder of the polynomial ring $F[x]$.

2. Preliminaries

In this section, an algorithm is presented for calculating the remainder on dividing $f(x)$ of degree n by the divisor $g(x) = \alpha(x - b)^m$, $\alpha = 1$ and $m < n$. The extension to the case $g(x)$ is nonmonic where $\alpha \neq 1$ is trivial, and the remainder term does not change when the divisor is changed from a nonmonic polynomial to the corresponding monic polynomial by taking out the coefficient of the highest power of x.

Lemma 2.1. *Let $f(x)$ and $g(x) = b_0 + b_1x + \ldots + b_{m-1}x^{m-1} + b_mx^m$ be polynomials in $F[x]$, $\deg g(x) = m$ and $b_m \neq 1$, if $g_1(x) = \frac{1}{b_m}g(x) = \beta_0 + \beta_1x + \ldots + \beta_{m-1}x^{m-1} + x^m$ where $\beta_i = \frac{b_i}{b_m}$, $i = 0, 1, \ldots, m - 1$, be the corresponding monic polynomial of $g(x)$, and*

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$q_1(x)$, $q(x)$ be the quotients and $r_1(x)$, $r(x)$ be the remainders on dividing $f(x)$ by $g_1(x)$, and by $g(x)$ respectively, then

$$q(x) = \frac{1}{b_m} q_1(x) \quad \textit{and} \quad r(x) = r_1(x). \tag{2.1}$$

Proof. Since

$$g(x) = b_0 + b_1 x + \ldots + b_{m-1}x^{m-1} + b_m x^m, \quad b_m \neq 1.$$

Let

$$\begin{array}{rcl} g_1(x) & = & \frac{1}{b_m} g(x) \\ & = & \frac{b_0}{b_m} + \frac{b_1}{b_m}x + \ldots + \frac{b_{m-1}}{b_m}x^{m-1} + x^m. \end{array}$$

By the Division Algorithm, there is a unique $q_1(x)$, and a unique $r_1(x)$ in $F[x]$ such that

$$f(x) = q_1(x)g_1(x) + r_1(x) \text{ whenever } r_1(x) = 0 \text{ or } \deg r_1(x) < \deg g_1(x),$$

that is

$$\begin{array}{rcl} f(x) & = & q_1(x)(\beta_0 + \beta_1 x + \ldots + \beta_{m-1}x^{m-1} + x^m) + r_1(x) \\ & = & q_1(x)(\frac{b_0}{b_m} + \frac{b_1}{b_m}x + \ldots + \frac{b_{m-1}}{b_m}x^{m-1} + x^m) + r_1(x) \\ & = & q_1(x)\frac{1}{b_m}(b_0 + b_1 x + \ldots + b_{m-1}x^{m-1} + b_m x^m) + r_1(x) \\ & = & \left\{\frac{1}{b_m}q_1(x)\right\} g(x) + r_1(x) \\ & = & q(x)g(x) + r(x). \end{array}$$

Thus

$$q(x) = \frac{1}{b_m} q_1(x), \text{ and } r(x) = r_1(x).$$

□

Now by Division Algorithm of the polynomial ring $F[x]$, we have

$$f(x) = (x-b)^m q(x) + c_0 + c_1 x + \ldots + c_{m-1}x^{m-1}, \tag{2.2}$$

where the remainder $r(x) = c_0 + c_1 x + \ldots + c_{m-1}x^{m-1}$. After differentiating $f(x)$ in (2.2) with respect to x at the point $x = b$, denote $\frac{d}{dx}f(x) = f^{(1)}(x)$, $\frac{d^2}{dx^2}f(x) = f^{(2)}(x)$ and so on, we have a system of linear equations of m equations and m unknowns over F as follows

$$\begin{array}{llllrcr} c_0 & + bc_1 & + b^2c_2 & + \ldots + & b^{m-1}c_{m-1} & = & f(b), \\ 0 & + 1c_1 & + 2bc_2 & + \ldots + & (m-1)b^{m-2}c_{m-1} & = & f^{(1)}(b), \\ 0 & + 0 & + 2!c_2 & + \ldots + & (m-1)(m-2)b^{m-3}c_{m-1} & = & f^{(2)}(b), \\ & & & & & \vdots & \\ 0 & + 0 & + 0 & + \ldots + & (m-1)!c_{m-1} & = & f^{(m-1)}(b). \end{array}$$

Write a matrix equation represent the system of linear equations as

$$WC = Y, \tag{2.3}$$

where $C^T = \begin{bmatrix} c_0 & c_1 & \ldots & c_{m-1} \end{bmatrix}$, $Y^T = \begin{bmatrix} f(b) & f^{(1)}(b) & \ldots & f^{(m-1)}(b) \end{bmatrix}$ and the coefficients matrix W is the Wronskian matrix

$$W = \begin{bmatrix} 1 & b & b^2 & \ldots & b^{m-1} \\ 0 & 1 & 2b & \ldots & (m-1)b^{m-2} \\ 0 & 0 & 2! & \ldots & (m-1)(m-2)b^{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (m-1)! \end{bmatrix}.$$

Using above symbols, we obtain the following lemma.

Lemma 2.2. *If $f(x) = a_0 + a_1x + \ldots + a_nx^n$, $g(x) = (x-b)^m$ are polynomials in $F[x]$, where $a_n \neq 0$, $m \leq n$, then the remainder on dividing $f(x)$ by $g(x)$ is*

$$r(x) = f(x) - \frac{\det B}{\det W}. \tag{2.4}$$

Proof. Let B be the result of bordering W, defined by $B = \begin{bmatrix} W & Y \\ X^T & f(x) \end{bmatrix}$, where $X^T = \begin{bmatrix} 1 & x & \ldots & x^{m-1} \end{bmatrix}$, $Y^T = \begin{bmatrix} f(b) & f^{(1)}(b) & \ldots & f^{(m-1)}(b) \end{bmatrix}$ by ([1], p. 417) asserts that

$$\det B = -X^T \cdot \operatorname{adj} W \cdot Y + f(x) \det W. \tag{2.5}$$

It is obvious that $\det W = \prod_{k=0}^{m-1} k! \neq 0$, so that W is a invertible matrix and its inverse is

$$W^{-1} = \frac{1}{\det W} \operatorname{adj} W. \tag{2.6}$$

From (2.3), we get

$$\frac{1}{\det W} \operatorname{adj} W \cdot Y = C. \tag{2.7}$$

Multiplied (2.7) both sides on the left side by X^T, we get

$$X^T \frac{1}{\det W} \operatorname{adj} W \cdot Y = X^T C = r(x).$$

Therefore

$$r(x) = X^T \frac{1}{\det W} \operatorname{adj} W \cdot Y. \tag{2.8}$$

From (2.5) and (2.8), we have

$$r(x) = f(x) - \frac{\det B}{\det W}.$$

□

We need the following result concerning a special remainder theorem.

Theorem 2.3. *If $f(x) = a_0 + a_1x + \ldots + a_nx^n$ and $g(x) = (x-b)^m$ are polynomials in $F[x]$, where $a_n \neq 0$, $m \leq n$, then the remainder on dividing $f(x)$ by $g(x)$ is*

$$r(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i f^{(i)}(b)}{i!}. \tag{2.9}$$

Proof. The theorem can be proved by mathematical induction on m, we define $f^{(0)}(b) = f(b)$.

For $m = 1$, it is well known by "Remainder Theorem" asserts that the remainder $r(x) = f(b)$, thus (2.9) is true.

Suppose that (2.9) is true for all $k < m$. We must show that it is true for m. From Lemma 2.2, calculate the determinant of the matrix B of order $m+1$ as the following. Since

$$B = \begin{bmatrix} W & Y \\ X^T & f(x) \end{bmatrix} = \begin{bmatrix} 1 & b & b^2 & \ldots & b^{m-1} & f^{(0)}(b) \\ 0 & 1 & 2b & \ldots & (m-1)b^{m-2} & f^{(1)}(b) \\ 0 & 0 & 2! & \ldots & (m-1)(m-2)b^{m-3} & f^{(2)}(b) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & (m-1)! & f^{(m-1)}(b) \\ 1 & x & x^2 & \ldots & x^{m-1} & f(x) \end{bmatrix},$$

we apply the elementary column operations on $\det B$ by multiplying $-b$ to the $(j-1)^{th}$ column and then adds to the j^{th} column $j = 2, 3, \dots, m-1$, and multiplying $-f^{(i)}(b)$, $i = 0, 1, \dots, m-1$ to the 1^{st} column and then add to the last column and then expanding the determinant on the first row. In each step we reduce a common factor in every row out of the determinant, after m-steps we have

$$\begin{aligned}
\det B &= \begin{array}{r|c|} 0! & \\ 1! & \\ 2! & \\ \vdots & \\ (m-1)! & \\ (x-b)^{m-1} & \left\{\dfrac{f(x)}{(x-b)^m} - \displaystyle\sum_{i=0}^{m-1} \dfrac{f^{(i)}(b)}{i!(x-b)^{m-i}}\right\} \end{array} \\
&= 0!1!2!\dots(m-1)!(x-b)^m \left\{\frac{f(x)}{(x-b)^m} - \sum_{i=0}^{m-1} \frac{f^{(i)}(b)}{i!(x-b)^{m-i}}\right\} \\
&= (\prod_{k=0}^{m-1} k!)(x-b)^m \left\{\frac{f(x)}{(x-b)^m} - \sum_{i=0}^{m-1} \frac{f^{(i)}(b)}{i!(x-b)^{m-i}}\right\} \\
&= \prod_{k=0}^{m-1} k! \left\{f(x) - (x-b)^m \sum_{i=0}^{m-1} \frac{f^{(i)}(b)}{i!(x-b)^{m-i}}\right\}.
\end{aligned}$$

By (2.4), we see that

$$\begin{aligned}
r(x) &= f(x) - \frac{1}{\det W} \det B \\
&= f(x) - \frac{1}{\prod_{k=0}^{m-1} k!} \prod_{k=0}^{m-1} k! \left\{f(x) - (x-b)^m \sum_{i=0}^{m-1} \frac{f^{(i)}(b)}{i!(x-b)^{m-i}}\right\} \\
&= \sum_{i=0}^{m-1} \frac{(x-b)^m f^{(i)}(b)}{i!(x-b)^{m-i}}.
\end{aligned}$$

Therefore we obtain at once the formula (2.9). ∎

Lemma 2.4 (Taylor's Polynomials). *If $f(x)$ is a polynomial of degree n in $F[x]$, and $b \in F$ then there is a unique representation*

$$f(x) = c_0 + c_1(x-b) + c_2(x-b)^2 + \dots + c_n(x-b)^n \tag{2.10}$$

with coefficient $c_i \in F$, $i = 0, 1, 2, \dots, n$.

Proof. In the polynomial ring $F[x]$, it is well known that, if $f(x), g(x) \in F[x]$, then there is a unique $q(x) \in F[x]$ and a unique $r(x) \in F[x]$ such that

$$f(x) = q(x)g(x) + r(x), \text{ whenever } r(x) = 0 \text{ or } \deg r(x) < \deg g(x).$$

Dividing $f(x)$ by $g(x) = x - b$, we obtain the unique quotient $q_1(x)$ in $F[x]$ of degree $n-1$, and the unique remainder is $c_0 \in F$ such that

$$f(x) = (x-b)q_1(x) + c_0.$$

Similarly, we have

$$q_1(x) = (x-b)q_2(x) + c_1.$$

The process can be continued through $n-2$ more stages to yield

$$\begin{aligned} q_{n-2}(x) &= (x-b)q_{n-1}(x)+c_{n-2} \\ q_{n-1}(x) &= (x-b)q_n(x)+c_{n-1}. \end{aligned}$$

Here $q_n(x)$ is just a polynomial of degree 0, so we can write

$$q_n(x)=c_n.$$

By combining these equalities, we obtain the formula

$$f(x)=c_0+c_1(x-b)+c_2(x-b)^2+\ldots+c_n(x-b)^n.$$

The uniqueness of $c_i \in F$, $i=1,2,\ldots,n$ was shown in each step. □

Theorem 2.5. *If $f(x)=a_0+a_1x+\ldots+a_nx^n$ and $g(x)=(x-b)^m$ where $a_n \neq 0$, $m<n$, are polynomials in $F[x]$, then the quotient on dividing $f(x)$ by $g(x)$ is*

$$q(x)=a_n(x-b)^{n-m}+\sum_{i=0}^{n-m-1}\frac{f^{(m+i)}(b)}{(m+i)!}(x-b)^i. \tag{2.11}$$

Proof. We extend $g(x)$ to $g_1(x)$ by $g_1(x)=g(x)(x-b)^{n-m}$. That is $g_1(x)=(x-b)^n$. Theorem 2.3 asserts that the remainder on dividing $f(x)$ by $g_1(x)$ is

$$r_1(x)=\sum_{i=0}^{n-1}\frac{(x-b)^i f^{(i)}(b)}{i!}. \tag{2.12}$$

From (2.10), we have a unique representation

(2.13)
$$f(x)=c_0+c_1(x-b)+c_2(x-b)^2+\ldots+c_{m-1}(x-b)^{m-1}+c_m(x-b)^m+\ldots+c_n(x-b)^n.$$

We see that the leading coefficient of (2.13) is c_n and $f(x)=a_0+a_1x+\ldots+a_nx^n$, by equating the coefficient, we get

$$c_n=a_n. \tag{2.14}$$

The equation (2.13) becomes

(2.15)
$$\begin{aligned} f(x) &= c_0+c_1(x-b)+c_2(x-b)^2+\ldots+c_{m-1}(x-b)^{m-1} \\ &\quad +c_m(x-b)^m+c_{m+1}(x-b)^{m+1}+\ldots+c_{n-1}(x-b)^{n-1}+a_n(x-b)^n. \end{aligned}$$

Rearranging (2.15) and grouping, we get

$$\begin{aligned} f(x) &= a_n(x-b)^n+\left\{c_0+c_1(x-b)+c_2(x-b)^2+\ldots+c_{n-1}(x-b)^{n-1}\right\} \\ &= a_ng_1(x)+r_1(x), \end{aligned}$$

where $r_1(x)=c_0+c_1(x-b)+c_2(x-b)^2+\ldots+c_{n-1}(x-b)^{n-1}$. Then, the uniqueness of remainder $r_1(x)$, from (2.12), equating the coefficients of $r_1(x)$ we obtain

$$c_i=\frac{f^{(i)}(b)}{i!},\ i=0,1,2,\ldots,n-1. \tag{2.16}$$

Rearranging (2.15) again and regrouping, we have

$$\begin{aligned} f(x) &= \left\{c_m(x-b)^m+c_{m+1}(x-b)^{m+1}+\ldots+c_{n-1}(x-b)^{n-1}+a_n(x-b)^n\right\} \\ &\quad + \left\{c_0+c_1(x-b)+c_2(x-b)^2+\ldots+c_{m-1}(x-b)^{m-1}\right\} \\ &= \left\{c_m+c_{m+1}(x-b)+\ldots+c_{n-1}(x-b)^{n-m-1}+a_n(x-b)^{n-m}\right\}(x-b)^m \\ &\quad + \left\{c_0+c_1(x-b)+c_2(x-b)^2+\ldots+c_{m-1}(x-b)^{m-1}\right\} \\ &= q(x)g(x)+r(x), \end{aligned}$$

where

$$r(x) = c_0 + c_1(x-b) + c_2(x-b)^2 + \ldots + c_{m-1}(x-b)^{m-1}, \tag{2.17}$$

and

$$\begin{aligned} q(x) &= c_m + c_{m+1}(x-b) + \ldots + c_{n-1}(x-b)^{n-m-1} + a_n(x-b)^{n-m} \\ &= a_n(x-b)^{n-m} + \sum_{i=0}^{n-m-1} c_{m+i}(x-b)^i. \end{aligned} \tag{2.18}$$

Replacing all c_i in equation (2.18) by (2.16), we obtain at once the formula states in (2.11), that is

$$q(x) = a_n(x-b)^{n-m} + \sum_{i=0}^{n-m-1} \frac{f^{(m+i)}(b)}{(m+i)!}(x-b)^i.$$

□

Remark, in above theorem if $m = n$, it is easy to see that $q(x) = a_n$.

The well known "Remainder Theorem" asserts that the remainder on dividing $f(x)$ by $x-b$ is $f(b)$, now we have the following corollary is called "Quotient Theorem".

Corollary 2.6 (Quotient Theorem)**.** *If $f(x) = a_0 + a_1x + \ldots + a_nx^n$ and $g(x) = x - b$ where $a_n \neq 0$ are polynomials in $F[x]$, then the quotient on dividing $f(x)$ by $x - b$ is*

$$q(x) = a_n(x-b)^{n-1} + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{(i+1)!}(x-b)^i.$$

Proof. From Theorem 2.5, we replace m by 1, to obtain

$$q(x) = a_n(x-b)^{n-1} + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{(i+1)!}(x-b)^i.$$

□

Resulting polynomials can be evaluated efficiently using Horner's Rule, from (2.17),

$$\begin{aligned} r(x) &= c_0 + c_1(x-b) + c_2(x-b)^2 + \ldots + c_{m-1}(x-b)^{m-1} \\ &= c_0 + (x-b)(c_1 + (x-b)(c_2 + \ldots + (x-b)(c_{m-1})\ldots)), \end{aligned}$$

it is clear that, (see [2], p. 84) this algorithm required just $m-1$ multiplications and m additions and/or subtractions.

From (2.18),

$$\begin{aligned} q(x) &= c_m + c_{m+1}(x-b) + \ldots + c_{n-1}(x-b)^{n-m-1} + a_n(x-b)^{n-m} \\ &= c_m + (x-b)(c_{m+1} + \ldots + (x-b)(c_{n-1} + (x-b)(a_n))\ldots). \end{aligned}$$

By this method ([3], and [5]), the remainder and the quotient without the renumbering is about half as expensive. This is quite efficient in computational cost.

The classical Taylor expansion of a smooth function in terms of a polynomial, (see [2] p. 15) asserts that, if $f(x) \in C^n[a,b]$ there exists $a \leq \xi_i \leq b$ such that

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(x-a)^j}{j!} + \frac{f^{(n)}(\xi_i)(x-a)^n}{n!},$$

where $f^{(j)}(a)$ is the jth derivatives of $f(x)$ at $x = a$. The proof uses integrating by parts $m-1$ times, and the remainder term is obtained by using the integral mean value theorem.

From Lemma 2.4 and (2.16) we obtain the "Taylor expansion" for any polynomial $f(x) \in F[x]$.

Corollary 2.7 (Taylor Expansion). *If $f(x) = a_0 + a_1x + a_2x_2 + \ldots + a_nx^n$ is a polynomials of degree n in $F[x]$, and $b \in F$ then there is a unique representation*

$$\begin{aligned} f(x) &= \frac{f^{(0)}(b)}{0!} + \frac{f^{(1)}(b)}{1!}(x-b) + \ldots + \frac{f^{(n-1)}(b)}{n-1!}(x-b)^{n-1} + a_n(x-b)^n \\ &= \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(x-a)^j}{j!} + a_n(x-a)^n. \end{aligned}$$

Proof. Replacing all $c_i,\ i = 0, 1, \ldots, n-1$ from (2.16) into the equation (2.10), and $c_n = a_n$ from (2.14). □

3. A Hermite's Interpolation Formula

Theorem 3.1 (Hermite's Interpolation Formula). *Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial over a field F of degree n, and $m_i, i = 1, 2, \ldots, s$ be positive integers such that $n \le m_1 + m_2 + \ldots + m_s = m$, and let $b_1, b_2, \ldots, b_s$ be s distinct elements in F. Then there is a unique representation*

$$\begin{aligned} f(x) &= r_0(x) + r_1(x)(x-b_1)^{m_1} + r_2(x)(x-b_1)^{m_1}(x-b_2)^{m_2} + \ldots \\ &\quad + r_s(x)(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s}, \end{aligned} \tag{3.1}$$

where $r_j(x) = \sum_{i=0}^{m_{j+1}-1} \frac{(x-b_{j+1})^i q_j^{(i)}(b_{j+1})}{i!},\ j = 0, 1, 2, \ldots, k-1.$

Proof. Dividing $(x-b_1)^{m_1}$ into $f(x)$ there exist the quotient $q_1(x) \in F[x]$ of degree $n - m_1$, and the remainder is $r_0(x) \in F[x]$

$$f(x) = (x-b_1)^{m_1} q_1(x) + r_0(x),$$

we denote $f(x) = q_0(x)$, from (2.9) and (2.11), then $r_0(x) = \sum_{i=0}^{m_1-1} \frac{(x-b_1)^i q_0^{(i)}(b_1)}{i!}$, and $q_1(x) = a_n(x-b_1)^{n-m_1} + \sum_{i=0}^{n-m_1-1} \frac{q_0^{(m_1+i)}(b_1)}{(m_1+i)!}(x-b_1)^i$.

Therefore the leading coefficient of $q_1(x)$ is a_n.

From $(x-b_2)^{m_2}$ and $q_2(x)$, obtain similarly

$$q_1(x) = (x-b_2)^{m_2} q_2(x) + r_1(x),$$

where $\deg q_1(x) = n - m_1$, by (2.9), $r_1(x) = \sum_{i=0}^{m_2-1} \frac{(x-b_2)^i q_1^{(i)}(b_2)}{i!}$ and by (2.11), $q_2(x) = a_n(x-b_2)^{n-m_1-m_2} + \sum_{i=0}^{n-m_1-m_2-1} \frac{q_1^{(m_2+i)}(b_2)}{(m_2+i)!}(x-b_2)^i$.

Since $n \le m$ there is some positive integer k, $1 \le k \le s$ such that the process can be continued through $k-2$ stages to yield

$$\begin{aligned} q_{k-2}(x) &= (x-b_{k-1})^{m_{k-1}} q_{k-1}(x) + r_{k-2}(x) \\ q_{k-1}(x) &= (x-b_k)^{m_k} q_k(x) + r_{k-1}(x), \end{aligned}$$

where $r_j(x) = \sum_{i=0}^{m_{j+1}-1} \frac{(x-b_{j+1})^i q_j^{(i)}(b_{j+1})}{i!}$, $j = 0, 1, 2, \ldots, k-1$ and $q_j(x) = a_n(x-b_j)^{n-m_1-\ldots-m_j} + \sum_{i=0}^{n-m_1-\ldots-m_j-1} \frac{q_{j-1}^{(m_j+i)}(b_j)}{(m_j+i)!}(x-b_j)^i$, $j = 1, 2, \ldots, k$. Here $q_k(x)$ is just a polynomial of degree less than m_k, the $q_k(x)$ is the remainder on dividing $q_{k-1}(x)$ by $(x-b_k)^{m_k}$, so we can write

$$q_k(x) = r_k(x)$$

and define $r_{k+1}(x) = \ldots = r_s(x) = 0$.

The polynomials $r_0(x), r_1(x), \ldots, r_s(x)$ are all in $F[x]$. By combining these equalities, we obtain at once the formula (3.1) stated.

The uniqueness of $r_i(x), i = 1, 2, \ldots, s$, was shown by the uniqueness of division algorithm in the Euclidean domain $F[x]$ in each step. □

Corollary 3.2 (Newton's Interpolation Formula). *If $f(x)$ is a polynomial of degree $n \leq m$, and $b_1, b_2, \ldots, b_m$ are m distinct elements in the field F, then there is a unique representation*

$$f(x) = c_0 + c_1(x - b_1) + c_2(x - b_1)(x - b_2) + \ldots + c_m(x - b_1)(x - b_2)\ldots(x - b_m)$$

with coefficient c_i in F, $i = 0, 1, \ldots, m$.

Proof. This Corollary is a special case of Theorem 3.1, when $m_i = 1$ for all $i = 1, 2, \ldots, s$, and $s = m$. Then we have $r_0(x) = c_0, r_1(x) = c_1, \ldots, r_m(x) = c_m$, for all $c_0, c_1, \ldots, c_m \in F$. □

4. **Main Results**

The author [4] was proved the generalized quotient theorem and the generalized remainder theorem when the dividend is $f(x) = a_0 + a_1x + \ldots + a_nx^n$, and the divisor is $g(x) = b_mx^m - b_{m-1}x^{m-1} - \ldots - b_1x - b_0$ are polynomials in $F[x]$.

Now if the field F is an algebraically closed field of the divisor polynomial $g(x)$ then it is factored completely over F, let $g(x) = (x - b_1)^{m_1}(x - b_2)^{m_2}\ldots(x - b_s)^{m_s}$ then we have another form of "Generalized Quotient and Remainder Theorem" this is an application of the Hermite's Interpolation Formula. It is sufficient to consider divisors which are monic polynomial, because in the another case it is obvious by Lemma 2.1.

Theorem 4.1 (Generalized Quotient and Remainder Theorem). *If $f(x) = a_0 + a_1x + \ldots + a_nx^n$, and $g(x) = (x - b_1)^{m_1}(x - b_2)^{m_2}\ldots(x - b_s)^{m_s}$ are polynomials in $F[x]$, where $b_1, b_2, \ldots, b_s$ are all nonzero distinct elements of F, and $\deg g(x) = m \leq n = \deg f(x)$, then the quotient and the remainder on dividing $f(x)$ by $g(x)$ is*

$$q(x) = r_s(x) + a_nx^{n-m} \quad \text{and} \quad r(x) = r_0(x) + \sum_{j=1}^{s-1} r_j(x)\prod_{i=1}^{j}(x - b_i)^{m_i}, \tag{4.1}$$

respectively, where $r_j(x) = \sum_{i=0}^{m_{j+1}-1} \frac{(x-b_{j+1})^i q_j^{(i)}(b_{j+1})}{i!}$, $j = 0, 1, 2, \ldots, s$.

Proof. Since $b_1, b_2, \ldots, b_s$ are all nonzero distinct elements of F. Let $g_1(x) = (x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x - b_s)^{m_s}(x - 0)^{m_{s+1}}$, where $m_{s+1} = n - m$, we can apply by the formula (2.18), to obtain

$$\begin{aligned} f(x) \quad = \quad & r_0(x) + r_1(x)(x - b_1)^{m_1} + r_2(x)(x - b_1)^{m_1}(x - b_2)^{m_2} + \ldots \\ & + r_s(x)(x - b_1)^{m_1}(x - b_2)^{m_2}\ldots(x - b_s)^{m_s} \\ & + r_{s+1}(x)(x - b_1)^{m_1}(x - b_2)^{m_2}\ldots(x - b_s)^{m_s}(x - 0)^{m_{s+1}}, \end{aligned} \tag{4.2}$$

where $r_j(x) = \sum_{i=0}^{m_{j+1}-1} \frac{(x-b_{j+1})^i q_j^{(i)}(b_{j+1})}{i!}$ $j = 0, 1, 2, \ldots, s$, and $q_j(x) = a_n(x-b_j)^{n-m_1-\ldots-m_j} + \sum_{i=0}^{n-m_1-\ldots-m_j-1} \frac{q_{j-1}^{(m_j+i)}(b_j)}{(m_j+i)!}(x - b_j)^i$, $j = 1, 2, \ldots, s + 1$. Thus the leading coefficient of $f(x)$ is $r_{s+1}(x) = a_n$.

Rearranging (4.2) and grouping, we have

$$\begin{aligned} f(x) &= \{r_s(x)(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s} \\ &\quad + a_n(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s}(x-0)^{m_{s+1}}\} \\ &\quad + \{r_0(x)+r_1(x)(x-b_1)^{m_1}+\ldots \\ &\quad + r_{s-1}(x)(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_{s-1}}\} \\ &= \{r_s(x)+a_nx^{m_{s+1}}\}(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s} \\ &\quad + \{r_0(x)+r_1(x)\prod_{i=1}^{1}(x-b_i)^{m_i}+\ldots+r_{s-1}(x)\prod_{i=1}^{s-1}(x-b_i)^{m_i}\} \\ &= \{r_s(x)+a_nx^{n-m}\}(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s} \\ &\quad + \{r_0(x)+\sum_{j=1}^{s-1}r_j(x)\prod_{i=1}^{j}(x-b_i)^{m_i}\}. \end{aligned}$$

□

If the divisor $g(x)=(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s}$ has some zero roots, $b_s=0$ say, then the formula of quotient on dividing $f(x)$ by $g(x)$ in (4.1) is changed to $q(x)=a_nx^{n-m}$, but the formula of the remainder is unchanged, as follows.

Corollary 4.2. *If $f(x)=a_0+a_1x+\ldots+a_nx^n$, and $g(x)=(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s}$ are polynomials in $F[x]$, where $b_1,b_2,\ldots,b_s$ are all distinct elements of F, $b_s=0$, say, and $\deg g(x)=m\le n=\deg f(x)$, then the quotient and the remainder on dividing $f(x)$ by $g(x)$ is*

$$q(x)=a_nx^{n-m+m_s} \quad \textit{and} \quad r(x)=r_0(x)+\sum_{j=1}^{s-1}r_j(x)\prod_{i=1}^{j}(x-b_i)^{m_i} \tag{4.3}$$

respectively, where $r_j(x)=\sum_{i=0}^{m_{j+1}-1}\frac{(x-b_{j+1})^i q_j^{(i)}(b_{j+1})}{i!}$, $j=0,1,2,\ldots,s-1$.

Proof. Let $g_1(x)=(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s}(x-0)^{m_{s+1}}$, where $m_{s+1}=n-m$, where $b_s=0$, write $g_1(x)=(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s+n-m}$ applying by the formula (3.1), we have

$$\begin{aligned} f(x) &= r_0(x)+r_1(x)(x-b_1)^{m_1}+r_2(x)(x-b_1)^{m_1}(x-b_2)^{m_2}+\ldots \\ &\quad + r_s(x)(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_s)^{m_s+n-m}. \end{aligned} \tag{4.4}$$

where $r_j(x)=\sum_{i=0}^{m_{j+1}-1}\frac{(x-b_{j+1})^i q_j^{(i)}(b_{j+1})}{i!}$ $j=0,1,2,\ldots,s-1$, and $q_j(x)=a_n(x-b_j)^{n-m_1-\ldots-m_j}+\sum_{i=0}^{n-m_1-\ldots-m_j-1}\frac{q_{j-1}^{(m_j+i)}(b_j)}{(m_j+i)!}(x-b_j)^i$, $j=1,2,\ldots,s$.

Consider, in this case the leading coefficient of $f(x)$ is $r_s(x)=a_n$. Then equation (4.4) become

$$f(x)=\left\{a_nx^{n-m+m_s}\right\}(x-b_1)^{m_1}\ldots(x-b_s)^{m_s}+\left\{r_0(x)+\sum_{j=1}^{s-1}r_j(x)\prod_{i=1}^{j}(x-b_i)^{m_i}\right\},$$

we obtain at once the formula (4.3). □

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