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ON A CERTAIN REPEATING PROCESSES PROBLEM IN ARITHMETIC

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Abstract. Our objective is to show all natural numbers $m = \sum_{i=0}^{n}(a_i \cdot 10^i)$, $(0 \le a_i \le 9,\ a_i \in \mathbb{N},\ a_n \neq 0)$ goes to 1 or 169 by taking finitely many successive operations of Δ such as $\Delta(m) = m/3$ (if $3|m$) or $\Delta(m) = \sum_{i=0}^{n}(a_i)^2$ (otherwise). It is easy to see that $\Delta(169) = 16^2 = 256$ and $\Delta(256) = 13^2 = 169$. We prove that for any $m \in \mathbb{N}$, either $\Delta^k(m) = 1$, or $\Delta^k(m) = 169$ for some $k \in \mathbb{N}$.

When the author's colleague Professor K. Kunimoto read a certain paper in the area of Mathematics Education, he found the following problem and asked the authors how to prove it. Almost all cases can be established instantly. But four cases are computed by a personal computer. Professor O. Nakamura helped the authors by programming what the authors required. The authors are grateful to them for their help.

Let $m = \sum_{i=0}^{n}(a_i \cdot 10^i) \in \mathbb{N}$ with $0 \le a_i \le 9, a_i \in \mathbb{N},\ a_n \neq 0$, and let $Dig(m)$ be the digit of m, that is, $n+1$.

Definition 1. *For any $m \in \mathbb{N}$, let*

$$\Delta(m) = \begin{cases} \dfrac{m}{3} & (\text{if } 3|m) \\ (a_n + \cdots + a_0)^2 & (\text{otherwise}) \end{cases}$$

Define $\Delta^k(m) := \Delta(\Delta^{k-1}(m))$ for $k,\ m \in \mathbb{N}$.

Remark 2. *If $m = \sum_{i=0}^{n}(a_i \cdot 10^i) \in \mathbb{N}$ with $0 \le a_i \le 9, a_i \in \mathbb{N}, a_n \neq 0$ is not a multiple of 3, then $\Delta(m) = (a_n + \cdots + a_0)^2 = (a_{\sigma(n)} + \cdots + a_{\sigma(0)})^2 = \Delta(m')$ ($\forall \sigma \in S_n$: the symmetric group of degree n). So considering $m' = \sum_{i=0}^{n}(a_{\sigma(i)} \cdot 10^i)$, we have $\Delta(m') = \Delta(m)$ ($\forall \sigma \in S_n$). This means that considering $\Delta(m)$ instead of m, we have only to consider natural numbers $\sum_{i=0}^{n}(a_i \cdot 10^i) \in \mathbb{N}$ with $0 \le a_i \le 9, a_i \in \mathbb{N}, a_n \neq 0$, which ensures that we may assume*

$$a_n \le a_{n-1} \le \cdots \le a_0 \qquad (*).$$

Let

$$T = \{m \in \mathbb{N} | \Delta^k(m) \text{ equals neither 1 nor 169 for all } k \in \mathbb{N})\}.$$

It is easy to see that for $m \in \mathbb{N}$,

$$m \in T \Longrightarrow \Delta(m) \in T \qquad (**).$$

We shall show $T = \emptyset$, that is, the following Theorem.

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Theorem 3. *For any $m \in \mathbb{N}$, either $\Delta^k(m) = 1$ or $\Delta^k(m) = 169$ for some $k \in \mathbb{N}$.*

Suppose $T \neq \emptyset$ and let $s \in T$ be the minimum number. Let $s = \sum_{i=0}^{t}(b_i \cdot 10^i) \in \mathbb{N}$ with $0 \leq b_i \leq 9, b_i \in \mathbb{N}, b_t \neq 0$. It is easy to see $3 \not| s$. For if $3|s$ then $\Delta(s) = \dfrac{s}{3} < s$, which contradicts the minimality of s.

Remark 4. *Let $m = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \in \mathbb{N}$ with $Dig(m) \leq 4$, where $0 \leq a_i \leq 9,\ a_i \in \mathbb{N}$.*
(i) *If $3|m$ then $\Delta(m) = \dfrac{m}{3}$. If moreover $s|\Delta(m)$ then $\Delta^2(m) = \dfrac{\Delta(m)}{3} = \dfrac{m}{9} \leq \dfrac{9999}{9} = 1111$.*
(ii) *If $3 \not| m$ then $\Delta(m) = (a_3 + a_2 + a_1 + a_0)^2 \leq 9^2(3+1)^2 = 1296$.*

Proposition 5. *The direct computations below implies $s \notin T \cap \{\ell \in \mathbb{N} | t+1 = Dig(\ell) \leq 4\}$.*

Proof. Suppose that $s \in T \cap \{\ell \in \mathbb{N} | t+1 = Dig(\ell) \leq 4\}$. Then $3 \not| s$ as mentioned above. Since $m \in \mathbb{N}$ such that $Dig(m) \leq 4$ which we consider can be assumed less than 1296 replacing m by $\Delta(m)$ or $\Delta^2(m)$ by Remark 4 and (**). So we have only to prove that s does not appear in the numbers $\{m \in \mathbb{N} | 1 \leq m \leq 1296,\ a_3 \leq a_2 \leq a_1 \leq a_0 \leq 9\}$. This will be checked as follows:

$\underline{m = 1 \sim 1299}$

$m \to \Delta(m)$
1 → 1
2 → 4 → 16 → 49 → 169
3 → 1
4 → 16 → 49 → 169
5 → 25 → 49 → 169
6 → 2 → 4 → 1 6 → 49 → 169
7 → 49 → 169
8 → 64 → 100 → 1 9 → 3 → 1
10 → 1
11 → 4 → 16 → 49 → 169
12 → 4 → 16 → 49 → 169
13 → 16 → 49 → 169
14 → 25 → 49 → 169
15 → 5 → 25 → 49 → 169
16 → 49 → 169
17 → 64 → 100 → 1
18 → 6 → 2 → 4 → 16 → 49 → 169
19 → 100 → 1
22 → 16 → 49 → 169
23 → 25 → 49 → 169
24 → 8 → 64 → 100 → 1
25 → 49 → 169
26 → 64 → 100 → 1
27 → 9 → 3 → 1
28 → 100 → 1
29 → 121 → 16 → 49 → 169
33 → 11 → 4 → 16 → 49 → 169
34 → 49 → 169
35 → 64 → 100 → 1
36 → 12 → 4 → 16 → 49 → 169
37 → 100 → 1
38 → 121 → 16 → 49 → 169
39 → 13 → 16 → 49 → 169
44 → 64 → 100 → 1
45 → 15 → 5 → 25 → 49 → 169
46 → 100 → 1
47 → 121 → 16 → 49 → 169
48 → 16 → 49 → 169
49 → 169
50 → 25 → 49 → 169
51 → 17 → 64 → 100 → 1
52 → 49 → 169
53 → 64 → 100 → 1
54 → 18 → 6 → 2 → 4 → 16 → 49 → 169
55 → 100 → 1
60 → 20 → 4 → 16 → 49 → 169
61 → 49 → 169
62 → 64 → 100 → 1
63 → 21 → 7 → 49 → 169
64 → 100 → 1
65 → 121 → 16 → 49 → 169
66 → 22 → 16 → 49 → 169
77 → 196 → 256 → 169
78 → 26 → 64 → 100 → 1
79 → 256 → 169
80 → 64 → 100 → 1
81 → 27 → 9 → 3 → 1
82 → 100 → 1
83 → 121 → 16 → 49 → 169
84 → 28 → 100 → 1
85 → 169
86 → 196 → 256 → 169
87 → 29 → 121 → 16 → 49 → 169
88 → 256 → 169
90 → 30 → 10 → 1
91 → 100 → 1
92 → 121 → 16 → 49 → 169
93 → 31 → 16 → 49 → 169
94 → 169
95 → 196 → 256 → 169

$96 \to 32 \to 25 \to 49 \to 169$
$97 \to 256 \to 169$
$98 \to 289 \to 361 \to 100 \to 1$
$99 \to 33 \to 11 \to 4 \to 16 \to 49 \to 169$
$100 \to 1$
$111 \to 37 \to 100 \to 1$
$112 \to 16 \to 49 \to 169$
$113 \to 25 \to 49 \to 169$
$114 \to 38 \to 121 \to 16 \to 49 \to 169$
$115 \to 49 \to 169$
$116 \to 64 \to 100 \to 1$
$117 \to 39 \to 13 \to 16 \to 49 \to 169$
$118 \to 100 \to 1$
$119 \to 121 \to 16 \to 49 \to 169$
$122 \to 25 \to 49 \to 169$
$123 \to 41 \to 25 \to 49 \to 169$
$124 \to 49 \to 169$
$125 \to 64 \to 100 \to 1$
$126 \to 42 \to 14 \to 25 \to 49 \to 169$
$127 \to 100 \to 1$
$128 \to 121 \to 16 \to 49 \to 169$
$129 \to 43 \to 49 \to 169$
$133 \to 49 \to 169$
$134 \to 64 \to 100 \to 1$
$135 \to 45 \to 15 \to 5 \to 25 \to 49 \to 169$
$136 \to 100 \to 1$
$137 \to 121 \to 16 \to 49 \to 169$
$138 \to 46 \to 100 \to 1$
$139 \to 169$
$144 \to 48 \to 16 \to 49 \to 169$
$145 \to 100 \to 1$
$146 \to 121 \to 16 \to 49 \to 169$
$147 \to 49 \to 169$
$148 \to 169$
$149 \to 196 \to 256 \to 169$
$155 \to 121 \to 16 \to 49 \to 169$
$156 \to 52 \to 49 \to 169$
$157 \to 169$
$158 \to 196 \to 256 \to 169$
$159 \to 53 \to 64 \to 100 \to 1$
$166 \to 169$
$167 \to 196 \to 256 \to 169$
$168 \to 56 \to 121 \to 16 \to 49 \to 169$
$169 \to 169$
$177 \to 59 \to 196 \to 256 \to 169$
$178 \to 256 \to 169$
$179 \to 289 \to 361 \to 100 \to 1$
$188 \to 289 \to 361 \to 100 \to 1$
$189 \to 63 \to 21 \to 7 \to 49 \to 169$
$199 \to 361 \to 100 \to 1$
$222 \to 74 \to 121 \to 16 \to 49 \to 169$
$223 \to 49 \to 169$
$224 \to 64 \to 100 \to 1$
$225 \to 75 \to 25 \to 49 \to 169$
$226 \to 100 \to 1$
$227 \to 121 \to 16 \to 49 \to 169$
$228 \to 76 \to 169$
$229 \to 169$
$244 \to 100 \to 1$
$245 \to 121 \to 16 \to 49 \to 169$
$246 \to 82 \to 100 \to 1$
$247 \to 169$
$248 \to 196 \to 256 \to 169$
$249 \to 83 \to 121 \to 16 \to 49 \to 169$
$255 \to 85 \to 169$
$256 \to 169$
$257 \to 196 \to 256 \to 169$
$258 \to 86 \to 196 \to 256 \to 169$
$259 \to 256 \to 169$
$266 \to 196 \to 256 \to 169$
$267 \to 89 \to 289 \to 361 \to 100 \to 1$
$268 \to 256 \to 169$
$269 \to 289 \to 361 \to 100 \to 1$
$277 \to 256 \to 169$
$278 \to 289 \to 361 \to 100 \to 1$
$279 \to 93 \to 31 \to 16 \to 49 \to 169$
$288 \to 96 \to 32 \to 25 \to 49 \to 169$
$289 \to 361 \to 100 \to 1$
$299 \to 400 \to 16 \to 49 \to 169$
$333 \to 111 \to 37 \to 100 \to 1$
$334 \to 100 \to 1$
$335 \to 121 \to 16 \to 49 \to 169$
$336 \to 112 \to 16 \to 49 \to 169$
$337 \to 169$
$338 \to 196 \to 256 \to 169$
$339 \to 113 \to 25 \to 49 \to 169$
$340 \to 49 \to 169$
$341 \to 64 \to 100 \to 1$
$342 \to 114 \to 38 \to 121 \to 16 \to 49 \to 169$
$343 \to 100 \to 1$
$344 \to 121 \to 16 \to 49 \to 169$
$345 \to 115 \to 49 \to 169$
$346 \to 169$
$347 \to 196 \to 256 \to 169$
$348 \to 116 \to 64 \to 100 \to 1$
$349 \to 256 \to 169$
$355 \to 169$
$356 \to 196 \to 256 \to 169$
$357 \to 119 \to 121 \to 16 \to 49 \to 169$
$358 \to 256 \to 169$
$359 \to 289 \to 361 \to 100 \to 1$
$366 \to 122 \to 25 \to 49 \to 169$
$367 \to 256 \to 169$
$368 \to 289 \to 361 \to 100 \to 1$
$369 \to 123 \to 41 \to 25 \to 49 \to 169$
$370 \to 100 \to 1$
$377 \to 289 \to 361 \to 100 \to 1$
$378 \to 126 \to 42 \to 14 \to 25 \to 49 \to 169$
$379 \to 361 \to 100 \to 1$
$388 \to 361 \to 100 \to 1$
$389 \to 400 \to 16 \to 49 \to 169$
$399 \to 133 \to 49 \to 169$
$444 \to 148 \to 169$
$445 \to 169$
$446 \to 196 \to 256 \to 169$
$447 \to 149 \to 196 \to 256 \to 169$
$448 \to 256 \to 169$
$449 \to 289 \to 361 \to 100 \to 1$
$455 \to 196 \to 256 \to 169$
$456 \to 152 \to 64 \to 100 \to 1$

457 → 256 → 169
458 → 289 → 361 → 100 → 1
459 → 153 → 51 → 17 → 64 → 100 → 1
466 → 256 → 169
467 → 289 → 361 → 100 → 1
468 → 156 → 52 → 49 → 169
469 → 361 → 100 → 1
477 → 159 → 53 → 64 → 100 → 1
478 → 361 → 100 → 1
479 → 400 → 16 → 49 → 169
486 → 162 → 54 → 18 → 6 → 2 → 4 → 16 → 49 → 169
487 → 361 → 100 → 1
488 → 400 → 16 → 49 → 169
489 → 163 → 100 → 1
499 → 484 → 256 → 169
555 → 185 → 196 → 256 → 169
556 → 256 → 169
557 → 289 → 361 → 100 → 1
558 → 186 → 62 → 64 → 100 → 1
559 → 361 → 100 → 1
560 → 121 → 16 → 49 → 169
561 → 187 → 256 → 169
562 → 169
563 → 196 → 256 → 169
564 → 188 → 289 → 361 → 100 → 1
565 → 256 → 169
566 → 289 → 361 → 100 → 1
567 → 189 → 63 → 21 → 7 → 49 → 169
568 → 361 → 100 → 1
569 → 400 → 16 → 49 → 169
570 → 190 → 100 → 1
571 → 169
572 → 196 → 256 → 169
573 → 191 → 121 → 16 → 49 → 169
574 → 256 → 169
575 → 289 → 361 → 100 → 1
576 → 192 → 64 → 100 → 1
577 → 361 → 100 → 1
578 → 400 → 16 → 49 → 169
579 → 193 → 169
588 → 196 → 256 → 169
589 → 484 → 256 → 169
599 → 529 → 256 → 169
600 → 200 → 4 → 16 → 49 → 169
601 → 49 → 169
602 → 64 → 100 → 1
603 → 201 → 67 → 169
604 → 100 → 1
605 → 121 → 16 → 49 → 169
606 → 202 → 16 → 49 → 169
666 → 222 → 74 → 121 → 16 → 49 → 169
667 → 361 → 100 → 1
668 → 400 → 16 → 49 → 169
669 → 223 → 49 → 169
677 → 400 → 16 → 49 → 169
678 → 226 → 100 → 1
679 → 484 → 256 → 169
688 → 484 → 256 → 169
689 → 529 → 256 → 169
699 → 233 → 64 → 100 → 1
777 → 259 → 256 → 169
778 → 484 → 256 → 169
779 → 529 → 256 → 169
788 → 529 → 256 → 169
789 → 263 → 121 → 16 → 49 → 169
799 → 625 → 169
888 → 296 → 289 → 361 → 100 → 1
889 → 625 → 169
899 → 676 → 361 → 100 → 1
999 → 333 → 111 → 37 → 100 → 1
1111 → 16 → 49 → 169
1112 → 25 → 49 → 169
1113 → 371 → 121 → 16 → 49 → 169
1114 → 49 → 169
1115 → 64 → 100 → 1
1116 → 372 → 124 → 49 → 169
1117 → 100 → 1
1118 → 121 → 16 → 49 → 169
1119 → 373 → 169
1122 → 374 → 196 → 256 → 169
1123 → 49 → 169
1124 → 64 → 100 → 1
1125 → 375 → 125 → 64 → 100 → 1
1126 → 100 → 1
1127 → 121 → 16 → 49 → 169
1128 → 376 → 256 → 169
1129 → 169
1133 → 64 → 100 → 1
1134 → 378 → 126 → 42 → 14 → 25 → 49 → 169
1135 → 100 → 1
1136 → 121 → 16 → 49 → 169
1137 → 379 → 361 → 100 → 1
1138 → 169
1139 → 196 → 256 → 169
1144 → 100 → 1
1145 → 121 → 16 → 49 → 169
1146 → 382 → 169
1147 → 169
1148 → 196 → 256 → 169
1149 → 383 → 196 → 256 → 169
1155 → 385 → 256 → 169
1156 → 169
1157 → 196 → 256 → 169
1158 → 386 → 289 → 361 → 100 → 1
1159 → 256 → 169
1166 → 196 → 256 → 169
1167 → 389 → 400 → 16 → 49 → 169
1168 → 256 → 169
1169 → 289 → 361 → 100 → 1
1177 → 256 → 169
1178 → 289 → 361 → 100 → 1
1179 → 393 → 131 → 25 → 49 → 169
1188 → 396 → 132 → 44 → 64 → 100 → 1
1189 → 361 → 100 → 1
1199 → 400 → 16 → 49 → 169
1222 → 49 → 169
1223 → 64 → 100 → 1
1224 → 408 → 136 → 100 → 1
1225 → 100 → 1
1226 → 121 → 16 → 49 → 169

$1227 \to 409 \to 169$
$1228 \to 169$
$1229 \to 196 \to 256 \to 169$
$1233 \to 411 \to 137 \to 121 \to 16 \to 49 \to 169$
$1234 \to 100 \to 1$
$1235 \to 121 \to 16 \to 49 \to 169$
$1236 \to 412 \to 49 \to 169$
$1237 \to 169$
$1238 \to 196 \to 256 \to 169$
$1239 \to 413 \to 64 \to 100 \to 1$
$1244 \to 121 \to 16 \to 49 \to 169$
$1245 \to 415 \to 100 \to 1$
$1246 \to 169$
$1247 \to 196 \to 256 \to 169$
$1248 \to 416 \to 121 \to 16 \to 49 \to 169$
$1249 \to 256 \to 169$
$1255 \to 169$
$1256 \to 196 \to 256 \to 169$
$1257 \to 419 \to 196 \to 256 \to 169$
$1258 \to 256 \to 169$
$1259 \to 289 \to 361 \to 100 \to 1$
$1266 \to 422 \to 64 \to 100 \to 1$
$1267 \to 256 \to 169$
$1268 \to 289 \to 361 \to 100 \to 1$
$1269 \to 423 \to 141 \to 47 \to 121 \to 16 \to 49 \to 169$
$1277 \to 289 \to 361 \to 100 \to 1$
$1278 \to 426 \to 142 \to 49 \to 169$
$1279 \to 361 \to 100 \to 1$
$1288 \to 361 \to 100 \to 1$
$1289 \to 400 \to 16 \to 49 \to 169$
$1299 \to 433 \to 100 \to 1$

□

Lemma 6. *Let $P(u) = 10^u - 9^2(u+1)^2$ for any $u \in \mathbb{N}$. Then $P(u) > 0$ for all natural number $u \geq 4$*

Proof. We shall show that $P(u) > 0$ for all natural numbers $u \geq 4$ by induction on u,

(i) If $u = 4$, Then $P(4) = 10^4 - 9^2((4+1)^2 = 10000 - 2025 > 0$,

(ii) Suppose that $10^u - 9^2(u+1)^2 > 0$ holds for all $1 \leq u \leq 4$. Then $P(u+1) = 10^{u+1} - 9^2(u+1+1)^2 = 10^{u+1} - 81(u+1)^2 - 9^2 \cdot 2(u+1) - 9^2 > 10^{u+1} - 2 \cdot 10^u - 81$ by induction. It is easy to see $10^{u+1} - 2 \cdot 10^u - 81 > 0$. Hence $P(u+1) > 0$. □

Proposition 7. $T = \emptyset$.

Proof. By Proposition 5, we can assume $Dig(s) \geq 5$. Note that $Dig(s) = t + 1 \geq 5$, that is $t \geq 4$. If $3|s$, then $\Delta(s) = \dfrac{s}{3} < s$, which contradicts the minimality of s. So assume that s is not a multiple of 3. Let $P(u)$ be the polynomial defined in Lemma 6. Since $s - \Delta(s) = \sum_{i=0}^{n}(b_i \cdot 10^i) - (\sum_{i=0}^{t} b_i)^2 \geq 10^t - 9^2(t+1)^2 = P(t) > 0$ by Lemma 6, we have $s > \Delta(s)$ for all $t \geq 4$, that is, $s > \Delta(s)$, which contradicts the minimality of s. Thus $T = \emptyset$. □

Therefore Theorem 3 follows Proposition 7.

Finally we pose a problem on the analogy of our result. The author does not know it is true or not.

Problem Let $m \in \mathbb{N}$ and let $t_0, t_1 \in \mathbb{N}$ be two fixed numbers greater than 2 with $t_0 > t_1$. For any $m \in \mathbb{N}$, let

$$\Delta_0(m) = \begin{cases} \dfrac{m}{t_0} & \text{if } t_0 | m \\ (a_n + \cdots + a_0)^{t_1} & if \text{otherwise} \end{cases}$$

Then does there exists $M \in \mathbb{N}$ such that

$$\Delta_0{}^k(m) \leq M, \text{ for } \forall m, k \in \mathbb{N}?$$

Of course, M depends on only t_0.

If this problem has an affirmative solution, then there exists $m_0 \in \mathbb{N}$ such that ${\Delta_0}^v(m_0) = m_0$ for some $v \in \mathbb{N}$ with $m_0 \leq M$.

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