ON ORDERED Γ-SEMIGROUPS

Niovi Kehayopulu

Received December 27, 2009

Abstract. In this paper we show the way we pass from ordered semigroups to ordered Γ-semigroups. Moreover we show that, exactly as in ordered semigroups, in the results of ordered Γ-semigroups based on right (left) ideals, quasi-ideals and bi-ideals, points do not play any essential role, but the sets, which shows their pointless character. Under the methodology using in this paper, all the results of ordered semigroups can be transferred into ordered Γ-semigroups.

1. Introduction and prerequisites. The notion of a Γ-ring, a generalization of the notion of associative rings, has been introduced and studied by N. Nobusawa in [6]. Γ-rings has been also studied by W.E. Barnes in [1]. J. Luh studied many properties of simple Γ-rings and primitive Γ-rings in [5]. The concept of a Γ-semigroup has been introduced by M.K. Sen in 1981 as follows: A nonempty set S is called a Γ-semigroup if the following assertions are satisfied: (1) $a\alpha b \in S$ and $\alpha a\beta \in \Gamma$ and (2) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$ [8]. In 1986, M.K. Sen and N.K. Saha changed that definition considering the following more general definition: Given two nonempty sets M and Γ, M is called a Γ-semigroup if (1) $a\alpha b \in M$ and (2) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$ [9]. One can find that definition of Γ-semigroups in [12], where the notion of radical in Γ-semigroups and the notion of $\Gamma\mathcal{S}$-act over a Γ-semigroup has been introduced, in [10] and [11], where the notions of regular and orthodox Γ-semigroups have been introduced and studied. With that second definition, a semigroup $(S, .)$ can be viewed as a particular case of a Γ-semigroup, considering $\Gamma = \{\gamma\}$ $(\gamma \notin M)$ and defining $a\gamma b := a.b$. Moreover, let M be a Γ-semigroup, take a (fixed) $\gamma \in \Gamma$, and define $a.\ b := a\gamma b$, then $(M, .)$ is a semigroup. Later, in [7], Saha calls a nonempty set S a Γ-semigroup if there is a mapping $S \times \Gamma \times S \to S \mid (a\gamma b) \to a\gamma b$ such that $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$, and remarks that most usual semigroup concepts, in particular regular and inverse Γ-semigroups have their analogous in Γ-semigroups. Although his first definition with Sen, where Γ plays the role of binary relations was better than his second one given by means of a mapping, the uniqueness condition was missing from the first one. Many authors tried to transfer results of semigroups or ordered semigroups to Γ-semigroups or ordered Γ-semigroups (shortly *po*-Γ-semigroups) some of them using the definition of a Γ-semigroup introduced by Sen in 1981, others the second one given by Sen in 1986. The reason that they used the two definitions is that in an expression of the form $a\gamma b\mu c\nu d\rho e$, for example, where $a, b, c, d, e \in M$ and $\gamma, \mu, \nu, \rho \in \Gamma$, it was not clear were to put the parenthesis. On the other hand, it is more convenient, of course, to define the Γ-semigroup M via a set Γ of binary relations than to define it as a mapping of $M \times \Gamma \times M$ into M. Adding the uniqueness condition in the definition given by Sen and Saha in [9], we avoid to define it via

2000 *Mathematics Subject Classification*. 06F99.
Key words and phrases. Γ-semigroup, ordered Γ-semigroup (*po*- Γ-semigroup), regular, intra-regular *po*- Γ-semigroup, right (left) ideal, bi-ideal, quasi-ideal in a *po*-Γ-semigroup, idempotent subset of a *po*-Γ-semigroup.

mappings. An ordered Γ-semigroup (shortly *po*-Γ-semigroup) defined by Sen and Seth in [13], is a Γ-semigroup together with an order relation $\leq$ such that $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $c \in M$ and all $\gamma \in \Gamma$.

A nonempty subset A of a *po*-Γ-semigroup M is called a right (resp. left) ideal of M if (1) $A\Gamma M \subseteq A$ (resp. $M\Gamma A \subseteq A$) and (2) If $a \in A$ and $M \ni b \leq a$, then $b \in A$. A is called an ideal of M if it is both a right and a left ideal of M. For $H \subseteq M$, we denote $(H] = \{t \in M \mid t \leq a$ for some $a \in H\}$. Using that notation, the second property of a right (left) ideal A can be obviously written as $(A] = A$. A *po*-Γ-semigroup M is called right (resp. left) duo if its right (resp. left) ideals of M are two-sided. It is called duo if it is right duo and left duo. A *po*-Γ-semigroup M is called commutative if $A\Gamma B = B\Gamma A$ for all $A, B \subseteq M$. A subset A of an ordered Γ-semigroup M is called idempotent if $A = (A\Gamma A]$.

L. Kovács was the first who observed that the regular rings (introduced by J.v. Neumann) can be characterized by the property $A \cap B = AB$ for every right ideal A and every left ideal B, where AB is the set of all finite sums of the form $\sum a_i b_i$; $a_i \in A$, $b_i \in B$ [4]. K. Iséki studied the same for semigroups and characterized the regular semigroups as semigroups satisfying the property $A \cap B = AB$ for all right ideals A and all left ideals B [3]. Theorem 3 below is the analogous of the characterization of regular semigroups given by J. Calais in [2].

In this paper we consider the definition of Γ-semigroup defined by Sen and Saha in 1986 in which we add the uniqueness condition (which is absolutely necessary for the investigation). The aim is to show the way we pass from ordered semigroups to ordered Γ-semigroups, to point out that the results of ordered semigroups can be transferred into ordered Γ-semigroups, and emphasize the fact that, exactly as in ordered semigroups, in many results of *po*-Γ-semigroups points do not play any essential role but the sets. In this purpose, some well known results of ordered semigroups in case of ordered Γ-semigroups are examined.

2. Main results.

For two nonempty sets M and Γ, define $M\Gamma M$ as the set of all elements of the form $m_1\gamma m_2$, where $m_1, m_2 \in M$, $\gamma \in \Gamma$. That is,

$M\Gamma M := \{m_1\gamma m_2 \mid m_1, m_2 \in M, \gamma \in \Gamma\}$.

Definition 1. Let M and Γ be two nonempty sets. The set M is called a Γ-semigroup if the following assertions are satisfied:

(1) $M\Gamma M \subseteq M$.

(2) If $m_1, m_2, m_3, m_4 \in M$, $\gamma_1, \gamma_2 \in \Gamma$ such that $m_1 = m_3$, $\gamma_1 = \gamma_2$ and $m_2 = m_4$, then $m_1\gamma_1 m_2 = m_3\gamma_2 m_4$.

(3) $(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$ for all $m_1, m_2, m_3 \in M$ and all $\gamma_1, \gamma_2 \in \Gamma$.

In other words, Γ is a set of binary operations on M and the following condition is satisfied:

$(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$ for all $m_1, m_2, m_3 \in M$ and all $\gamma_1, \gamma_2 \in \Gamma$.

According to that "associativity" relation, each of the elements $(m_1\gamma_1 m_2)\gamma_2 m_3$, and $m_1\gamma_1(m_2\gamma_2 m_3)$ is denoted as $m_1\gamma_1 m_2\gamma_2 m_3$.
That is,

$(3')$ $m_1\gamma_1 m_2\gamma_2 m_3 := (m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$.

There are several examples of Γ-semigroups in the bibliography. However, the example below based on Definition 1 above, shows clearly what a Γ-semigroup is.

Example 2. (A) Consider the two-elements set $M := \{a, b\}$, and let $\Gamma = \{\gamma, \mu\}$ be the set of two binary operations on M defined in the tables below:

γ	a	b
a	a	b
b	b	a

μ	a	b
a	b	a
b	a	b

One can check that $(x\rho y)\omega z = x\rho(y\omega z)$ for all $x, y, z \in M$ and all $\rho, \omega \in \Gamma$. So M is a Γ-semigroup.

(B) Consider the set $M := \{a, b, c\}$, and let $\Gamma = \{\gamma, \mu\}$ be the set of two binary operations on M defined in the tables below:

γ	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

μ	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

Since $(x\rho y)\omega z = x\rho(y\omega z)$ for all $x, y, z \in M$ and all $\rho, \omega \in \Gamma$, M is a Γ-semigroup.

Using conditions (1)–(3) one can prove that for an element of the form $m_1\gamma_1 m_2\gamma_2 m_3\gamma_3 m_4$ one can put a parenthesis in any expression beginning with some m_i and ending in some m_j, that is,

$$\begin{aligned} m_1\gamma_1 m_2\gamma_2 m_3\gamma_3 m_4 : &= (m_1\gamma_1 m_2)\gamma_2 m_3\gamma_3 m_4 = m_1\gamma_1(m_2\gamma_2 m_3)\gamma_3 m_4 \\ &= m_1\gamma_1 m_2\gamma_2(m_3\gamma_3 m_4) = (m_1\gamma_1 m_2\gamma_2 m_3)\gamma_3 m_4 \\ &= m_1\gamma_1(m_2\gamma_2 m_3\gamma_3 m_4) = (m_1\gamma_1 m_2)\gamma_2(m_3\gamma_3 m_4). \end{aligned}$$

In general, for any element of the form

$m_1\gamma_1 m_2\gamma_2 m_3\gamma_3 m_4 ....\gamma_{n-1} m_n\gamma_n m_{n+1}$,

one can put a parenthesis in any expression beginning with some m_i and ending in some m_j, that is, for example, in

$m_1\gamma_1(m_2\gamma_2 m_3)\gamma_3 m_4 ....\gamma_{n-1} m_n\gamma_n m_{n+1}$ or

$m_1\gamma_1 m_2\gamma_2(m_3\gamma_3 m_4 ....\gamma_{n-1} m_n)\gamma_n m_{n+1}$

In this paper we will mainly use expressions of type $M_1\Gamma_1 M_2\Gamma_2 M_3$ and $M_1\Gamma_1 M_2\Gamma_2 M_3\Gamma_3 M_4$. For expressions with more that seven words, we work in a similar way.

One can easily prove that condition $(3')$ can be equivalently defined as follows:

$M_1\Gamma_1 M_2\Gamma_2 M_3 := (M_1\Gamma_1 M_2)\Gamma_2 M_3 = M_1\Gamma_1(M_2\Gamma_2 M_3)$ for all $M_1, M_2, M_3 \subseteq M$ and all $\Gamma_1, \Gamma_2 \subseteq \Gamma$.

As far as the $M_1\Gamma_1 M_2\Gamma_2 M_3\Gamma_3 M_4$ is concerned, the following are equal to each other:

$(M_1\Gamma_1 M_2)\Gamma_2 M_3\Gamma_3 M_4$, $M_1\Gamma_1(M_2\Gamma_2 M_3)\Gamma_3 M_4$, $M_1\Gamma_1 M_2\Gamma_2(M_3\Gamma_3 M_4)$,
$(M_1\Gamma_1 M_2\Gamma_2 M_3)\Gamma_3 M_4$, $M_1\Gamma_1(M_2\Gamma_2 M_3\Gamma_3 M_4)$, $(M_1\Gamma_1 M_2)\Gamma_2(M_3\Gamma_3 M_4)$.

For a nonempty subset A of M, we denote by $R(A)$ (resp. $L(A)$) the right (resp. left) ideal of M generated by A and by $I(A)$ the ideal of M generated by A. We have $R(A) = (A \cup A\Gamma M]$. Indeed: First of all, the set $(A \cup A\Gamma M]$ contains A, and it is nonempty. Moreover,

$$\begin{aligned} (A \cup A\Gamma M]\Gamma M &= (A \cup A\Gamma M]\Gamma(M] \subseteq ((A \cup A\Gamma M)\Gamma M] \\ &= (A\Gamma M \cup (A\Gamma M)\Gamma M] = (A\Gamma M \cup A\Gamma(M\Gamma M)] \\ &= (A\Gamma M] \subseteq (A \cup A\Gamma M], \end{aligned}$$

and $((A \cup A\Gamma M]] = (A \cup A\Gamma M]$, that is, $(A \cup A\Gamma M]$ is a right ideal of M containing A. Now, if T is a right ideal of M containing A, then $(A \cup A\Gamma M] \subseteq (T \cup T\Gamma M] = (T] = T$. Similarly, $L(A) = (A \cup M\Gamma A]$, $I(A) = (A \cup A\Gamma M \cup M\Gamma A \cup M\Gamma A\Gamma M]$.

An ordered semigroup S is called regular if for every $a \in S$ there exists $x \in S$ such that $a \le axa$. An ordered semigroup S is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \le xa^2y$. An ordered Γ-semigroup M is called regular if for every $a \in M$ there exist $x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a \le a\gamma_1 x\gamma_2 a$. An ordered Γ-semigroup M is called intra-regular if for every $a \in M$ there exist $x, y \in M$ and $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that $a \le x\gamma_1 a\gamma_2 a\gamma_3 y$. As one can easily see, a *po*-Γ-semigroup M is regular if and only if $a \in (a\Gamma M\Gamma a]$ for all $a \in M$, equivalently, if $A \subseteq (A\Gamma M\Gamma A]$ for all $A \subseteq M$. It is intra-regular if and only if $a \in (M\Gamma a\Gamma a\Gamma M]$ for all $a \in M$, equivalently, if $A \subseteq (M\Gamma A\Gamma A\Gamma M]$ for all $A \subseteq M$.

On the other hand, an ordered semigroup S is regular if and only if for every right ideal A and every left ideal B of S, $A \cap B \subseteq (AB]$, equivalently, $A \cap B = (AB]$. This result can be naturally transferred into ordered Γ-semigroups as follows: An ordered Γ-semigroup M is regular if and only if for every right ideal A and every left ideal B of M, we have $A \cap B \subseteq (A\Gamma B]$, equivalently, $A \cap B = (A\Gamma B]$. In fact, suppose A be a right ideal and B a left ideal of M. If M is regular, we have

$$\begin{aligned} A \cap B &\subseteq ((A \cap B)\Gamma M\Gamma(A \cap B)] \subseteq ((A\Gamma M)\Gamma B] \subseteq (A\Gamma B] \\ &\subseteq (A\Gamma M] \cap (M\Gamma B] \subseteq (A] \cap (B] = A \cap B. \end{aligned}$$

For the converse statement, we see that

$$\begin{aligned} A &\subseteq R(A) \cap L(A) \subseteq (R(A)\Gamma L(A)] = ((A \cup A\Gamma M]\Gamma(A \cup M\Gamma A]] \\ &= ((A \cup A\Gamma M)\Gamma(A \cup M\Gamma A)] = (A\Gamma A \cup A\Gamma M\Gamma A \cup (A\Gamma(M\Gamma M)\Gamma A] \\ &= (A\Gamma A \cup A\Gamma M\Gamma A], \end{aligned}$$

from which

$$\begin{aligned} A\Gamma A &\subseteq (A\Gamma A \cup A\Gamma M\Gamma A]\Gamma(A] \subseteq (A\Gamma A\Gamma A \cup A\Gamma(M\Gamma A)\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A]. \end{aligned}$$

Then $A \subseteq ((A\Gamma M\Gamma A] \cup A\Gamma M\Gamma A] = ((A\Gamma M\Gamma A]] = (A\Gamma M\Gamma A]$, so M is regular.

If M is a duo *po*-Γ-semigroup and the ideals of M are idempotent, then M is regular. In fact, let A be a right ideal and B a left ideal of M. Since M is a duo A and B are ideals of M. Then $A \cap B$ is an ideal of M and
$A \cap B = ((A \cap B)\Gamma(A \cap B)] \subseteq (A\Gamma A \cap B\Gamma A \cap A\Gamma B \cap B\Gamma B] \subseteq (A\Gamma B]$,
so M is regular.

An ordered semigroup S is intra-regular if and only if for every right ideal A and every left ideal B of S, we have $A \cap B \subseteq (BA]$. An *le*-semigroup is intra-regular if and only if for every right ideal element a and every left ideal element b of S, we have $b \wedge a \le ba$. The corresponding result on ordered Γ-semigroups is the following: An ordered Γ-semigroup M is intra-regular if and only if for every right ideal A and every left ideal B of M, we have $A \cap B \subseteq (B\Gamma A]$. Exactly as in the case of regular ordered Γ-semigroups, this result can be proved using only sets.

A nonempty subset B of an ordered semigroup S is called a bi-ideal of S if (1) $BSB \subseteq B$ and (2) If $a \in B$ and $S \ni b \le a$, then $b \in B$. A nonempty subset Q if an ordered semigroup S is called a quasi-ideal of S if (1) $(QS] \cap (SQ] \subseteq Q$ and (2) If $a \in Q$ and $S \ni b \le a$, then $b \in Q$. A nonempty subset B of an ordered Γ-semigroup M is called a bi-ideal of M if (1) $B\Gamma M\Gamma B \subseteq B$ and (2) If $a \in B$ and $M \ni b \le a$, then $b \in B$. A nonempty subset Q of an ordered Γ-semigroup M is called a quasi-ideal of M if (1) $(Q\Gamma M] \cap (M\Gamma Q] \subseteq Q$ and (2) If $a \in Q$ and $M \ni b \le a$, then $b \in Q$.

Each quasi-ideal of an ordered semigroup S is also a bi-ideal of S. Analogous result for an ordered Γ-semigroup M also holds. In fact, if Q be a quasi-ideal of M, then

$$\begin{aligned} Q\Gamma M\Gamma Q &\subseteq Q\Gamma(M\Gamma M)\cap(M\Gamma M)\Gamma Q \\ &\subseteq Q\Gamma M\cap M\Gamma Q\subseteq(Q\Gamma M]\cap(M\Gamma Q]\subseteq Q. \end{aligned}$$

In regular *po*-Γ-semigroups the concepts of bi-ideals and quasi-ideals coincide. In fact: Let M be a regular *po*-Γ-semigroup and B a bi-ideal of M. Then $B\Gamma M\Gamma B\subseteq B$ and $B\subseteq(B\Gamma M\Gamma B]$. Since $B\Gamma M\Gamma B\subseteq B$, we get $(B\Gamma M\Gamma B]\subseteq(B]=B$, so $B=(B\Gamma M\Gamma B]$. Moreover, M is idempotent. Indeed, since M is regular, we have $M\subseteq((M\Gamma M)\Gamma M]\subseteq(M\Gamma M]\subseteq(M]=M$, so $M=(M\Gamma M]$. On the other hand, since M is regular and $(B\Gamma M]$ (resp. $(M\Gamma B]$) is a right (resp. left) ideal of M, we have

$$\begin{aligned} (B\Gamma M]\cap(M\Gamma B] &= ((B\Gamma M]\Gamma(M\Gamma B]]=(B\Gamma M\Gamma M\Gamma B] \\ &= (B\Gamma(M\Gamma M]\Gamma M\Gamma B]\ \text{(since } M \text{ is idempotent)} \\ &= (B\Gamma(M\Gamma M)\Gamma M\Gamma B] \\ &\subseteq (B\Gamma M\Gamma B]=B. \end{aligned}$$

Theorem 3. *An ordered* Γ*-semigroup* M *is regular if and only if the right and the left ideals of* M *are idempotent and for every right ideal* A *and every left ideal* B *of* M*,* $(A\Gamma B]$ *is a quasi-ideal of* M*.*

Proof. $\Longrightarrow$. Let A be a right ideal of M. Since M is regular, we have $A\subseteq(A\Gamma(M\Gamma A)]\subseteq(A\Gamma A]\subseteq(A\Gamma M]$. Since A is a right ideal of M, $(A\Gamma M]\subseteq(A]=A$. Thus $A=(A\Gamma A]$, and A is idempotent. Similarly, the left ideals of M are idempotent.

Let now A be a right ideal and B a left ideal of M. Then $(A\Gamma B]$ is a quasi-ideal of M. In fact, since M is regular, we have $A\cap B=(A\Gamma B]$, so it is enough to prove that $A\cap B$ is a quasi-ideal of M. First of all, $A\cap B\neq\emptyset$. Indeed, let $a\in A$, $b\in B$, $\gamma\in\Gamma$ $(A,B,\Gamma\neq\emptyset)$, then $a\gamma b\in A\Gamma M\subseteq A$ and $a\gamma b\in M\Gamma B\subseteq B$, thus $a\gamma b\in A\cap B$. Moreover,

$$((A\cap B)\Gamma M]\cap(M\Gamma(A\cap B)]\subseteq(A\Gamma M]\cap(M\Gamma B]\subseteq(A]\cap(B]=A\cap B,$$

and if $x\in A\cap B$ and $M\ni y\leq x$, then $y\in A\cap B$. Hence $A\cap B$ is a quasi-ideal of M.
$\Longleftarrow$. Let $A\subseteq M$. Since $R(A)$ is a right ideal of M, it is idempotent, so

$$\begin{aligned} A\subseteq R(A) &= (R(A)\Gamma R(A)]=((A\cup A\Gamma M]\Gamma(A\cup A\Gamma M]] \\ &= ((A\cup A\Gamma M)\Gamma(A\cup A\Gamma M)] \\ &= (A\Gamma A\cup A\Gamma(M\Gamma A)\cup A\Gamma(A\Gamma M)\cup A\Gamma(M\Gamma A)\Gamma M] \\ &\subseteq (A\Gamma M]. \end{aligned}$$

Similarly, $A\subseteq(M\Gamma A]$, so $A\subseteq(A\Gamma M]\cap(M\Gamma A]$. As $(A\Gamma M]$, $(M\Gamma A]$) are right and left ideals of M, respectively, they are idempotent, that is,

$$(A\Gamma M]=((A\Gamma M]\Gamma(A\Gamma M]]\ \text{and}\ (M\Gamma A]=((M\Gamma A]\Gamma(M\Gamma A]].$$

Therefore, we have

$$\begin{aligned} A &\subseteq ((A\Gamma M]\Gamma(A\Gamma M]]\cap((M\Gamma A]\Gamma(M\Gamma A]] \\ &= ((A\Gamma M\Gamma A]\Gamma M]\cap(M\Gamma(A\Gamma M\Gamma A]]. \end{aligned}$$

Since M is a right ideal of M, by hypothesis, it is idempotent i.e. $M=(M\Gamma M]$. Thus we have $(A\Gamma M\Gamma A]=(A\Gamma(M\Gamma M]\Gamma A]=((A\Gamma M]\Gamma(M\Gamma A]]$. Since $(A\Gamma M]$ is a right ideal and $(M\Gamma A]$ a left ideal of M, by hypothesis, $((A\Gamma M]\Gamma(M\Gamma A]]$ is a quasi-ideal of M. Then $(A\Gamma M\Gamma A]$ is a quasi-ideal of M, that is,

$$((A\Gamma M\Gamma A]\Gamma M] \cap (M\Gamma(A\Gamma M\Gamma A]] \subseteq (A\Gamma M\Gamma A].$$

Thus we obtain $A \subseteq (A\Gamma M\Gamma A]$, and M is regular. $\square$

As a consequence, a duo (and so a commutative) *po*-Γ-semigroup M is regular if and only if the ideals of M are idempotent.

Proposition 4. *Let M be a po- Γ-semigroup, R (resp. L) a right (resp. left) ideal of M and $\emptyset \neq A \subseteq M$. Then $(R\Gamma A]$ (resp. $(A\Gamma L]$) is a bi-ideal of M.*

Proof. First of all, $(R\Gamma A]$ is a nonempty subset of M. Moreover,

$$\begin{aligned}(R\Gamma A]\Gamma M\Gamma(R\Gamma A] &\subseteq (R\Gamma A]\Gamma(M]\Gamma(R\Gamma A] \subseteq ((R\Gamma A)\Gamma M\Gamma(R\Gamma A)]\\ &= (R\Gamma(A\Gamma M)\Gamma R\Gamma A] \subseteq ((R\Gamma M)\Gamma R\Gamma A]\\ &\subseteq (R\Gamma R\Gamma A] \subseteq ((R\Gamma M)\Gamma A) \subseteq (R\Gamma A],\end{aligned}$$

so $(R\Gamma A]$ is a bi-ideal of M. If $a \in (R\Gamma A]$ and $b \in S$ such that $b \le a$, then $b \in (R\Gamma A]$. Similarly, $(A\Gamma L]$ is a bi-ideal of M. $\square$

Theorem 5. *Let M be a regular ordered Γ-semigroup. Then B is a bi-ideal of M if and only if there exists a right ideal R and a left ideal L of M such that $B = (R\Gamma L]$.*

Proof. $\Longrightarrow$. Let B be a bi-ideal of M. Since M is regular, we have $B = (B\Gamma M\Gamma B]$, and $M = (M\Gamma M]$. Hence we obtain

$$B = (B\Gamma M\Gamma B] = (B\Gamma(M\Gamma M]\Gamma B] = ((B\Gamma M]\Gamma(M\Gamma B]],$$

where $(B\Gamma M]$ is a right ideal and $(M\Gamma B]$ a left ideal of M.
$\Longleftarrow$. If R is a right ideal and L a left ideal of M then, by Proposition 4, $(R\Gamma L]$ is a bi-ideal of M. $\square$

Proposition 6. *A po-Γ-semigroup M is regular if and only if for every right ideal R, every left ideal L and every bi-ideal B of M, we have $R \cap B \cap L \subseteq (R\Gamma B\Gamma L]$.*

Proof. We first remark that the property

$$A \subseteq (A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A] \text{ for every } A \subseteq M$$

characterizes the regular *po*-Γ-semigroups. In fact: Since M is regular, we have

$$\begin{aligned}A &\subseteq (A\Gamma M\Gamma A] \subseteq ((A\Gamma M\Gamma A]\Gamma M\Gamma(A\Gamma M\Gamma A]]\\ &= (A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A].\end{aligned}$$

Conversely, if $A \subseteq (A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A]$, then we have

$$\begin{aligned}A &\subseteq (A\Gamma(M\Gamma A)\Gamma(M\Gamma A)\Gamma M\Gamma A] \subseteq (A\Gamma M\Gamma M\Gamma M\Gamma A]\\ &= (A\Gamma(M\Gamma M)\Gamma M\Gamma A] \subseteq (A\Gamma M\Gamma M\Gamma A]\\ &= (A\Gamma(M\Gamma M)\Gamma A] \subseteq (A\Gamma M\Gamma A],\end{aligned}$$

and M is regular.
$\Longrightarrow$. Let R be a right ideal, L a left ideal, B a bi-ideal of M. Since M is regular, we have

$$\begin{aligned}R \cap B \cap L &\subseteq ((R \cap B \cap L)\Gamma M\Gamma(R \cap B \cap L)\Gamma M\Gamma(R \cap B \cap L)\Gamma M\Gamma(R \cap B \cap L)]\\ &\subseteq (R\Gamma M\Gamma B\Gamma M\Gamma B\Gamma M\Gamma L] = ((R\Gamma M)\Gamma(B\Gamma M\Gamma B)\Gamma(M\Gamma L)]\\ &\subseteq (R\Gamma B\Gamma L].\end{aligned}$$

$\Longleftarrow$. Let R be a right ideal, L a left ideal of M. Then $R \cap L \subseteq (R\Gamma L]$. Indeed: Since M is a bi-ideal of M, by hypothesis, we have $R \cap L \subseteq (R\Gamma M\Gamma L] \subseteq (R\Gamma L]$. □
In a similar way we prove the following

Proposition 7. *A po-Γ-semigroup M is intra-regular if and only if for every right ideal R, every left ideal L and every bi-ideal B of M, we have*
$R \cap B \cap L \subseteq (L\Gamma B\Gamma R]$.

Proposition 8. *A po-Γ-semigroup M is both regular and intra-regular if and only if for every right ideal R, every left ideal L and every bi-ideal B of M, we have $R \cap B \cap L \subseteq (B\Gamma R\Gamma L]$.*

Proposition 9. *A po-Γ-semigroup M is intra-regular and the left ideals of M are idempotent if and only if for every right ideal R, every left ideal L and every bi-ideal B of M, we have $R \cap B \cap L \subseteq (L\Gamma R\Gamma B]$.*

Proposition 10. *A po-Γ-semigroup M is intra-regular and the right ideals of M are idempotent if and only if for every right ideal R, every left ideal L and every bi-ideal B of M, we have $R \cap B \cap L \subseteq (B\Gamma L\Gamma R]$.*

Acknowledgment. I would like to express my warmest thanks to the editor of the journal Professor Kiyoshi Iséki, for editing, communicating the paper, and his prompt reply.

References

[1] W.E. Barnes, *On the Γ-rings of Nobusawa*, Pacific J. Math. **18** (1966), 411–422.

[2] J. Calais, *Demi-groupes quasi-inversifs*, C.R. Acad. Sci. Paris **252** (1961), 2357–2359.

[3] K. Iséki, *A characterization of regular semi-group*, Proc. Japan Acad. **32** (1956), 676–677.

[4] L. Kovács, *A note on regular rings*, Publ. Math. Debrecen **4** (1956), 465–468.

[5] J. Luh, *On the theory of simple Γ-rings*, Michigan Math. J. **16** (1969), 65–75.

[6] N. Nobusawa, *On a generalization of the ring theory*, Osaka J. Math. **1** (1964), 81–89.

[7] N.K. Saha, *The maximum idempotent separating congruence on an inverse Γ-semigroup*, Kyungpook Math. J. **34**, no. 1 (1964), 59–66.

[8] M.K. Sen, *On Γ-semigroups*, Algebra and its applications (New Delhi, 1981), 301–308, Lecture Notes in Pure and Appl. Math. **91**, Dekker, New York, 1984.

[9] M.K. Sen, N.K. Saha, *On Γ-semigroup*-I. Bull. Calcutta Math. Soc. **78**, no. 3 (1986), 180–186.

[10] M.K. Sen, N.K. Saha, *The maximum idempotent-separating congruence on an orthodox Γ-semigroup*, J. Pure Math. **7** (1990), 39–47.

[11] M.K. Sen, N.K. Saha, *Orthodox Γ-semigroups*, Internat. J. Math. Math. Sci. **13**, no. 3 (1990), 537–534.

[12] M.K. Sen, A. Seth, *Radical of Γ-semigroup*, Bull. Calcutta Math. Soc. **80**, no. 3 (1988), 189–196.

[13] M.K. Sen, A. Seth, *On po-Γ-semigroups*, Bull. Calcutta Math. Soc. **85**, no. 5 (1993), 445–450.

University of Athens
Department of Mathematics
157 84 Panepistimiopolis
Athens, Greece