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INTERVAL VALUED INTUITIONISTIC (S^*, T^*)-FUZZY BI-IDEALS OF SEMIGROUPS

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Abstract. We introduce the notion of interval-valued fuzzy bi-ideals with respect to t-norm T^* and s-norm S^* and investigate some of the properties. The homomorphism image and inverse image are investigated. In particular, by the congruence relations on semigroups, new interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideals are constructed.

1. **Introduction.** Fuzzy sets were initiated by Zadeh [9]. After that fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics. The notion of intuitionistic fuzzy sets was introduced by Atanassove [1]. In [8], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. Interval-valued fuzzy sets have many applications in several areas. For example, Zadeh [8] constructed a method of approximate inference using his interval-valued fuzzy set, Gorzalczany [4] studied the interval-valued fuzzy sets for approximate reasoning, and Roy and Biswas [7] studied interval-valued fuzzy relations and applied these in Sanxhez's approach for medical diagnosis. In this paper, we introduce the notion of interval-valued fuzzy bi-ideals with respect to t-norm T^* and s-norm S^*, and investigate some of the properties. The homomorphism image and inverse image are investigated. In particular, by the congruence relations on semigroups, new interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideals are constructed.

2. **Preliminaries.** We first recall some basic concepts which are used to present the paper.

Let X be a semigroup. By a *subsemigroup* of X we mean a non-empty subset A of X such that $A^2 \subseteq A$, and by a *left* (*right*) *ideal* of X we mean a non-empty subset A of X such that $XA \subseteq A$ ($AX \subseteq A$). By *two-sided ideal* or simply *ideal*, we mean a non-empty subset of X which is both a left and a right ideal of X. A subsemigroup A of a semigroup X is called a *bi-ideal* of X if $AXA \subseteq A$.

An *interval number* on $[0, 1]$, say $\bar{a}$, is a closed subinterval of $[0, 1]$, i.e., $\bar{a} = [a^-, a^+]$ where $0 \le a^- \le a^+ \le 1$. The set of all interval numbers is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$.

For any interval numbers $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ on $[0, 1]$, we define

(i) $\bar{a} \le \bar{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$,

(ii) $\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$, whenever $a^- + b^- \le 1$ and $a^+ + b^+ \le 1$.

(iii) $\inf \bar{a}_i = [\bigwedge_{i\in I} {a_i}^-, \bigwedge_{i\in I} {a_i}^+]$, $\sup \bar{a}_i = [\bigvee_{i\in I} {a_i}^-, \bigvee_{i\in I} {a_i}^+]$ for interval numbers $\bar{a}_i = [{a_i}^-, {a_i}^+] \in D[0, 1], i \in I$.

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It is clear that $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $0 = [0,0]$ as the least element and $1 = [1,1]$ as the greatest element.

By an *interval valued fuzzy set* F on X we mean

$$F = \{(x, [\mu_F^-(x), \mu_F^+(y)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy sets of X such that $\mu_F^- \leq \mu_F^+$ for all $x \in X$. Putting $\mu_F(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, [\mu_F(x) \mid x \in X\}$, where $\mu_F : X \to D[0,1]$.

3. **Interval valued intuitionistic** (S^*, T^*)**-fuzzy bi-ideals.**

In what follows, we use X to denote a semigroup unless otherwise specified.

Definition 3.1. A mapping $T^* : D[0,1] \times D[0,1] \to D[0,1]$ is called an *interval valued t-norm* (briefly, *i-v t-norm*) if it is satisfying the following condition :

(IT1) $T^*(\bar{x}, 1) = \bar{x}$,
(IT2) $T^*(\bar{x}, \bar{y}) \leq T^*(\bar{x}, \bar{z})$ if $\bar{y} \leq \bar{z}$,
(IT3) $T^*(\bar{x}, \bar{y}) = T^*(\bar{y}, \bar{x})$,
(IT4) $T^*(\bar{x}, T^*(\bar{y}, \bar{z})) = T^*(T^*(\bar{x}, \bar{y}), \bar{z})$

for every $\bar{x}, \bar{y}, \bar{z} \in D[0,1]$. An i-v t-norm T^* is said to be *idempotent* if $T^*(\bar{x}, \bar{x}) = \bar{x}$ for every $\bar{x} \in D[0,1]$.

If T is an idempotent t-norm, then the mapping $\triangle : D[0,1] \times D[0,1] \to D[0,1]$ defined by $\triangle(\bar{a}_1, \bar{a}_2) = [T({a_1}^-, {a_2}^-), T({a_1}^+, {a_2}^+)]$ is an idempotent i-v t-norm and is called an *idempotent interval t-norm.*

Proposition 3.2. *Every t-norm T^* has a useful property:*

$$T^*(\alpha, \beta) \leq \min(\alpha, \beta)$$

for all $\alpha, \beta \in [0,1]$.

Definition 3.3. A mapping $S^* : D[0,1] \times D[0,1] \to D[0,1]$ is called an *interval valued s-norm* (briefly, *i-v s-norm*) if it is satisfying the following condition :

(IS1) $S^*(\bar{x}, 0) = \bar{x}$,
(IS2) $S^*(\bar{x}, \bar{y}) \leq S^*(\bar{x}, \bar{z})$ if $\bar{y} \leq \bar{z}$,
(IS3) $S^*(\bar{x}, \bar{y}) = S^*(\bar{y}, \bar{x})$,
(IS4) $S^*(\bar{x}, S^*(\bar{y}, \bar{z})) = S^*(S^*(\bar{x}, \bar{y}), \bar{z})$

for every $\bar{x}, \bar{y}, \bar{z} \in D[0,1]$. An i-v s-norm S^* is said to be *idempotent* if $S^*(\bar{x}, \bar{x}) = \bar{x}$ for every $\bar{x} \in D[0,1]$.

If S is an idempotent s-norm, then the mapping $\triangle : D[0,1] \times D[0,1] \to D[0,1]$ defined by $\triangle(\bar{a}_1, \bar{a}_2) = [S({a_1}^-, {a_2}^-), S({a_1}^+, {a_2}^+)]$ is an idempotent i-v s-norm and is called an *idempotent interval s-norm.*

Proposition 3.4. *Every s-norm S^* has a useful property:*

$$\max(\alpha, \beta) \leq S^*(\alpha, \beta)$$

for all $\alpha, \beta \in [0,1]$.

According to Atanassove ([1], [2], [3]), an *interval valued intuitionistic fuzzy set* on X is defined as an object of the form $A = \{(x, \widetilde{M_A}(x), \widetilde{N_A}(x))\} \mid x \in X\}$, where $\widetilde{M_A}(x)$ and $\widetilde{N_A}(x)$ are interval valued fuzzy sets on X such that $0 \leq \sup \widetilde{M_A}(x) + \sup \widetilde{N_A}(x) \leq 1$ for all $x \in X$. For the sake of simplicity, interval valued intuitionistic fuzzy set will be denoted by $A = (\widetilde{M_A}, \widetilde{N_A})$.

Definition 3.5. An interval-valued intuitionistic fuzzy set $A = (\widetilde{M_A}, \widetilde{N_A})$ of X is called an *interval-valued intuitionistic* (S, T)*-fuzzy bi-ideal* of X if

(IV1) $\widetilde{M_A}(xy) \geq T^*(\widetilde{M_A}(x), \widetilde{M_A}(y))$, $\widetilde{M_A}(xay) \geq T^*(\widetilde{M_A}(x), \widetilde{M_A}(y))$.

(IV2) $\widetilde{N_A}(xy) \leq S^*(\widetilde{N_A}(x), \widetilde{N_A}(y))$, $\widetilde{N_A}(xay) \leq S^*(\widetilde{N_A}(x), \widetilde{N_A}(y))$ for all $x, a, y \in X$.

Example 3.1. Let $X := \{a, b, c, d, e\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

We define an interval valued intuitionstic fuzzy set $A = (\widetilde{M_A}, \widetilde{N_A})$ by

$$\widetilde{M_A}(x) = \begin{cases} [0.7,\ 0.8] & \text{if } x = a \\ [0.5,\ 0.7] & \text{if } x = b \\ [0.4,\ 0.5] & \text{if } x = c \\ [0.3,\ 0.5] & \text{if } x = d = e, \end{cases} \quad \text{and} \quad \widetilde{N_A}(x) = \begin{cases} [0.1,\ 0.2] & \text{if } x = a \\ [0.2,\ 0.3] & \text{if } x = b \\ [0.3,\ 0.4] & \text{if } x = c \\ [0.4,\ 0.5] & \text{if } x = d = e. \end{cases}$$

Let $T^* : D[0, 1] \times D[0, 1] \to D[0, 1]$ be a function defined by

$$T^*(\bar{x}, \bar{y}) = \bar{x} \wedge \bar{y}$$

and $S^* : D[0, 1] \times D[0, 1] \to D[0, 1]$ be a function defined by

$$S^*(\bar{x}, \bar{y}) = \bar{x} \vee \bar{y}$$

for all $\bar{x}, \bar{y} \in D[0, 1]$. Then T^* is a t-norm and S^* is a s-norm. It is easy to verify that $A = (\widetilde{M_A}, \widetilde{N_A})$ is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of semigroup X.

Let χ_H denote the characteristic function of a non-empty subset H in a semigroup X.

Theorem 3.6. *If H is a bi-ideal of a semigroup X, then the $I = (\chi_H, \bar{\chi}_H)$ is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X.*

Proof. Let $x, y \in X$. If $x, y \in H$, then $xy \in H$ since H is a bi-ideal of X. Hence we have $\chi_H(xy) = 1 \geq T^*(\chi_H(x), \chi_H(y))$ and $0 = 1 - \chi_H(xy) = \bar{\chi}_H(xy) \leq S^*(\bar{\chi}_H(x), \bar{\chi}_H(y))$. Now let $H \in X$. Then we get $xay \in H$. Thus $\chi_H(xay) = 1 \geq T^*(\chi_H(x), \chi_H(y))$ and $0 = 1 - \chi_H(xay) = \bar{\chi}_H(xay) \leq S^*(\bar{\chi}_H(x), \bar{\chi}_H(y))$. If $x \in H$ and $y \notin H$, (or, $x \notin H$ and $y \in H$), then $\chi_H(x) = 1$, or $\chi_H(y) = 0$. Thus we have $\chi_H(xy) \geq T^*(\chi_H(x), \chi_H(y)) = T^*(1, 0) = T^*(0, 1) = 0$ and $S^*(\bar{\chi}_H(x), \bar{\chi}_H(y)) = S^*(1 - \chi_H(x), 1 - \chi_H(y)) = S^*(0, 1) = 1 \geq \bar{\chi}_H(xy)$. Let $a \in X$. Then we obtain $\chi_H(xay) \geq T^*(\chi_H(x), \chi_H(y)) = T^*(1, 0) = T^*(0, 1) = 0$ and $S^*(\bar{\chi}_H(x), \bar{\chi}_H(y)) = S^*(1 - \chi_H(x), 1 - \chi_H(y)) = S^*(0, 1) = 1 \geq \bar{\chi}_H(xay)$. This proves the theorem. □

Theorem 3.7. *Let H be a nonempty subset of a semigroup X. If $I = (\chi_H, \bar{\chi}_H)$ satisfies (IV1) or (IV2), then H is a bi-ideal of a semigroup X.*

Proof. Suppose that $I = (\chi_H, \bar{\chi}_H)$ satisfy (IV1). Let $x, y \in H$ and $a \in X$. Then it follows from (IV1) that

$$\chi_H(xy) \geq T^*(\chi_H(x), \chi_H(y)) = T^*(1, 1) = 1$$

so that $\chi_H(xy) = 1$, i.e., $xy \in H$. Also we obtain

$$\chi_H(xay) \geq T^*(\chi_H(x), \chi_H(y)) = T^*(1, 1) = 1,$$

which imply $\chi_H(xay) = 1$ so that $xay \in H$. Hence H is a bi-ideal of X.

Now suppose that $I = (\chi_H, \bar{\chi}_H)$ satisfy (IV2). Let $x, y \in H$ and $a \in X$. Then from (IV2), we have

$$\bar{\chi}_H(xy) \leq S^*(\bar{\chi}_H(x), \bar{\chi}_H(y)) = S^*(1 - \chi_H(x), 1 - \chi_H(y)) = S^*(0, 0) = 0,$$

which imply $\bar{\chi}_H(xy) = 1 - \chi_H(xy) = 0$ so that $\chi_H(xy) = 1$, i.e., $xy \in H$. Also we get from (IV2),

$$\bar{\chi}_H(xay) \leq S^*(\bar{\chi}_H(x), \bar{\chi}_H(y)) = S^*(1 - \chi_H(x), 1 - \chi_H(y)) = S^*(0, 0) = 0,$$

which imply $\bar{\chi}_H(xay) = 1 - \chi_H(xay) = 0$ so that $\chi_H(xay) = 1$, i.e., $xay \in H$. This proves the theorem. □

For an interval valued intuitionistic fuzzy set $A = (\widetilde{M_A}, \widetilde{N_A})$, we define two levels as following,

$$U(\widetilde{M_A}; [t, s]) = \{x \in X \mid \widetilde{M_A}(x) \geq [t, s]\}$$

and

$$L(\widetilde{N_A}; [t, s]) = \{x \in X \mid \widetilde{N_A}(x) \leq [t, s]\}.$$

Theorem 3.8. *Let T^* and S^* be idempotent interval t-norm and s-norm respectively. Then $A = (\widetilde{M_A}, \widetilde{N_A})$ is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X if and only if for all $t, s \in [0, 1]$ and $t \leq s$, $U(\widetilde{M_A}; [t, s])$ and $L(\widetilde{N_A}; [t, s])$ are bi-ideals of X.*

Proof. Let $A = (\widetilde{M_A}, \widetilde{N_A})$ be an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X. For every $x, y \in U(\widetilde{M_A}; [t, s])$ and $a \in X$, we have $\widetilde{M_A}(x) \geq [t, s]$ and $\widetilde{M_A}(y) \geq [t, s]$. Hence $T^*(\widetilde{M_A}(x), \widetilde{M_A}(y)) \geq T^*([t, s], [t, s]) = [t, s]$, and so $\widetilde{M_A}(xy) \geq T^*(\widetilde{M_A}(x), \widetilde{M_A}(y)) \geq [t, s]$. This means $xy \in U(\widetilde{M_A}; [t, s])$. Also, we have $\widetilde{M_A}(xay) \geq T^*(\widetilde{M_A}(x), \widetilde{M_A}(y)) \geq T^*([t, s], [t, s]) = [t, s]$, which imply $\widetilde{M_A}(xay) \geq T^*(\widetilde{M_A}(x), \widetilde{M_A}(y)) \geq [t, s]$ so that $xay \in U(\widetilde{M_A}; [t, s])$. Therefore $U(\widetilde{M_A}; [t, s])$ is a bi-ideal of X. The proof is similar in case of $L(\widetilde{N_A}; [t, s])$.

Conversely, assume that for every $[t, s] \in D[0, 1]$ any non-empty $U(\widetilde{M_A}; [t, s])$ is a bi-ideal of X. If $[t_0, s_0] = T^*(\widetilde{M_A}(x), \widetilde{M_A}(y))$ for some $x, y \in X$. Then we have $x, y \in \widetilde{M_A}; [t_0, s_0])$, which imply $xy \in \widetilde{M_A}; [t_0, s_0])$. Therefore $\widetilde{M_A}(xy) \geq [t_0, s_0] = T^*(\widetilde{M_A}(x), \widetilde{M_A}(y))$. Also suppose that there exist $x_1, a_1, y_1 \in X$ such that

$$\widetilde{M_A}(x_1 a_1 y_1) < T^*(\widetilde{M_A}(x_1), \widetilde{M_A}(y_1)).$$

Let $T^*(\widetilde{M_A}(x_1), \widetilde{M_A}(y_1)) = [\gamma_1, \gamma_2]$, and $\widetilde{M_A}(x_1 a_1 y_1) = [\delta_1, \delta_2]$. Then

$$[\delta_1, \delta_2] < T^*(\widetilde{M_A}(x_1), \widetilde{M_A}(y_1)) = [\gamma_1, \gamma_2].$$

Let $[\lambda_1, \lambda_2] = \frac{1}{2}(\widetilde{M_A}(x_1 a_1 y_1) + T^*(\widetilde{M_A}(x), \widetilde{M_A}(y))$. Then

$$[\lambda_1, \lambda_2] = \frac{1}{2}([\delta_1, \delta_2] + [\gamma_1, \gamma_2]) = [\frac{\delta_1 + \gamma_1}{2}, \frac{\delta_2 + \gamma_2}{2}].$$

It follows that $[\lambda_1, \lambda_2] > [\delta_1, \delta_2] = \widetilde{M_A}(x_1 a_1 y_1)$ since $\gamma_1 > \lambda_1 > \delta_1$ and $\gamma_2 > \lambda_2 > \delta_2$. Hence $x_1 a_1 y_1 \notin U(\widetilde{M_A}; [\lambda_1, \lambda_2])$. On the other hand, noticing that

$$\min(\widetilde{M_A}(x), \widetilde{M_A}(y)) \geq T^*(\widetilde{M_A}(x), \widetilde{M_A}(y)),$$

we get $\widetilde{M_A}(x_1) \geq [\gamma_1, \gamma_2] \geq [\lambda_1,\ \lambda_2]$ and $\widetilde{M_A}(y_1) \geq [\gamma_1, \gamma_2)] > [\lambda_1,\ \lambda_2]$, and so $x_1, y_1 \in U(\widetilde{M_A}; [\lambda_1, \lambda_2])$. It contradicts that $U(\widetilde{M_A}; [\lambda_1, \lambda_2])$ is a bi-ideal of X. Hence $\widetilde{M_A}(xay) \geq T^*(\widetilde{M_A}(x), \widetilde{M_A}(y))$. for all $x, y, a \in X$. This completes the proof. □

Let $A = (\widetilde{M_A}, \widetilde{N_A})$ be an interval valued intuitionistic fuzzy set of X and let $t_1, t_2, s_1, s_2 \in [0,1]$ such that $t_1 \leq s_1$ and $t_2 \leq s_2$. Put

$$H_{[t_2,s_2]}^{[t_1,s_1]} = \{x \in X \mid \widetilde{M}_A(x) \geq [t_1, s_1] \text{ and } \widetilde{N}_A(x) \leq [t_2, s_2]\}.$$

Then we have

$$H_{[t_2,s_2]}^{[t_1,s_1]} = U(\widetilde{M_A}; [t_1, s_1]) \cap L(\widetilde{N_A}; [t_2, s_2]).$$

Corollary 3.9. *Let T^* and S^* be idempotent interval t-norm and s-norm respectively. Then $A = (\widetilde{M_A}, \widetilde{N_A})$ is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X if and only if for all $t_1, s_1, t_2, s_2 \in [0,1], t_1 \leq s_1$ and $t_2 \leq s_2$, $H_{[t_2,s_2]}^{[t_1,s_1]}$ is a bi-ideal of X.*

Proof. It is immediately followed by Theorem 3.9. □

Definition 3.10. Let $f : X \to Y$ be a mapping and $A = (\widetilde{M_A}, \widetilde{N_A})$ and let $B = (\widetilde{M_B}, \widetilde{N_B})$ be interval valued intuitionistic fuzzy sets X and Y, respectively. Then *the image* of $f[A] = (f(\widetilde{M_A}), f(\widetilde{N}_A)$ of A is the interval valued intuitionistic fuzzy set of Y defined by

$$f(\widetilde{M_A}(y) := \begin{cases} \sup\limits_{z \in f^{-1}(y)} \widetilde{M}_A(z), & f^{-1}(y) \neq \emptyset, \\ [0,\ 0] & \text{otherwise}, \end{cases}$$

and

$$f(\widetilde{N}_A(y)) := \begin{cases} \inf\limits_{z \in f^{-1}(y)} \widetilde{N}_A(z), & f^{-1}(y) \neq \emptyset, \\ [1,\ 1] & \text{otherwise}, \end{cases}$$

for all $y \in Y$. The *inverse image* $f^{-1}(B) = (f^{-1}(\widetilde{M_B}), f^{-1}(\widetilde{N}_B))$ of B is an interval valued intuitionistic fuzzy set defined by

$$f^{-1}(\widetilde{M_B})(x) = \widetilde{M}_{f^{-1}(B)}(x) = \widetilde{M}_B(f(x)),$$

and

$$f^{-1}(\widetilde{N}_B)(x) = \widetilde{N}_{f^{-1}(B)}(x) = \widetilde{N}_B(f(x)),$$

for all $x \in X$.

Lemma 3.11. *Let X_1 and X_2 be two semigroups and $f : X_1 \to X_2$ an epimorphism.*
(1) If H is a bi-ideal of X_1, then $f(H)$ is a bi-ideal of X_2.
(2) If J is a bi-ideal of X_2, then $f^{-1}(J)$ is a bi-ideal of X_1.

Proof. It is immediately straightforward. □

Theorem 3.12. *Let X_1 and X_2 be two maps and $f : X_1 \to X_2$ an epimorphism and T^* and S^* idempotent interval valued t-norm and s-norm, respectively.*

(1) If $A = (\widetilde{M_A}, \widetilde{N_A})$ is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X_1, then the image $f[A] = (f(\widetilde{M_A}), f(\widetilde{N}_A)$ of A is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X_2.

(2) If $B = (\widetilde{M_B}, \widetilde{N_B})$ is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X_2, then the inverse image $f^{-1}(B) = (f^{-1}(\widetilde{M_B}), f^{-1}(\widetilde{N}_B))$ is an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X_1.

Proof. (1) Let $A = (\widetilde{M_A}, \widetilde{N_A})$ be an interval valued intuitionistic (S^*, T^*)-fuzzy bi-ideal of X_1. By Theorem 3.9, $U(\widetilde{M_A}; [t,s])$ and $L(\widetilde{N_A}; [t,s])$ are bi-ideals of X_1 for every $[t,s] \in D[0,1]$. Therefore, by Lemma 3.12, $f(U(\widetilde{M_A}; [t,s]))$ and $f(L(\widetilde{N_A}; [t,s]))$ are bi-ideal of X_2. But $U(f(\widetilde{M_A}); [t,s]) = f(U(\widetilde{M_A}; [t,s]))$ and $L(f(\widetilde{N_A}); [t,s]) = f(L(\widetilde{N_A}; [t,s]))$. Hence

$U(f(\widetilde{M_A});[t,s])$ and $L(f(\widetilde{N_A});[t,s])$ are bi-ideals of X_2. Therefore $f[A]$ is an interval valued intuitionistic (S^*,T^*)-fuzzy bi-ideal of X_2.

(2) For any $x,y,a\in X$, we have

$$\begin{aligned}\widetilde{M}_{f^{-1}(B)}(xy) =&\widetilde{M}_B(f(xy))\geq T^*(\widetilde{M}_{(B)}(f(x)),\widetilde{M}_{(B)}(f(y))\\ &= T^*(\widetilde{M}_{f^{-1}(B)}(x),\widetilde{M}_{f^{-1}(B)}(y)).\end{aligned}$$

and

$$\begin{aligned}\widetilde{N}_{f^{-1}(B)}(xy) =&\widetilde{N}_B(f(xy))\leq S^*(\widetilde{N}_{(B)}(f(x)),\widetilde{N}_{(B)}(f(y))\\ &= S^*(\widetilde{M}_{f^{-1}(B)}(x),\widetilde{N}_{f^{-1}(B)}(y)).\end{aligned}$$

Also we have

$$\begin{aligned}\widetilde{M}_{f^{-1}(B)}(xay) =&\widetilde{M}_B(f(xay))\geq T^*(\widetilde{M}_{(B)}(f(x)),\widetilde{M}_{(B)}(f(y))\\ &= T^*(\widetilde{M}_{f^{-1}(B)}(x),\widetilde{M}_{f^{-1}(B)}(y))\end{aligned}$$

and

$$\begin{aligned}\widetilde{N}_{f^{-1}(B)}(xay) =&\widetilde{N}_B(f(xay))\leq S^*(\widetilde{N}_{(B)}(f(x)),\widetilde{N}_{(B)}(f(y))\\ &= S^*(\widetilde{N}_{f^{-1}(B)}(x),\widetilde{N}_{f^{-1}(B)}(y)).\end{aligned}$$

This completes the proof. □

Let θ be a congruence relation on a semigroup X. It is easy to check that $(X/\theta,\odot)$ is a semigroup where $\theta(x)\odot\theta(y)=\theta(xy)$ for every $\theta(x),\theta(y)\in X/\theta$.

Definition 3.13. Let $A=(\widetilde{M_A},\widetilde{N_A})$ be an interval valued intuitionistic fuzzy set. The intuitionistic fuzzy set $A/\theta=(\widetilde{M}_{A/\theta},\widetilde{N}_{A/\theta})$ is defined as a pair of maps

$$\begin{cases}\widetilde{M}_{A/\theta}:X/\theta\to D[0,1]\\ \widetilde{N}_{A/\theta}:X/\theta\to D[0,1]\end{cases}$$

where $\widetilde{M}_{A/\theta}(\theta(x))=\sup\limits_{a\in\theta(x)}\widetilde{M}_A(a)$ and $\widetilde{N}_{A/\theta}(\theta(x))=\inf\limits_{a\in\theta(x)}\widetilde{N}_A(a)$.

Theorem 3.14. *Let X be a semigroup. If $A=(\widetilde{M_A},\widetilde{N_A})$ is an interval valued intuitionistic (S^*,T^*)-fuzzy bi-ideal of X, then $A/\theta=(\widetilde{M}_{A/\theta},\widetilde{N}_{A/\theta})$ is an interval valued intuitionistic (S^*,T^*)-fuzzy bi-ideal of X/θ.*

Proof. Let $A=(\widetilde{M_A},\widetilde{N_A})$ is an interval valued intuitionistic (S^*,T^*)-fuzzy bi-ideal of X and let $\theta(x),\theta(y)\in X/\theta$. Then we have

$$\begin{aligned}T^*(\widetilde{M}_{A/\theta}(\theta(x)),\widetilde{M}_{A/\theta}(\theta(y))) &= T^*(\sup_{a\in\theta(x)}\widetilde{M}_A(a),\sup_{b\in\theta(y)}\widetilde{M}_A(b))\\ &= \sup_{a\in\theta(x),b\in\theta(y)}T^*(\widetilde{M}_A(a),\widetilde{M}_A(b))\leq\sup_{a\in\theta(x),b\in\theta(y)}\widetilde{M}_A(ab)\\ &\leq\sup_{a\in\theta(x),b\in\theta(y)}(\sup_{t\in\theta(ab)}\widetilde{M}_A(t))=\sup_{a\in\theta(x),b\in\theta(y)}\widetilde{M}_{A/\theta}(\theta(ab))\\ &=\widetilde{M}_{A/\theta}(\theta(ab)),\end{aligned}$$

for all $a\in\theta(x),b\in\theta(y)$. On the other hand, we get

$$\begin{aligned}\widetilde{M}_{A/\theta}(\theta(ab)) &= \widetilde{M}_{A/\theta}(\theta(a)\odot\theta(b))=\widetilde{M}_{A/\theta}(\theta(x)\odot\theta(y))\\ &=\widetilde{M}_{A/\theta}(\theta(xy)),\end{aligned}$$

which imply

$$T^*(\widetilde{M}_{A/\theta}(\theta(x)),\widetilde{M}_{A/\theta}(\theta(y)))\leq\widetilde{M}_{A/\theta}(\theta(xy)).$$

Also we obtain

$$\begin{aligned} S^*(\widetilde{N}_{A/\theta}(\theta(x)), \widetilde{N}_{A/\theta}(\theta(y))) &= S^*(\inf_{a\in\theta(x)} \widetilde{N}_A(a), \inf_{b\in\theta(y)} \widetilde{N}_A(b)) \\ &= \inf_{a\in\theta(x), b\in\theta(y)} S^*(\widetilde{N}_A(a), \widetilde{N}_A(b)) \geq \inf_{a\in\theta(x), b\in\theta(y)} \widetilde{N}_A(ab) \\ &\geq \inf_{a\in\theta(x), b\in\theta(y)} (\inf_{t\in\theta(ab)} \widetilde{N}_A(t)) = \inf_{a\in\theta(x), b\in\theta(y)} \widetilde{N}_{A/\theta}(\theta(ab)) \\ &= \widetilde{N}_{A/\theta}(\theta(ab)), \end{aligned}$$

for all $a \in \theta(x), b \in \theta(y)$. On the other hand, we get

$$\begin{aligned} \widetilde{N}_{A/\theta}(\theta(ab)) &= \widetilde{N}_{A/\theta}(\theta(a) \odot \theta(b)) = \widetilde{N}_{A/\theta}(\theta(x) \odot \theta(y)) \\ &= \widetilde{N}_{A/\theta}(\theta(xy)), \end{aligned}$$

which imply

$$S^*(\widetilde{N}_{A/\theta}(\theta(x)), \widetilde{N}_{A/\theta}(\theta(y))) \geq \widetilde{N}_{A/\theta}(\theta(xy)).$$

Similarly, we can show that

$$T^*(\widetilde{M}_{A/\theta}(\theta(x)), \widetilde{M}_{A/\theta}(\theta(y))) \leq \widetilde{M}_{A/\theta}(\theta(xay))$$

and

$$S^*(\widetilde{N}_{A/\theta}(\theta(x)), \widetilde{N}_{A/\theta}(\theta(y))) \geq \widetilde{N}_{A/\theta}(\theta(xay))$$

for all $a \in X$. This completes the proof. □

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