ON PETRI NETS GENERATING ALL THE BINARY n-VECTORS

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Abstract. In this paper, we construct a 1-safe Petri net that generates all the binary n-vectors, possibly with repetitions.

1. Introduction

Carl Adam Petri[7] proposed that a discrete event system can be modeled effectively as a net, which is a 4-tuple $C = (P, T, F, \mu^0)$
where,
(a) P is a finite set of $'places'$
(b) T is a finite set of $'transitions'$
(c) $P \cup T \neq \emptyset$, $P \cap T = \emptyset$,
(d) $F \subseteq (P \times T) \cup (T \times P)$, called *flow relation* or *causal dependency* relation, is chosen such that $\mathtt{dom}\, F \cup \mathtt{codom}\, F = P \cup T$, where

$$\mathtt{dom}\, F = \{x \in P \cup T \mid y \in P \cup T \,:\, (x, y) \in F\},$$

$$\mathtt{codom}\, F = \{x \in P \cup T \mid y \in P \cup T :\, (y, x) \in F\},\ and$$

(e) $\mu^0 : P \longrightarrow N$, called the *initial marking*, is a function that assigns to each place p_i a non-negative integer $\mu^0(p_i)$ often referred to as being the number of *tokens* placed at p_i. Most of the work on Petri nets since then has been based on the final report of Holt et.al.[4] to the U.S. Air Force which translated Petri's dissertation into English as well as extended the work considerably (also see [7]). Petri net can be used as a visual communication tool similar to flowchart, block diagram or a network. Theoretically, Petri nets have been used as a powerful and convenient tool for representing and studying the structure of decision making processes which can often be tricky or complex. The development of high speed computers has greatly enhanced the use of Petri nets in diverse fields(e.g., see [7]). In this paper, we construct a Petri net that generates all the binary n-vectors, possibly with repetitions. Such a system is often required in practice to construct control systems (e.g., see [1,2]).

2. Preliminaries

For standard terminology and notation on Petri nets, we refer the reader to Peterson [8]. Jenson [6] has given the following more operative definition of a Petri net, which we shall adopt in this paper.
A Petri net is a 5-tuple $C = (P, T, I^-, I^+, \mu^0)$, where
(a) P is a nonempty set of $'places'$,
(b) T is a nonempty set of $'transitions'$,
(c) $P \cap T = \emptyset$,

(d) I^-, $I^+ : P \times T \longrightarrow N$, where N is the set of nonnegative integers, are called the negative and the positive incidence functions respectively,

(e) $\forall\ p \in P,\ \exists\ t \in T\ :\ I^-(p,t) \neq 0\ or\ I^+(p,t) \neq 0$

and

$$\forall\ t \in T, \exists\, p \in P\ \ :\ I^-(p,t) \neq 0\ or\ I^+(p,t) \neq 0$$

(f) $\mu^0 : P \to N$ is the *initial marking.*

In fact, $I^-(p,t)$ and $I^+(p,t)$ represent the number of arcs from p to t and t to p respectively. (I^-, I^+, μ^0 can be viewed as matrices of size $|P| \times |T|$, $|P| \times |T|$, $|P| \times 1$, respectively). Petri nets have a well known graphical representation in which transitions are represented as boxes and places as circles with directed arcs interconnecting places and transitions to represent the flow relation. The initial marking is represented by placing a token in the circle representing a place p_i as a black dot whenever $\mu^0(p_i) = 1$, $1 \leq i \leq n = |\ P\ |$.

In general, a marking μ is a mapping $\mu : P \longrightarrow N$. A marking μ can hence be represented as a vector $\mu \in N^n$, $n = |P|$, such that the ith component of μ is the value $\mu(p_i)$.

Let $C = (P,\ T,\ I^-,\ I^+,\ \mu)$ be a Petri net. A transition $t \in T$ '$fires$' at the marking μ (or it is *enabled* at μ) iff $I^-(p,t) \leq \mu(p)$, $\forall\, p \in P$. After firing at μ, the new marking μ' is given by the rule

$$\mu'(p) = \mu(p) - I^-(p,t) + I^+(p,t), \text{ for all } \quad p \in P.$$

We say t fires at μ to yield μ' (or t *fires* μ to μ'), and we write $\mu \xrightarrow{t} \mu'$, whence μ' is said to be directly reachable from μ. Hence, it is clear, what is meant by a sequence like

$$\mu^0 \xrightarrow{t_1} \mu^1 \xrightarrow{t_2} \mu^2 \xrightarrow{t_2} \mu^3 \cdots \xrightarrow{t_k} \mu^k,$$

which simply represents the fact that the transitions $t_1, t_2, t_3, \ldots, t_k$ have been successively fired to transform the marking μ^0 into the marking μ^k. The whole of this transformation is also written in short as $\mu^0 \xrightarrow{\sigma} \mu^k$, where $\sigma = t_1,\ t_2,\ t_3,\ \ldots,\ t_k$.

A marking μ is said to be reachable from μ^0, if there exists a sequence of transitions which can be successively fired to obtain μ from μ^0. The set of all markings of a Petri net C reachable from a given marking μ is denoted by $R(C, \mu)$.

A place in a Petri net is *safe* if the number of tokens in that place never exceeds one. A Petri net is safe if all its places are safe.

3.1-Safe Star Petri Net

The notion of a *star* is from graph theory[3]; it is the *complete bipartite graph* $K_{1,n}$ which consists of exactly one vertex c, called the *center*, joined by a single edge cv_i to the pendant vertex v_i (i.e. the degree of v_i is 1) for each $i \in \{1, 2, \ldots, n\}, n \geq 1$. A *1-safe star Petri net* S_n is obtained by *subdividing* every edge of the graph $K_{1,n}, n \geq 1$, so that every subdividing vertex is a place node and the original vertices of $K_{1,n}, n \geq 1$, are the $(n+1)$ transition nodes, $(n+1)^{th}$ being the central node. Further, every arc incident to the central node is directed towards it, and every arc incident to a pendent node is directed towards the pendent node. The general configuration of a *1-safe star* Petri net is shown in Fig.1.
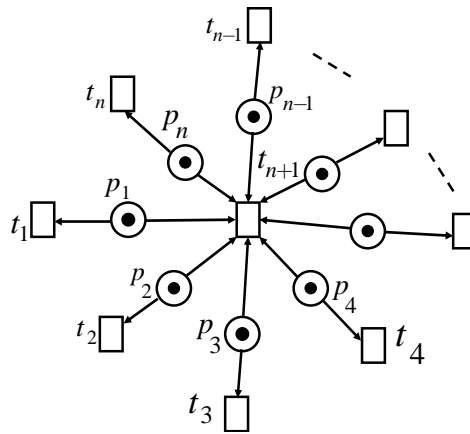

Figure 1: 1-safe star Petri net S_n

Now we prove the following theorem which is the main result of this paper.

Theorem : *The reachability tree of S_n with $\mu_0 = (1, 1, 1, 1, \ldots, 1)$ as the initial marking contains every binary n-vector $(a_1, a_2, a_3, \ldots, a_n), a_i \in \{0, 1\}$.*

Proof: We shall prove this result by using the Principle of Mathematical Induction (PMI). Clearly, the reachability tree $R(S_1, \mu^0)$ of S_1 generates both the binary 1-vectors (1) and (0) as shown in Figure 2. Next, consider the 1-safe star Petri net S_2 as shown in Figure 3 and its $R(S_2, \mu^0)$ displayed in Figure 4.
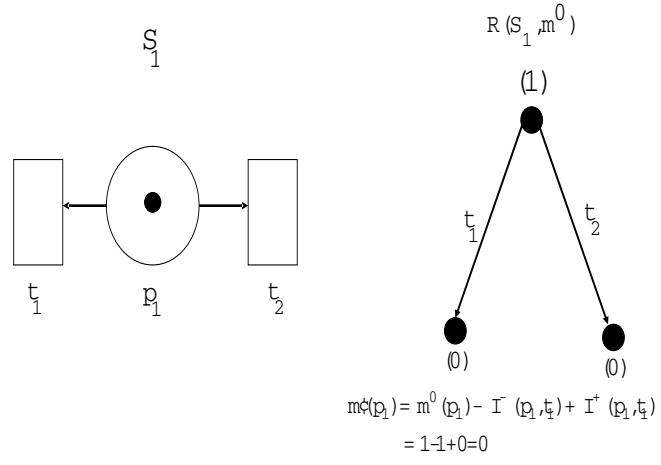
S_1

$R(S_1, m^0)$

(1)

t_1 t_2

(0) (0)

$$m¢(p_1) = m^0(p_1) - I^-(p_1, t_1) + I^+(p_1, t_1)$$
$$= 1 - 1 + 0 = 0$$

t_1 p_1 t_2

Figure 2:

t_1 t_2

S_2 : p_1 p_2

t_3
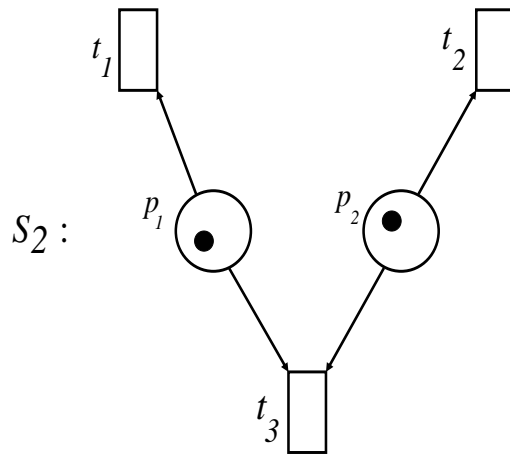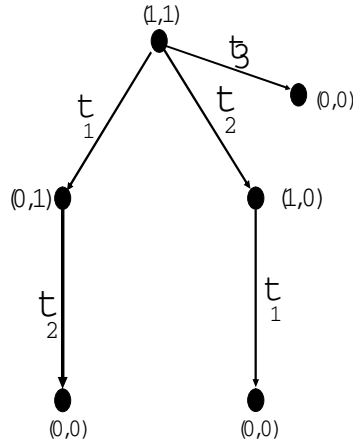
Figure 3: 1-safe star Petri net S_2

Figure 4: $R(S_2,\ \mu^0),\ \mu^0 = (1,1)$

It is clear from Figure 4 that $R(S_2,\mu^0)$ has all the $4 = 2^2$, binary 2-vectors (a_1, a_2), a_1, $a_2 \in \{0,1\}$. We can construct $R(S_2,\mu^0)$ from $R(S_1,\mu^0)$ as follows. Take two copies of $R(S_1,\mu^0)$. In the first copy, augment each vector of $R(S_1,\mu^0)$, by putting a 0 entry at the second position of every marking vector and denote the resulting labeled tree as $R_0(S_1,\mu^0)$. Similarly, in the second copy, augment each vector by putting 1 at the second position of every marking and let $R_1(S_1,\mu^0)$ be the resulting labeled tree (See Figure 5).
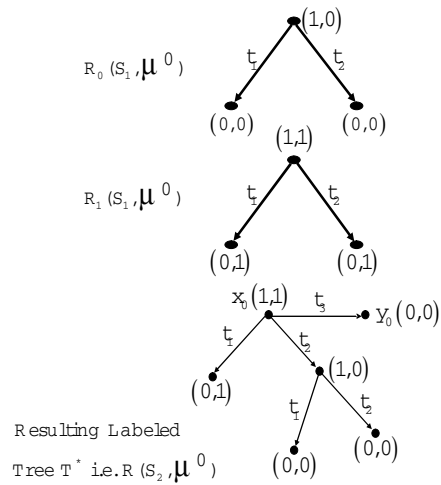

Figure 5:

Now, using the following steps we construct the reachability tree $R(S_2, \mu^0)$ of S_2 from $R_0(S_1, \mu^0)$ and $R_1(S_1, \mu^0)$.

1. Clearly, the set of binary 2-vectors in $R_0(S_1, \mu^0)$ is disjoint with the set of those appearing in $R_1(S_1, \mu^0)$ and together they contain all the binary 2-vectors.

2. In $R_0(S_1, \mu^0)$, transition t_2 does not satisfy the enabling condition, since $I^-(p_i, t) \leq \mu(p_i)$, for each $p_i \in S_1$ is violated. So, we can ignore this transition at this stage.

3. In $R_1(S_1, \mu^0)$, transition t_2 is enabled and the marking obtained after firing of t_2 is actually $(1, 0)$ whereas the augmented vector attached to this node is $(0,1)$. So, we concatenate $R_0(S_1, \mu^0)$ by fusing the node labeled $(1, 0)$ with the node labeled $(0, 1)$ in $R_1(S_1, \mu^0)$ and replacing $(0, 1)$ by the label $(1, 0)$ which is the initial marking of $R_0(S_1, \mu^0)$.

4. We then augment an extra pendent node labeled y_0 joined to the new root node x_0 labeled by the 2-vector $(1, 1)$ by the new arc (x_0, y_0) labeled as t_3. The resulting labeled tree T^* has all the binary 2-vectors as its node labels, possibly with repetitions. It remains to show that it is the reachability tree $R(S_2, \mu^0)$ of S_2 with 2-vector $(1, 1)$ as its initial marking μ^0. For this, consider an arbitrary 2-vector $\mu = (a_1, 1)$, where $a_1 \in \{0, 1\}$. When transition t_2 is enabled, this yields

$$\mu'(p_i) = \mu(p_i) - I^-(p_i, t_2) + I^+(p_i, t_2)$$

$$= 1 - 1 + 0 = 0$$

Then, we get a new marking $\mu' = (a_1, 0)$, where $a_1 \in \{0, 1\}$. The marking μ' is found in $R_0(S_2, \mu^0)$. If all a_i's are zero then μ' is a dead marking. Hence, suppose some $a_i \neq 0$. In this case, t_i is enabled and in the next new marking μ'', the ith component is reduced to zero. Eventually, this process will lead to a dead marking. Further, the marking vectors of the form $\mu = (a_1, 0)$ are already obtained as a result of firing t_1, t_2, through some subsequences. Thus, T is indeed the reachability tree $R(S_2, \mu^0)$ of S_2.

Now, we assume that the result is true for all the 1-safe star Petri nets S_k having k-places, $k \leq n$. We will prove the result for the 1-safe star Petri net S_{n+1} having $(n + 1)$ places. For this purpose, consider two copies of the reachability tree of $R(S_n, \mu^0)$ of S_n. In the first copy, we extend each vector by augmenting a 0 entry at the $(n + 1)^{th}$ position and let $R_0(S_n, \mu^0)$ denote the resulting labeled tree. Next, in the second copy of $R(S_n, \mu^0)$, we augment the entry 1 to the $(n + 1)^{th}$ position in every marking vector and let $R_1(S_n, \mu^0)$ be the resulting labeled tree. Hence, using the following steps we construct the reachability tree of the 1-safe star Petri net S_{n+1} having $(n + 1)$ places.

1. Clearly, the set of binary (n+1)-vectors in $R_0(S_n, \mu^0)$ is disjoint with the set of those appearing in $R_1(S_n, \mu^0)$ and together they contain all the binary (n+1)-vectors.

2. In $R_0(S_n, \mu^0)$, transition t_{n+1} does not satisfy the enabling condition, since $I^-(p_i, t) \leq \mu(p_i)$, for each $p_i \in S_n$ is violated. So, we can ignore this transition for the moment.

3. In $R_1(S_n, \mu^0)$, transition t_{n+1} is enabled and the marking obtained after firing of t_{n+1} is actually $(1, 1, 1, \ldots, 0)$. So we concatenate $R_0(S_n, \mu^0)$ at this node with the (n+1)-vector $(0, 0, 0, \ldots, 1)$ replaced by the actual marking $(1, 1, 1, \ldots, 0)$ being the initial marking of $R_0(S_n, \mu^0)$.

4. We then augment an extra pendent node labeled y_0 joined to the new root node x_0 labeled by the (n+1)-vector $(1, 1, 1, \ldots, 1)$ by the new arc (x_0, y_0) labeled as t_{n+2}. The tree T^* so obtained has all the binary (n+1)-vectors as its node labels, possibly with repetitions. It remains to show that T^* is indeed the reachability tree $R(S_{n+1}, \mu^0)$ of S_{n+1} with binary

(n+1)-vector $(1, 1, 1, \ldots, 1)$ as its initial marking μ^0. For this, consider an arbitrary (n+1)-vector

$\mu = (a_1, a_2, a_3, \ldots, a_n, 1)$, where $a_i \in \{0,1\}, \forall\ i$. When transition t_{n+1} is enabled, this yields

$$\mu'(p_i) = \mu(p_i) - I^-(p_i,\ t_{n+1}) + I^+(p_i,\ t_{n+1})\ = 1 - 1 + 0 = 0$$

Then, we get a new marking $\mu' = (a_1, a_2, a_3, \ldots, a_n, 0)$, where $a_i \in \{0, 1\}$. The marking μ' is found in $R_0(S_{n+1}, \mu^0)$. If all a_i's are zero, then μ' is a dead marking. Hence, suppose some $a_i \neq 0$. In this case, t_i is enabled and in the next new marking μ'', the ith component is reduced to zero. Eventually, this process will lead to a dead marking. Further, the marking vectors of the form $\mu = (a_1, a_2, a_3, \ldots, a_n, 0)$ are already obtained as a result of firing $t_1, t_2, t_3, \ldots, t_n$ through some subsequences by virtue of the hypothesis of the PMI. Thus, T^* is precisely the reachability tree $R(S_{n+1}, \mu^0)$ of S_{n+1}. Hence, the result follows by PMI.   □

Remark

We also have an alternative proof for the above theorem using incidence matrix of S_n. As we know when I^- and I^+ can be viewed, as matrices of size $|P| \times |T|$, the incidence matrix of any Petri net is defined by $I = I^+ - I^-$. The incidence matrix of S_n is

$$- \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

The rth column of the incidence matrix will have only one non-zero entry at the rth place and the last column of the incidence matrix is $(1, 1, 1, \ldots, 1)$. So the incidence matrix of S_n contains the column vectors $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$ $\ldots, (0, 0, 0, \ldots, 1)$, and $(1, 1, 1, \ldots, 1)$. The first of these n-vectors can be viewed as a basis, where the initial marking vector $(1, 1, 1, \ldots, 1)$ can be written as a linear combination of these vectors. Similarly, any binary n-vector can be expressed as a linear combination of these basis vectors. Here, the matrix I generates all the vectors having 0, 1 entries. Thus the incidence matrix of S_n generates every binary n-vector $(a_1, a_2, \ldots, a_n)$, $a_i \in \{0, 1\}$.

4.Conclusions

In this paper, we have constructed a 1-safe Petri net that generates all the binary n-vectors, possibly with repetitions. There do exist other 1-safe Petri nets that generate all the binary n-vectors. For instance, the 1-safe Petri net with three places and four transitions shown in Figure 6, also generates all the binary 3-vectors.This throws open not only the general problem of determining all such Petri nets but also raises the question of determining such optimal Petri nets ; for example, one can ask

1. Precisely which Petri nets produce the set of all binary n-vectors with minimum repetitions ?

2. Precisely which Petri nets produce all the binary n-vectors in the smallest possible number of steps ? As pointed out, these questions could be quite important from practical application point of view.
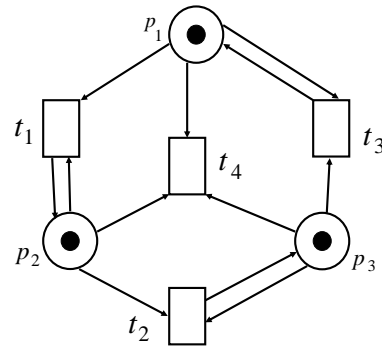
Figure 6

Acknowledgement.

The authors deeply acknowledge the valuable suggestions made by Dr. B. D. Acharya from time to time in carrying out the work of this paper.

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