ON FREE ORDERED SEMIGROUPS

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Abstract. In this paper, using the concept of pseudoorder, we first prove that each ordered semigroup can be considered as a quotient of a free ordered semigroup. Then we give a characterization of free ordered semigroups.

1 Introduction and prerequisites. Jean-Eric Pin and Pascal Weil defined the free ordered semigroup over an ordered set $(X, \leq_X)$ as the free semigroup (in the usual sense [1,2]) over X, that is the set $F_X := \{(x_1, x_2, ..., x_n) \mid n \in N \text{ and } x_i \in X, i = 1, 2, ..., n\}$ with the operation of juxtaposition, endowed with the order $(x_1, x_2, ..., x_n) \preceq (y_1, y_2, ..., y_m) \Longleftrightarrow n = m$ and $x_i \leq_X y_i \ \forall\, i = 1, 2, ..., n$ [6]. According to Proposition 2.1 in [6], this definition satisfies the "universal" property, and it is useful for studying the wreath product of ordered semigroups considered in [6]. In the present paper we give a more general definition of free ordered semigroups, which generalizes the concept of free semigroups, and it seems to be useful for studying the structure of free ordered semigroups. By definition, the free ordered semigroups have the "universal" property. According to this definition, there exists at least one free ordered semigroup over a set X (which is the free ordered semigroup considered by Pin-Weil), the others being isomorphic to each other. We first prove that the ordered semigroup of integers can be considered as a quotient of a free ordered semigroup. Then, using the concept of pseudoorder, we prove that each ordered semigroup is isomorphic to a quotient of a free ordered semigroup. It might be noted that the concept of pseudoorder, first introduced in [3], plays an essential role in studying the structure of ordered semigroups. We finally give a characterization of free ordered semigroups.

Let $(S, ., \leq)$ be an ordered semigroup. A relation σ on S is called *pseudoorder* if

1) $\leq \subseteq \sigma$.

2) $(a, b) \in \sigma$ and $(b, c) \in \sigma \Longrightarrow (a, c) \in \sigma$.

3) $(a, b) \in \sigma \Longrightarrow (ac, bc) \in \sigma$ and $(ca, cb) \in \sigma \ \forall\, c \in S$.

If σ is a pseudoorder on S, then the relation $\overline{\sigma}$ on S defined by

$$\overline{\sigma} := \{(a, b) \in S \times S \mid (a, b) \in \sigma \text{ and } (b, a) \in \sigma\} \ (= \sigma \cap \sigma^{-1})$$

is a congruence on S. The set of all $\overline{\sigma}$-classes $S/\overline{\sigma}$ of S with the multiplication "." and the order " $\preceq$ " on $S/\overline{\sigma}$ defined by:

$(x)_{\overline{\sigma}}.(y)_{\overline{\sigma}} := (xy)_{\overline{\sigma}}$

$(x)_{\overline{\sigma}} \preceq (y)_{\overline{\sigma}} \Longleftrightarrow \exists\, a \in (x)_{\overline{\sigma}},\ b \in (y)_{\overline{\sigma}} : (a, b) \in \sigma$

is an ordered semigroup. Two equivalent definitions of the order " $\preceq$ " on $S/\overline{\sigma}$ are the following:

(a) $(x)_{\overline{\sigma}} \preceq (y)_{\overline{\sigma}} \Longleftrightarrow \forall\, a \in (x)_{\overline{\sigma}}\ \forall\, b \in (y)_{\overline{\sigma}}$, we have $(a, b) \in \sigma$.

(b) $(x)_{\overline{\sigma}} \preceq (y)_{\overline{\sigma}} \Longleftrightarrow (x, y) \in \sigma$ [3].

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Notation 1.1. [5] For each homomorphism f of an ordered semigroup $(S, ., \leq_S)$ into an ordered semigroup $(T, ., \leq_T)$, denote by $f_{\triangleright}$ the pseudoorder on S defined by $f_{\triangleright} := \{(a, b) \in S \times S \mid f(a) \leq_T f(b)\}$. On the other hand, the set $Kerf$ is, by definition, the set $\{(a, b) \in S \times S \mid f(a) = f(b)\}$.

Lemma 1.2. *If S, T are ordered semigroups and f a homomorphism of S into T, then $Kerf = \overline{f_{\triangleright}}$.*

Lemma 1.3. (First Homomorphism Theorem) [5]. *Let $(S, ., \leq_S)$, $(T, ., \leq_T)$ be ordered semigroups and $f : S \to T$ a homomorphism. Then $S/Kerf \cong f(S)$. In particular, if the mapping f is onto, then $S/Kerf \cong T$.*

We denote by $N = \{1, 2, ...\}$ the set of natural numbers.

As usual, a nonempty set M is said to be a set of generators of S if for any $a \in S$ there exist $n \in N$ and $m_1, m_2, ..., m_n \in M$ such that $a = m_1.m_2 ...... m_n$.

2 Ordered semigroups as quotients of free ordered semigroups.

In this paragraph we first prove that the ordered semigroup of integers can be considered as a quotient of a free ordered semigroup. Then, using the concept of pseudoorder, we prove that each ordered semigroup can be considered as a quotient of a free ordered semigroup.

Definition 2.1. Let $(X, \leq_X)$ be an ordered set and $(F, ., \leq_F)$ an ordered semigroup. Suppose there exists an isotone mapping $\varepsilon : (X, \leq_X) \to (F, \leq_F)$ such that the following "universal" condition is satisfied: for any ordered semigroup $(S, *, \leq_S)$ and any isotone mapping $f : (X, \leq_X) \to (S, \leq_S)$ there exists a unique homomorphism $\varphi : (F, ., \leq_F) \to (S, *, \leq_S)$ such that $\varphi \circ \varepsilon = f$. Then we say that the ordered semigroup $(F, ., \leq_F)$ is a *free ordered semigroup over* $(X, \leq_X)$. Since the free ordered semigroup depends on ε, it is convenient to use the notation $((F, ., \leq_F), \varepsilon)$ instead of $(F, ., \leq_F)$.

Construction 2.2. (cf. also [6; section 2.1]) For each ordered set $(X, \leq_X)$, we can construct a free ordered semigroup over $(X, \leq_X)$. In fact, let $(X, \leq_X)$ be an ordered set. We consider the set

$$F_X := \{(x_1, x_2, ..., x_n) \mid n \in N \text{ and } x_i \in X, i = 1, 2, ..., n\}$$

with the operation and the order on F_X defined by:

$$(x_1, x_2, ..., x_n) * (y_1, y_2, ..., y_m) := (x_1, x_2, ..., x_n, y_1, y_2, ..., y_m)$$

$$(x_1, x_2, ..., x_n) \preceq (y_1, y_2, ..., y_m) \Longleftrightarrow n = m \text{ and } x_i \leq_X y_i \ \forall\, i = 1, 2, ..., n.$$

Let now ε be the isotone mapping defined by $\varepsilon : (X, \leq_X) \to (F_X, \preceq) \mid x \to (x)$.
Then the pair $((F, *, \preceq), \varepsilon)$ is a free ordered semigroup over $(X, \leq_X)$. In fact, if $(S, ., \leq_S)$ is an ordered semigroup and $f : (X, \leq_X) \to (S, \leq_S)$ an isotone mapping, then the mapping

$$\varphi : (F_X, *, \preceq) \to (S, ., \leq_S) \mid (x_1, x_2, ..., x_n) \to f(x_1)f(x_2).....f(x_n)$$

is the unique homomorphism of $(F_X, *, \preceq)$ into $(S, ., \leq_S)$ such that $\varphi \circ \varepsilon = f$. $\square$

Throughout the paper we denote by $((F_X, *, \preceq), \varepsilon)$ or simply by F_X the free ordered semigroup considered in Construction 2.2.

Remark 2.3. The ordered semigroup of integers is isomorphic to a quotient of a free ordered semigroup. In fact, we consider the ordered semigroup $(Z, +, \leq)$ of integers with the usual operation of addition and the usual order, and let $M := \{-1, 1\}$ be the subset of Z with the order on M induced by the order of Z, that is, $\leq_M := \leq \cap (M \times M) = \{(-1,-1), (1,1), (-1,1)\}$. Consider the free ordered semigroup $((F_M, *, \preceq), \varepsilon)$ over the ordered set $(M, \leq_M)$ considered in Construction 2.2, that is, the set

$F_M := \{(x_1, x_2, ..., x_n) \mid n \in N \text{ and } x_i \in M, i = 1, 2, ..., n\}$,

endowed with the multiplication and the order defined by

$(x_1, x_2, ..., x_n) * (y_1, y_2, ..., y_m) := (x_1, x_2, ..., x_n, y_1, y_2, ..., y_m)$

$(x_1, x_2, ..., x_n) \preceq (y_1, y_2, ..., y_m) \Longleftrightarrow n = m \text{ and } x_i \leq_X y_i \; \forall \, i = 1, 2, ..., n,$

and the mapping $\varepsilon : (M, \leq_M) \to (F_M, \preceq) \mid x \to (x)$. Let now f be the isotone mapping defined by $f : (M, \leq_M) \to (Z, \leq) \mid x \to x$. By the "universal" condition of Definition 2.1, there exists a unique homomorphism $\varphi : (F_M, *, \preceq) \to (Z, +, \leq)$ such that $\varphi \circ \varepsilon = f$. The mapping φ is onto. Indeed, let $n \in Z$. If $n \in \{-1, -2, ...\}$ then, for the element $a := \underbrace{(-1, -1, ..., -1)}_{|n|=-n}$ of F_M, we have

$$\begin{aligned}
\varphi(a) &= \varphi((-1, -1, ..., -1)) = \varphi((-1) * (-1) * ..... * (-1)) \\
&= \varphi((-1)) + \varphi((-1)) + ... + \varphi((-1)) \text{ (since } \varphi \text{ is a homomorphism)} \\
&= \varphi(\varepsilon(-1)) + \varphi(\varepsilon(-1)) + ... + \varphi(\varepsilon(-1)) \\
&= (\varphi \circ \varepsilon)(-1) + (\varphi \circ \varepsilon)(-1) + ... + (\varphi \circ \varepsilon)(-1) \\
&= f(-1) + f(-1) + ... + f(-1) \\
&= \underbrace{(-1) + (-1) + ... + (-1)}_{|n|=-n} = |n|(-1) = n \text{ (since } n < 0).
\end{aligned}$$

If $n = 0$ then, for the element $a := (-1, 1)$ of F_M, we have

$$\begin{aligned}
\varphi(a) &= \varphi((-1, 1)) = \varphi((-1) * (1)) \\
&= \varphi((-1)) + \varphi((1)) \text{ (since } \varphi \text{ is a homomorphism)} \\
&= \varphi(\varepsilon(-1)) + \varphi(\varepsilon(1)) \\
&= (\varphi \circ \varepsilon)(-1) + (\varphi \circ \varepsilon)(1) \\
&= f(-1) + f(1) = -1 + 1 = 0.
\end{aligned}$$

In a similar way we prove that if $n \in \{1, 2, ...\}$ then, for the element $a := \underbrace{(1, 1, ..., 1)}_{n}$ of F_M, we have $\varphi(a) = n$.

Since $(F_M, *, \preceq)$ and $(Z, +, \leq)$ are ordered semigroups and the mapping φ is an onto homomorphism, by Lemma 1.3, we have $F_M/Ker\varphi \cong Z$. We notice that $Ker\varphi = \overline{\varphi_\triangleright}$, where $\overline{\varphi_\triangleright}$ is the congruence of F_M which corresponds to the pseudoorder $\varphi_\triangleright$ on F_M defined by $\varphi_\triangleright := \{(a, b) \in F_M \times F_M \mid \varphi(a) \leq \varphi(b)\}$.

We finally remark that the set M is a set of generators of $(Z, +)$. Clearly, if $n \in \{-1, -2, ...\}$, then $n = \underbrace{(-1) + (-1) + ... + (-1)}_{|n|}$. If $n = 0$, then $n = (-1) + 1$. If $n \in N$, then $n = \underbrace{1 + 1 + ... + 1}_{n}$. □

Theorem 2.4. *Let $(S, ., \leq_S)$ be an ordered semigroup, M a set of generators of S, $(X, \leq_X)$ an ordered set and $f : (X, \leq_X) \to (M, \leq_M)$ (where $\leq_M = \leq_S \cap (M \times M)$) an isotone and onto mapping. Consider the free ordered semigroup $((F_X, *, \preceq), \varepsilon)$ over $(X, \leq_X)$ constructed* *in Construction 2.2. Then there exists a pseudoorder σ on $(F_X, *, \preceq)$ such that the ordered semigroups $F_X/\overline{\sigma}$ and S are isomorphic.*

Proof. Consider the mapping

$$g : (X, \leq_X) \to (S, \leq_S) \mid z \to f(z).$$

Since M is a subset of S, the mapping g is clearly well defined. The mapping g is isotone. Indeed, let $x \leq_X y$. Since f is isotone, we have
$(f(x), f(y)) \in \leq_M = \leq_S \cap (M \times M) \subseteq \leq_S$, that is $f(x) \leq_S f(y)$.
Since $((F_X, *, \preceq), \varepsilon)$ is a free ordered semigroup over $(X, \leq_X)$, $(S, ., \leq_S)$ an ordered semigroup and $g : (X, \leq_X) \to (S, \leq_S)$ an isotone mapping, by the "universal" property, there exists a homomorphism $\varphi : (F_X, *, \preceq) \to (S, ., \leq_S)$ such that $\varphi \circ \varepsilon = g$. The mapping φ is onto. In fact, let $b \in S$. Since M is a set of generators of S, there exist $n \in N$ and $m_1, m_2, ..., m_n \in M$ such that $b = m_1.m_2.....m_n$. Since $m_i \in M$ and the mapping $f : (X, \leq_X) \to (M, \leq_M)$ is onto, there exist $x_i \in X$ such that $f(x_i) = m_i$, $i = 1, 2, ..., n$. Then we have

$$\begin{aligned} b &= f(x_1) \;.....\; f(x_n) = g(x_1) \;.....\; g(x_n) = (\varphi \circ \varepsilon)(x_1) \;.....\; (\varphi \circ \varepsilon)(x_n) \\ &= \varphi(\varepsilon(x_1)) \;.....\; \varphi(\varepsilon(x_n)) = \varphi((x_1)) \;.....\; \varphi((x_n)) \\ &= \varphi((x_1) * ..... * (x_n)) \text{ (since } \varphi \text{ is a homomorphism)} \\ &= \varphi((x_1,\ .....\ x_n)), \text{ where } (x_1,\ .....\ x_n) \in F_X. \end{aligned}$$

Since the mapping φ is a homomorphism, the relation $\varphi_\triangleright$ is a pseudoorder on $(F_X, *, \preceq)$. Since φ is a homomorphism and onto, by Lemma 1.3, the ordered semigroups $F_X/Ker\varphi$ and S are isomorphic. In addition, $Ker\varphi = \overline{\varphi_\triangleright}$, where $\varphi_\triangleright$ is the pseudoorder on F_X defined by $\varphi_\triangleright = \{(u, v) \in F_X \times F_X \mid \varphi(u) \leq_S \varphi(v)\}$. Hence we have $F_X/\overline{\varphi_\triangleright} \cong S$, and the proof is complete. □

Theorem 2.5. *Let $(S, ., \leq_S)$ be an ordered semigroup. Then there exists an ordered set $(X, \leq_X)$ and a pseudoorder σ on $(F_X, *, \preceq)$ (where $((F_X, *, \preceq), \varepsilon)$ is the free ordered semigroup considered in Construction 2.2) such that the ordered semigroups $F_X/\overline{\sigma}$ and S are isomorphic.*

Proof. Let M be a set of generators of S (such an M exists since S itself generates S) endowed with the order $\leq_M := \leq_S \cap (M \times M)$. We consider the identity mapping 1_M of $(M, \leq_M)$ onto itself which is clearly isotone. Since $(S, ., \leq_S)$ is an ordered semigroup, M a set of generators of S, $(M, \leq_M)$ an ordered set and the identity mapping of $(M, \leq_M)$ onto $(M, \leq_M)$ is isotone, by Theorem 2.5, there exists a pseudoorder σ on $(F_X, *, \preceq)$ such that the ordered semigroups' $F_X/\overline{\sigma}$ and S are isomorphic. □

3 A characterization of free ordered semigroups.

Proposition 3.1. [4] *Let $((F_1, ., \leq_1), \varepsilon_1)$, $((F_2, *, \leq_2), \varepsilon_2)$ be free ordered semigroups over the ordered set $(X, \leq_X)$. Then there exists an isomorphism*

$$g : (F_1, ., \leq_1) \to (F_2, *, \leq_2)$$

such that $g \circ \varepsilon_1 = \varepsilon_2$.

Theorem 3.2. *Let $(X, \leq_X)$ be an ordered set, $(F, ., \leq_F)$ an ordered semigroup and $\overline{\varepsilon} : X \to F$ a mapping of X into F. Then the pair $((F, ., \leq_F), \overline{\varepsilon})$ is a free ordered semigroup over $(X, \leq_X)$ if and only if the semigroup $(F, .)$ is generated by $\overline{\varepsilon}(X)$ and for each $x_1, ..., x_n, y_1, ..., y_m \in X$ $(n, m \in N)$, we have*

$$\overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).....\overline{\varepsilon}(x_n) \leq_F \overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2).....\overline{\varepsilon}(y_m) \Longleftrightarrow n = m \text{ and } x_i \leq_X y_i \ \forall\, i = 1, ..., n.$$

Proof. $\Longrightarrow$. As we have already seen in Construction 2.2, the pair $((F_X, *, \preceq), \varepsilon)$ is a free ordered semigroup over $(X, \leq_X)$. By hypothesis, $((F, ., \leq_F), \overline{\varepsilon})$ is also a free ordered semigroup over $(X, \leq_X)$. Then, by Proposition 3.1, there exists an isomorphism $g : (F_X, *, \preceq) \to (F, ., \leq_F)$ such that $g \circ \varepsilon = \overline{\varepsilon}$.
Clearly $\emptyset \neq \overline{\varepsilon}(X) \subseteq F$. To prove that F is generated by $\overline{\varepsilon}(X)$, we have to prove that for each $w \in F$ there exist $x_1, x_2, ..., x_n \in X$ $(n \in N)$ such that

$$w = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).....\overline{\varepsilon}(x_n).$$

Let now $w \in F$. Since g is onto, there exists $u \in F_X$ such that $g(u) = w$. Since $u \in F_X$, there exist $n \in N$ and $x_1, x_2, ..., x_n \in X$ such that $u = (x_1, x_2, ..., x_n)$. Then we have

$$\begin{aligned} w &= g(u) = g((x_1, x_2, ..., x_n)) = g((x_1) * (x_2)...... * (x_n)) \\ &= g((x_1))g((x_2)).......g((x_n)) \text{ (since } g \text{ is a homomorphism)} \\ &= g(\varepsilon(x_1))g(\varepsilon(x_2)).......g(\varepsilon(x_n)) \text{ (see Construction 2.2)} \\ &= (g \circ \varepsilon)(x_1)(g \circ \varepsilon)(x_2).......(g \circ \varepsilon)(x_n) \\ &= \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2) ....\overline{\varepsilon}(x_n). \end{aligned}$$

Let now $x_1, ..., x_n, y_1, ..., y_m \in X$ $(n, m \in N)$. Then we have

$$\begin{aligned} & \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n) \leq_F \overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2).......\overline{\varepsilon}(y_m) \\ \Longleftrightarrow\ & (g \circ \varepsilon)(x_1).........(g \circ \varepsilon)(x_n) \leq_F (g \circ \varepsilon)(y_1).......(g \circ \varepsilon)(y_m) \\ \Longleftrightarrow\ & g(\varepsilon(x_1)).......g(\varepsilon(x_n)) \leq_F g(\varepsilon(y_1)).......g(\varepsilon(y_m)) \\ \Longleftrightarrow\ & g((x_1)).......g((x_n)) \leq_F g((y_1)).......g((y_m)) \text{ (by the definition of } \varepsilon) \\ \Longleftrightarrow\ & g((x_1) * .... * (x_n)) \leq_F g((y_1) * .... * (y_m)) \text{ (since } g \text{ is a homomorphism)} \\ \Longleftrightarrow\ & g((x_1, ..., x_n)) \leq_F g((y_1, ..., y_m)) \text{ (by the Definition of } * \text{)} \\ \Longleftrightarrow\ & (x_1, ..., x_n) \preceq (y_1, ..., y_m) \text{ (since } g \text{ is reverse isotone)} \\ \Longleftrightarrow\ & n = m \text{ and } x_i \leq_X y_i \ \forall\, i = 1, 2, ..., n \text{ (see Construction 2.2).} \end{aligned}$$

$\Longleftarrow$. First of all, the mapping $\overline{\varepsilon}$ is isotone. Indeed, if $a \leq_X b$ then, by hypothesis (taking $n = m = 1$), we have $\overline{\varepsilon}(a) \leq_F \overline{\varepsilon}(b)$. To prove that $((F, ., \leq_F), \overline{\varepsilon})$ is a free ordered semigroup over $(X, \leq_X)$, it remains to prove that the "universal" condition of Definition 2.1 is satisfied. In this respect, let $(S, *, \leq_S)$ be an ordered semigroup and $f : (X, \leq_X) \to (S, \leq_S)$ an isotone mapping. Then there exists a unique homomorphism

$$\varphi : (F, ., \leq_F) \to (S, *, \leq_S)$$

such that $\varphi \circ \overline{\varepsilon} = f$. In fact, we consider the mapping

$$\varphi : (F, ., \leq_F) \to (S, *, \leq_S) \mid w \to f(x_1) * f(x_2) * .... * f(x_n),$$
where $x_1, ..., x_n \in X$ $(n \in N)$ such that $w = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n)$.

1. The mapping φ is well defined. In fact: Let $w \in F$. Since $(F, .)$ is generated by $\overline{\varepsilon}(X)$, there exist $n \in N$ and $x_1, x_2, ..., x_n \in X$ such that

$$w = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n).$$

Since $x_i \in X$, $i = 1, 2, ..., n$, we have $f(x_i) \in S$, $i = 1, 2, ..., n$. Since $(S, *)$ is a semigroup, we have $f(x_1) * f(x_2) * .... * f(x_n) \in S$.
Let now $w, u \in F$, $w = u$. Suppose
$w = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n)$ for some $x_1, x_2, ..., x_n \in X$, $n \in N$ and
$u = \overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2).......\overline{\varepsilon}(y_m)$ for some $y_1, y_2, ..., y_m \in X$, $m \in N$.
Then $f(x_1) * f(x_2) * .... * f(x_n) = f(y_1) * f(y_2) * .... * f(y_m)$. Indeed:
Since $w = u$, we have $w \leq_F u$ and $u \leq_F w$. Since

$$\overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2) .... \overline{\varepsilon}(x_n) = w \leq_F u = \overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2) .... \overline{\varepsilon}(y_m),$$

by hypothesis, we have $n = m$ and $x_i \leq_X y_i$ for every $i = 1, 2, ..., n$. Since $x_i \leq_X y_i$ and the mapping $f : (X, \leq_X) \to (S, \leq_S)$ is isotone, we have $f(x_i) \leq_S f(y_i)$ for every $i = 1, 2, ..., n$. Since $(S, *, \leq_S)$ is an ordered semigroup, we have

$$f(x_1) * f(x_2) * .... * f(x_n) \leq_S f(y_1) * f(y_2) * .... * f(y_n).$$

Since $n = m$, we have

$$f(x_1) * f(x_2) * .... * f(x_n) \leq_S f(y_1) * f(y_2) * .... * f(y_m).$$

In a similar way, by $u \leq_F w$, we have

$$f(y_1) * f(y_2) * .... * f(y_m) \leq_S f(x_1) * f(x_2) * .... * f(x_n),$$

so $\varphi(w) = \varphi(u)$.
2. The mapping φ is a homomorphism. In fact, let $w, u \in F$. Suppose
$\varphi(w) := f(x_1) * f(x_2) * .... * f(x_n)$ for some $x_1, x_2, ..., x_n \in X$ $(n \in N)$
such that $w = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n)$ and
$\varphi(u) := f(y_1) * f(y_2) * .... * f(y_m)$ for some $y_1, y_2, ..., y_m \in X$ $(m \in N)$
such that $u = \overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2).......\overline{\varepsilon}(y_m)$.

Since $wu = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n)\overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2).......\overline{\varepsilon}(y_m)$, where $x_i, y_j \in X$, $i = 1, ..., n$, $j = 1, ..., m$ $(n, m \in N)$, we have

$$\begin{aligned} \varphi(wu) &:= f(x_1) * f(x_2) * .... * f(x_n) * f(y_1) * f(y_2) * .... * f(y_m) \\ &= \varphi(w) * \varphi(u). \end{aligned}$$

Let $w \leq_F u$. Then $\varphi(w) \leq_F \varphi(u)$. Indeed:
Suppose
$\varphi(w) := f(x_1) * f(x_2) * .... * f(x_n)$ for some $x_1, x_2, ..., x_n \in X$ $(n \in N)$
such that $w = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n)$ and
$\varphi(u) := f(y_1) * f(y_2) * .... * f(y_m)$ for some $y_1, y_2, ..., y_m \in X$ $(m \in N)$
such that $u = \overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2).......\overline{\varepsilon}(y_m)$.

Since $w \leq_F u$, we have $\overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n) \leq_F \overline{\varepsilon}(y_1)\overline{\varepsilon}(y_2).......\overline{\varepsilon}(y_m)$. Then, by hypothesis, we have $n = m$ and $x_i \leq_X y_i \ \ \forall\, i = 1, ..., n$. Since the mapping $f : (X, \leq_X) \to (F, \leq_S)$ is isotone and $x_i \leq_X y_i$, we have $f(x_i) \leq_S f(y_i)$ for every $i = 1, 2, ..., n$. Since $(S, *, \leq_S)$ is an ordered semigroup, we have $f(x_1) * f(x_2) * .... * f(x_n) \leq_S f(y_1) * f(y_2) * .... * f(y_n)$. Since $n = m$, we have $f(x_1) * f(x_2) * .... * f(x_n) \leq_S f(y_1) * f(y_2) * .... * f(y_m)$.

3. We have $\varphi \circ \overline{\varepsilon} = f$. Indeed, if $x \in S$ then, by the definition of φ, we have

$$(\varphi \circ \overline{\varepsilon})(x) = \varphi(\overline{\varepsilon}(x)) = f(x).$$

4. Let now $h : (F, ., \leq) \to (S, *, \leq_S)$ be a homomorphism such that $h \circ \overline{\varepsilon} = f$. Then $h = \varphi$. Indeed, let $u \in F$. Since $(F, .)$ is generated by $\overline{\varepsilon}(X)$, there exist $n \in N$ and $x_1, x_2, ..., x_n \in X$ such that $u = \overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n)$. Then
$\varphi(u) := f(x_1) * f(x_2) * .... * f(x_n)$ and

$$\begin{aligned} h(u) &= h(\overline{\varepsilon}(x_1)\overline{\varepsilon}(x_2).......\overline{\varepsilon}(x_n)) \\ &= h(\overline{\varepsilon}(x_1)) * h(\overline{\varepsilon}(x_2)) * ....h(\overline{\varepsilon}(x_n)) \text{ (since } h \text{ is a homomorphism)} \\ &= (h \circ \overline{\varepsilon})(x_1) * (h \circ \overline{\varepsilon})(x_2) * .... * (h \circ \overline{\varepsilon})(x_n) \\ &= f(x_1) * f(x_2) * .... * f(x_n) \\ &= \varphi(u). \end{aligned}$$

□

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