REMEMBERING JUN-ITI NAGATA

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Abstract. I share my impressions from Nagata's work and from his person. To me he represents a monumental figure in the history of General Topology in the second half of twentieth century.

For the first time I heard the name of Professor Jun-iti Nagata in 1955. That year Professor P.S. Alexandroff started a new seminar in General Topology at Moscow University, which at the early stage was addressed to undergraduates. More than thirty beginners, 17-19 years old, attended the first meetings of that seminar. I was among them; together with V.I. Ponomarev, B.A. Pasynkov, E.G. Sklyarenko, V.I. Kuzminov, and others. Yu. M. Smirnov was one of Alexandroff's assistants in the leadership of the seminar.

One of the first things we learned at that seminar was that the famous general metrization problem, about thirty years old by that time, was recently solved, in a brilliant manner, by three independent researchers, R.H. Bing (1951), Jun-iti Nagata (1950), and Ju.M. Smirnov (1951), an American, a Japanese, and a Russian.

Why was this solution considered by general topologists to be a great event? In the nature of general topological spaces nothing suggests that their topology should be generated by a metric. In fact, a metric is a rather sophisticated instrument which can be used effectively in studying topologies, but why should it be possible to express conditions for the existence of a metric generating a given topology in natural terms involving purely topological properties? Observe that in the definition of the concept of topology nothing suggests the relevance of any kind of numbers.

Only some partial, or very special, metrizability conditions were known before 1950, even though the general metrizability problem was posed quite early, at the beginning of 1920-ties. P.S. Urysohn's Theorem on metrizability of normal spaces with a countable base was among them.

The General Metrization Theorem of Bing, Nagata, and Smirnov was accepted as a great success because, after 30 years of work on general metrization problem of many ambitious experts, it did exactly what not so many topologists had hopes for: metrizability of a space had been reduced to a simple combination of natural topological properties: a restriction on a base and a restriction on separation in the space. Here it is, Nagata-Smirnov version of the metrizability criterion (see [19], [10]):

Theorem 1. *A topological space X is metrizable if and only if X is a regular T_1-space with a base which can be represented as the union of a countable collection of locally finite families of sets.*

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Nagata's paper [19] containing his solution of the general metrization problem was just his second research paper in topology. He was only 25 years old at that time. He was the youngest among the three co-solvers. He immediately became famous. And I believe that Alexandroff's decision to start a new general topology research seminar at Moscow University could have been influenced by the triple success in solving the metrization problem. This success has become a source of inspiration for Alexandroff's new young students.

I met Professor Nagata personally very early in my career. It was, I think, between 1958 and 1960; unfortunately, I do not remember the precise date. He has delivered a lecture at Moscow Mathematical Society, and we met him and talked to him at a dinner in Alexandroff's apartment at Moscow University.

Very soon I met Nagata again, this time at the First International Prague Symposium on General Topology and its Applications in Algebra and Analysis, in 1961. It was the first major conference in General Topology after the Second World War. Many famous mathematicians attended it. In particular, M.H. Stone, R.H. Bing, E.A. Michael, J. Isbell, M.G. Katětov, R. Anderson, D. Kurepa were there, and many others. This Symposium was very influential; it forged the international cooperation in General Topology which lasts until this day.

At the Symposium I have learned about another major interest of Nagata in Topology: Dimension Theory. In his talk Nagata introduced the concept of rank of a family of subsets. This concept generalized the notion of a base of rank 1 which was used earlier to characterize zero-dimensional metric spaces (in particular, by P. Alexandroff and P. Urysohn). The Nagata rank of a base $\mathcal{B}$ does not exceed a natural number n, if, for every subfamily of $\mathcal{B}$ consisting of $n+1$ elements of $\mathcal{B}$ with a non-empty intersection, at least one of them is contained in the union of the others. The general idea was to use bases of finite rank to study and characterize dimension of spaces. Nagata posed several interesting questions concerning bases of finite rank. I have answered one of his questions (see [1], [2], [3]). Nagata has also published his results (see [23]) in this direction.

This was the first of numerous examples of influence of Nagata's work on my research. Interestingly, Nagata himself did not pursue the idea of rank much further. I believe, this was typical of him: he just wanted to learn how the things are, and he didn't try to keep it all for himself. He often asked good questions, and influenced through that the progress in General Topology. And he was able to coin good new notions, which is indispensable for posing original problems and for opening new areas of research. The concept of a rank was just one of those new concepts. There was a rich following in this direction, including, P. Nyikos, P. Biriukov, and others. See also [26] and [27].

The main stream of Nagata's works after 1955 was in dimension theory and in the theory of generalized metric spaces. The techniques and ideas developed by Nagata and other topologists in their attempts to solve the general metrization problem were most relevant here. Indeed, the method of open coverings, and of other types of coverings, special bases, the concept of paracompactness - a jewel of the method of coverings, – star-refinements and star-finite coverings, strong paracompactness, and so on, play a key role in metrization theory as well as in the dimension theory. After all, the covering dimension, introduced by Lebesgue, is defined in terms of open covers.

Nagata's contribution to dimension theory was enormous. Often it was directly related to metrization theory. Here are some of his results of this kind:

Theorem 2. *[26] There is a universal space with respect to topological embeddings in the class of metrizable spaces with* $\dim \leq n$ *and the weight* $\leq \tau$*, for each natural number* n *and for each cardinal number* τ*.*

Theorem 3. *[23] A metrizable space X satisfies the condition* $\dim X \leq n$ *if and only if X has a base of rank $\leq n$.*

Theorem 4. *[31] A metrizable space X has* $\dim \leq n$ *if and only if X admits a metric ρ such that for every positive ϵ the family of all ϵ-balls is closure-preserving, and the dimension dim of the boundary of every ϵ-ball is $\leq n-1$.*

Theorem 5. *[24], [32], [31] A metrizable space X has* $\dim \leq n$ *if and only if X admits a metric ρ such that for every sequence $x, y_1, ..., y_{n+2}$ (of length $n+3$) of points of X there are distinct i and j satisfying the next condition: $\rho(y_i, y_j) \leq \rho(x, y_j)$.*

See Nagata's own comments on this Theorem in [31].

Theorem 6. *[20] Each metrizable space such that* $\dim X = n$ *is embeddable in the topological product of n+1 metrizable* 1*-dimensional spaces.*

Observe, in connection with this remarkable result, that the two-dimensional sphere cannot be embedded into the product of two 1-dimensional spaces (K. Borsuk, see [11], p. 275).

One of important general aspects of Dimension Theory is the study and classification of infinite-dimensional spaces. J. Nagata has been working very actively in this direction. Here is a result obtained by him jointly with J. de Groot (see [11], p. 49):

Theorem 7. *Suppose that X is an infinite-dimensional complete separable metric space, and that X is the union of countably many of finite-dimensional subspaces. Then for each $n \in \omega$ the space X contains a closed subspace M such that* $\mathrm{ind}(M) = n$.

Many deep results on dimension theory by Nagata himself and by many other mathematicians can be found in his very rich book on Dimension Theory. Its first edition appeared in 1965. This book and its second, revised, edition (see [26]) have made a strong impact on development of Dimension Theory, many topologists in all countries were strongly influenced by it in their research. I also believe that the book has stimulated the other mathematicians, P.S. Alexandroff, B. A. Pasynkov, R. Engelking, to write their monographs on Dimension Theory. I remember that in Moscow, at the main topological seminar unifying all Moscow topologists (called in Russian "kruzhok") Nagata's book has been extensively reviewed (probably, by Smirnov), it took several meetings of the seminar.

One common feature of the theory of metrizable spaces and of dimension theory is that both have a fundamental significance for General Topology. They represent generic roots of General Topology. The ideas of Dimension Theory were born in the works of Poincaré, Brouwer, Lebesgue, and were developed by P.S. Urysohn, W. Hurewicz and others. Here even the basic definitions are quite non-trivial, and an amazing fact is that the algebraic approach to dimension, the mixed topologo-algebraic approach, and the three approaches of General Topology to defining the dimension (one covering and two inductive) all lead to the same number in the very wide class of all separable metrizable spaces (see [26], [11]). The space does not even have to be complete!

Looking now at Jun-iti Nagata's research, we see that he was always working on the most fundamental problems belonging to the main stream of General Topology, that he dealt with most natural and basic concepts. On the whole, what he did was strategically important for the development of General Topology. See in this connection [5], [14], [8], and [6].

This is also confirmed by Nagata's remarkable contributions to another fundamental direction of research: to the theory of continuous mappings. We mention here just two of his results of this kind.

Theorem 8. *[25] A Tychonoff space X admits a perfect mapping onto a metrizable space if and only if X is homeomorphic to a closed subspace of the topological product $Y \times Z$, where Y is a metrizable space and X is a compact Hausdorff space.*

The next result was announced by Hurewicz in 1928 (see [11], p.71).

Theorem 9. *[21] A separable metrizable space X can be represented as a continuous image of a zero-dimensional separable metric space under a closed mapping with finite fibres if and only if X is the union of a countable family of finite-dimensional subspaces.*

There are many other results of Jun-iti Nagata, which are now well known and have become a part of textbooks on General Topology. They all deal with fundamental facts and properties, such as additivity of metrizability, strong paracompactness, the product operation, hereditary normality, and others. See [10] and [27] for a discussion of some of these results briefly mentioned below. Each of them had already a good following. See also [13], [15], and [33].

If a space X can be covered by a locally finite family of closed metrizable subspaces, then X is metrizable. If a Čech-complete paracompact space X has a G_δ-diagonal, then X is metrizable. The product space $[0,1] \times B(\aleph_1)$ is not strongly paracompact. A metrizable space which is the union of a locally finite countable family of strongly paracompact closed subspaces is not necessarily strongly paracompact. Notice that one of the first to recognize the importance of strongly paracompact spaces was Professor K. Morita [16].

Even though Nagata's interests in General Topology were very rich, I venture to suggest that the closest to his heart was the theme of metrizability. In the last period of his research, after 2000, he comes back to this theme, and publishes several interesting articles bringing forward some fresh ideas and techniques [29], [30], [31]. In particular, one finds there the next lemma concerning symmetric open assignments on a space. Since this concept is not widely known, we recall it.

A family $\{U(x) : x \in X\}$ is a *symmetric open neighborhood assignment* on a topological space X, if $U(x)$ is an open neighborhood of x, and $x \in U(y)$ if and only if $y \in U(x)$. Open (not necessarily symmetric) assignments were used by Eric van Douwen to introduce an original and important class of D-spaces delicately related to various general metrizability properties, such as paracompactness and stratifiability (see [9], [7]). Nagata's idea, probably, was to use the concept of a symmetric open neighborhood assignment to characterize metrizability and some other metrizability type properties.

The next statement is a valuable piece of technique.

Lemma 10. *[29] If $\eta = \{U(x) : x \in X\}$ is a point-finite symmetric open neighborhood assignment on a topological space X, then $U(x) = St(x, \gamma)$, for $x \in X$, where $\gamma = \{V(p,q) : q \in U(p)\}$, and $V(p,q) = \cap\{U(z) : p, q \in U(z)\}$.*

Let us call a regular space X *strongly metrizable* if it has a σ-star-finite base.

Theorem 11. *[29] A regular space X is strongly metrizable if and only if X has a sequence of symmetric point-finite open assignments $\{U_n(x) : x \in X\}$, $n \in \omega$, such that $\{U_n(x) : n \in \omega\}$ is a base of X at x, for each $x \in X$.*

An unusual feature of this criterion is that the key concept of local finiteness is not used. Instead, the much more elementary concept of point-finiteness is present.

In the last article [31] Nagata discussed some important classes of generalized metric spaces, such as $\aleph$-spaces, stratifiable spaces, some important instruments, like g-functions, k-networks, further sophisticated metrizability criteria, some basic results on them, and questions remaining open, despite his own effort. Nagata's comments in this paper provide a glimpse of his personal attitude to research. See, in particular, his comments at the end of the article and on Theorem 4.1.

Now let me come to the very first article of Nagata [18]. It was published even before his famous metrization paper [19]. And, in my opinion, it is also a very influential paper.

In [18], Jun-iti Nagata established the foundation for what later became known as C_p-theory, or theory of $C_p(X)$-spaces. Recall that $C_p(X)$ is the space of continuous real-valued functions on a Tychonoff space X in the topology of pointwise convergence. This space can be naturally considered as a topological ring. Nagata has proved the following basic result:

Theorem 12. *Suppose that X and Y are Tychonoff spaces such that $C_p(X)$ and $C_p(Y)$ are topologically isomorphic as topological rings. Then the spaces X and Y are homeomorphic.*

This result clearly shows that the topology of pointwise convergence has some great advantages over the topology of uniform convergence in certain matters. But its principal significance lies in the fact that the natural ring structure of $C(X)$ combined with the topology of pointwise convergence is responsible for every topological property of a Tychonoff space X. Therefore, we may look for natural characterizations of properties of X in terms of properties of $C_p(X)$ some of which can be topological, some algebraic. This approach is the door to C_p-theory (see [4]), and the key which opens this door is Nagata's theorem above. Nagata himself commented on Theorem 12 in [31], discussed some related results from [18], and posed some open questions, one of which is to characterize metrizability of X in terms of any natural structures on $C(X)$ (topology, order, algebraic operations). Thus, Nagata is one of the founding fathers of C_p-theory.

After 1961 I met Nagata quite a few times, mostly at various conferences. I was invited by him to his house in 1965, when I came for a brief visit to Pittsburgh. I remember that Nagata asked me about the topics on which I was working recently. I told him that I had characterized preimages of metrizable spaces under perfect mappings. He immediately replied that Kiiti Morita has published a paper in 1964 with a solution of the same problem. Fortunately, I could reply that my first paper on this matter, containing all essential components of my solution, was published in Doklady AN SSSR in 1963. This was the first paper on p-spaces. In this way I have learned that Professor Morita and myself have done, independently, some research in very close directions.

I met Nagata several times in Prague at the famous and important Prague Symposia in General Topology. On one occasion, we went together to drink beer in some bar (Nagata knew the place where to go), on another occasion I remember traveling with him on a vessel along a river in Prague with a crowd of topologists (we had a party on the boat), and Nagata asked me about Ju.M. Smirnov; in particular, he wanted to know whether it was true that Smirnov served during the war as an army sailor on some submarine. That was true, Smirnov did indeed serve for some time as a submarine sailor (a radio specialist) during the Second World War. Nagata told me that he also was an army sailor on a submarine during the World War. There was something strange in this coincidence. Nobody knew at that time that the two brilliant topologists who have discovered the Great Metrization Theorem almost simultaneously, working at great distance from each other, will die in the same year, 2007, just less than two months apart.

When Nagata came back from Amsterdam to Japan (he was a very conservative man, in my opinion, a man of tradition, as most of Japanese people are, I believe, so he must have been missing Japan all the time greatly), he has initiated Japanese-Soviet Symposia in General Topology, taking place every second year, alternatively, in Japan and in USSR. Nagata used these Symposia to invite general topologists from Soviet Union who he knew well because of numerous intersections in research interests. I have participated in these and other meetings with Japanese topologists several times. Once a few of us, including

Pasynkov, Ponomarev, Fedorchuk, Choban, and myself, came to such a meeting. We immediately went to the sea and had a good swim, despite the fact that it was quite windy, and the sea was a bit rough. The next day we all got a written note from Nagata of somewhat authoritarian nature, exposing dangers of swimming in a sea at an unknown place during a dubious weather. "We do not want you to become five Urysohns", wrote Nagata. On another occasion, Nagata gave each of five of us a fun certificate confirming that we are five samurais, this was to celebrate the fact that we were able to overcome all hurdles with traveling abroad that existed in Soviet Union at the time and were able to come to the conference in time.

This was related to another accident. To one of the first Soviet-Japanese conferences in General Topology, out of all Soviets invited, only myself was able to come (I have used for this travel a tourist agency, and others applied to some Ministry to pass the formalities, and failed to get a permission to travel).

Nevertheless, the conference took place and was a success. I had to make a few more talks, there were very interesting lectures by Japanese topologists, and many pictures were taken. I keep a few photo albums of the conference at home. At that conference, for the first time, I have met Professor K. Morita, a great Japanese mathematician famous among general topologists and algebraists alike. There was a small party at Professor Morita's house, were Professor Nagata was also present, and, during the dinner, at some point, Morita have asked for another supply of sake to be brought in.

Russian-Japanese Symposia on General Topology had a considerable impact on promoting General Topology and collaboration between scientists. The two major schools in General Topology, Japanese and Russian, came much closer, and Nagata was the leading figure in organizing the Symposia.

It is a pity that later, due to difficulties caused by "perestrojka" in Russia and to the lack of leadership on Russian side, these Symposia have stopped and were replaced by Japanese-Mexican Symposia in Topology.

When we now look back at Jun-iti Nagata's role in development of General Topology, he emerges as a monumental leading figure in this field in the second half of twenties century.

Nagata's great personal research achievements alone, briefly described above, would have been sufficient to make him a celebrated figure in the history of General Topology. But he has also written two original and influential books on General Topology (one of them in Dimension Theory). He has edited, with K. Morita, an advanced collection of surveys on General Topology [17], and in the very last years he has edited, in collaboration with K.P. Hart and J.E. Vaughan, an Encyclopedia of General Topology [12], to which many experts from all over the world have generously contributed, many Japanese topologists among them. Nagata has helped to organize many symposia and conferences on General Topology, he has been in the founding group of editors of the international journal "Topology and its Applications" (established in 1971). In later years, when Nagata returned to Japan, he has created the journal "Questions and Answers in General Topology" which has already gained strong international recognition and attracts general topologists, including top level experts, from Japan and from many foreign countries as well. All these deeds constitute a great service of Professor Nagata to the profession.

General Topology has always been Jun-iti Nagata's passion, and this passion has now become a part of General Topology itself.

References

[1] A.V. Arhangel'skii , *The ranks of systems of sets and the dimension of spaces*, Dokl. Akad. Nauk SSSR 143 (1962), 755-758 (English translation in Soviet Math. Dokl. 3 (1962), 456-459).

[2] A.V. Arhangel'skii , *k-dimensional metrizable spaces*, Vestnik Moskov. Univ. Ser. 1 Mat. Meh. 2 (1962), 3-6 (English translation in Mosc. Univ. Math. Bull.).

[3] A.V. Arhangel'skii , *Ranks of families of sets and dimension of spaces*, Fund Math., 52 (1963), 257-275 (in Russian).

[4] A. V. Arhangel'skii, *Topological function spaces*, Math. Appl., vol. 78, Kluwer Academic Publisher, Dordrecht, 1992.

[5] Z. Balogh, G. Gruenhage, *When the collection of ϵ-balls is locally finite*, Topology Appl. 124 (2002), 445-450.

[6] J. Bruijning, J. Nagata, *A characterization of covering dimension by use of* $\Delta_k(X)$, Pacific J. Math. 80 (1979), 1-8.

[7] R.Z. Buzyakova, *Hereditary D-property of Function Spaces Over Compacta*, Proc. Amer. Math. Soc. 132 (2004), 3433-3439.

[8] P.J. Collins, A.W. Roscoe, *Criteria for metrizability*, Proc. Amer. Math. Soc. 90 (1984), 631-640.

[9] E. van Douwen and W.F. Pfeffer, *Some properties of the Sorgenfrey line and related spaces.* Pacific J. Math. 81:2 (1979), 371-377.

[10] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[11] R. Engelking, *Dimension Theory*, PWN, Warszawa, North-Holland, Amsterdam, 1978.

[12] K.P. Hart, J. Nagata, and J.E. Vaughan (Eds.), *Encyclopedia of General Topology*, Elsevier, North-Holland, Amsterdam-London, 2004.

[13] Y. Hattori, *On special metrics characterizing topological properties*, Fund. Math. 126 (1986), 133-145.

[14] Y. Kanatani, N. Sasaki, J. Nagata, *New characterizations of some generalized metric spaces*, Math. Japon. 30 (1985), 805-820.

[15] T. Mizokami, N. Shimane, *On Fréchet M_3-spaces*, Math. Japon. 50 (1999), 391-399.

[16] K. Morita, *Star-finite coverings and star-finite property*, Math. Japonicae 1 (1948), 60-68.

[17] K. Morita, J. Nagata (Eds.), *Topics in General Topology*, North-Holland, Amsterdam, 1989.

[18] J. Nagata, *On lattices of functions on topological spaces and of functions on uniform spaces*, Osaka Math. J. 1(1949), 166-181.

[19] J. Nagata, *On a necessary and sufficient condition of metrisability*, J. Inst. Polytech. Osaka City Univ. 1 (1950), 93-100.

[20] J. Nagata, *Note on dimension theory for metric spaces*, Fund. Math. 45 (1958), 143-181.

[21] J. Nagata, *On countable sum of zero-dimensional spaces*, Fund. Math. 48 (1960), 1-14.

[22] J. Nagata, *On a universal n-dimensional set for metric spaces*, Journal fur die Reine und Angew. Math. 204 (1960), 132-138.

[23] J. Nagata, *Two theorems for the n-dimensionality of metric spaces*, Comp. Math. 15 (1963), 227-237.

[24] J. Nagata, *On a special metric characterizing a metric space of* $\dim \leq n$, Proc. Japan Acad. 39 (1963), 278-282.

[25] J. Nagata, *A note on M-space and topologically complete space*, Proc. Japan Acad. 45 (1969), 541-543.

[26] J. Nagata, *Modern Dimension Theory*, revised ed, Heldermann, Berlin, 1983.

[27] J. Nagata, *Modern General Topology*, 2-nd rev. ed., North-Holland, Amsterdam, 1985 .

[28] J. Nagata, *Generalized metric spaces 1*, in: K. Morita, J. Nagata (Eds.), Topics in General Topology, North-Holland, Amsterdam, 1989

[29] J. Nagata, *Symmetric Neighbourhood assignments and metrizability*, Q and A in General Topology 22 (2004), 181-184.

[30] J. Nagata, *Some questions concerning symmetric neighborhood assignments*, Q and A in General Topology 22 (2004), 5-8.

[31] J. Nagata, *Open problems left in the wake of my research*, Topology and Appl. 146-147 (2005), 5-13.

[32] P.A. Ostrand, *A conjecture of J. Nagata on dimension and metrization*, Bull. Amer. Math. Soc. 71 (1965), 623-625.

[33] M. Sakai, K. Tamano, Y. Yajima, *Regular networks for metrizable spaces and Lashnev spaces*, Bull. Pol. Acad. Sci. Math. 46 (1998), 121-133.

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