# GEOMETRIC MEANS OF POSITIVE OPERATORS II 

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#### Abstract

Borrowing a technique due to Ando-Li-Mathias, we define a geometric mean of $(k+1)$ (positive invertible) operators from that of $k$ (or a $k$-tuple of) operators with a parameter $\lambda \in(0,1]$. If $\lambda=1$, then the corresponding geometric mean $G_{\lambda}\left(=G_{1}\right)$ of $(k+1)$ operators is one defined by Ando-Li-Mathias, and if $\lambda=2 / 3$, then $G_{\lambda}$ is one given by one of the authors in the preceding paper. We also show that a formula due to Yamazaki of the geometric mean for a 3-tuple of $2 \times 2$ matrices satisfying a trace condition does not depend on any choice of a parameter in construction.


## 1. Introduction

Borrowing a technique presented in Ando-Li-Mathias [2], based on the Riccati equation, one of the authors defined new geometric means of more than two (positive invertible) operators on a Hilbert space [9]. In succession of the paper we shall define such geometric means with parameters.

Let $A$ and $B$ be operators on a Hilbert space $H$. Then the weighted geometric (or $\alpha$ power) mean $A \nVdash_{\alpha} B$ for $\alpha \in(0,1]$ is defined [6] by

$$
A \not \sharp_{\alpha} B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}} .
$$

The usual geometric mean $A \sharp B$ is given as the case $\alpha=1 / 2$. For three operators $A, B, C$ and for $\lambda \in(0,1]$, let us define three sequences $\left\{A_{\lambda, n}\right\},\left\{B_{\lambda, n}\right\}$ and $\left\{C_{\lambda, n}\right\}$, by $A_{\lambda, 1}=$ $A, B_{\lambda, 1}=B, C_{\lambda, 1}=C$,

$$
\begin{align*}
& A_{\lambda, n+1}=A_{\lambda, n} \sharp \lambda\left(B_{\lambda, n} \sharp C_{\lambda, n}\right), \\
& B_{\lambda, n+1}=B_{\lambda, n} \sharp \lambda\left(C_{\lambda, n} \sharp A_{\lambda, n}\right) \quad \text { and }  \tag{1.1}\\
& C_{\lambda, n+1}=C_{\lambda, n} \sharp \lambda\left(A_{\lambda, n} \sharp B_{\lambda, n}\right) \quad \text { for } n \geq 1 .
\end{align*}
$$

Then we shall obtain a common limit of them, which we define a geometric mean $G_{\lambda}=$ $G_{\lambda}(A, B, C)$ with a parameter $\lambda$. If $\lambda=1$, then the geometric mean $G_{\lambda}\left(=G_{1}\right)$ is one defined in [2], and if $\lambda=2 / 3$, then $G_{\lambda}$ is one defined in [9].

The above technical device using sequential limits due to Ando-Li-Mathias causes the geometric mean $G_{\lambda}$ to satisfy a property, permutation invariance ( P 3 , below) for three operators. In [2], Ando-Li-Mathias stated the following ten postulates for a geometric mean $G\left(A_{1}, \ldots, A_{k}\right)$ of $k$ (or a $k$-tuple of) operators $A_{1}, \ldots, A_{k}$ to be a reasonable one, (the usual geometric mean $G\left(A_{1}, A_{2}\right)=A_{1} \sharp A_{2}$ is reasonable):

P1 Consistency with scalars. If $A_{1}, A_{2}, \ldots, A_{k}$ commute then

$$
G\left(A_{1}, A_{2}, \ldots, A_{k}\right)=\left(A_{1} A_{2} \ldots A_{k}\right)^{\frac{1}{k}} .
$$

[^0]P1' This implies $G(\overbrace{A, \ldots, A}^{k})=A$.
P2 Joint homogeneity. $G\left(a_{1} A_{1}, a_{2} A_{2}, \ldots, a_{k} A_{k}\right)=\left(a_{1} a_{2} \cdots a_{k}\right)^{\frac{1}{k}} G\left(A_{1}, A_{2}, \ldots, A_{k}\right) \quad$ for $a_{i} \geq 0$ with $i=1, \ldots, k$.
P2' This implies $G\left(a A_{1}, a A_{2}, \ldots, a A_{k}\right)=a G\left(A_{1}, A_{2}, \ldots, A_{k}\right) \quad(a \geq 0)$.
P3 Permutation invariance. For any permutation $\pi\left(A_{1}, A_{2}, \ldots, A_{k}\right)$ of $\left(A_{1}, A_{2}, \ldots, A_{k}\right)$, $G\left(A_{1}, A_{2}, \ldots, A_{k}\right)=G\left(\pi\left(A_{1}, A_{2}, \ldots, A_{k}\right)\right)$.
P4 Monotonicity. The map $\left(A_{1}, A_{2}, \ldots, A_{n}\right) \mapsto G\left(A_{1}, A_{2}, \ldots, A_{n}\right)$ is monotone, i.e., if $A_{i} \geq B_{i}$ for $i=1, \ldots, k$, then $G\left(A_{1}, A_{2}, \ldots, A_{k}\right) \geq G\left(B_{1}, B_{2}, \ldots, B_{k}\right)$.
P5 Continuity from above. If $\left\{A_{1}^{(n)}\right\},\left\{A_{2}^{(n)}\right\}, \ldots,\left\{A_{k}^{(n)}\right\}$ are monotonic decreasing sequences converging to $A_{1}, A_{2}, \ldots, A_{k}$, respectively, then $\left\{G\left(A_{1}^{(n)}, A_{2}^{(n)}, \ldots, A_{k}^{(n)}\right)\right\}$ converges to $G\left(A_{1}, A_{2}, \ldots, A_{k}\right)$.
P6 Congruence invariance. For any invertible $S$,

$$
G\left(S^{*} A_{1} S, S^{*} A_{2} S, \ldots, S^{*} A_{k} S\right)=S^{*} G\left(A_{1}, A_{2}, \ldots, A_{k}\right) S
$$

P7 Joint concavity. The map $\left(A_{1}, A_{2}, \ldots, A_{k}\right) \mapsto G\left(A_{1}, A_{2}, \ldots, A_{k}\right)$ is jointly concave:

$$
\begin{aligned}
& G\left(\lambda A_{1}+(1-\lambda) A_{1}^{\prime}, \lambda A_{2}+(1-\lambda) A_{2}^{\prime}, \ldots, \lambda A_{k}+(1-\lambda) A_{k}^{\prime}\right) \\
\geq & \lambda G\left(A_{1}, A_{2}, \ldots, A_{k}\right)+(1-\lambda) G\left(A_{1}^{\prime}, A_{2}^{\prime}, \ldots, A_{k}^{\prime}\right)(0<\lambda<1)
\end{aligned}
$$

P8 Self-duality. $G\left(A_{1}, A_{2}, \ldots, A_{k}\right)^{*}=G\left(A_{1}, A_{2}, \ldots, A_{k}\right)$. The dual $G\left(A_{1}, A_{2}, \ldots, A_{k}\right)^{*}$ is defined by

$$
G\left(A_{1}, A_{2}, \ldots, A_{k}\right)^{*}=G\left(A_{1}^{-1}, A_{2}^{-1}, \ldots, A_{k}^{-1}\right)^{-1}
$$

P9 (In case $A_{1}, A_{2}, \ldots, A_{k}$ are matrices.) Determinant identity.

$$
\operatorname{det} G\left(A_{1}, A_{2}, \ldots, A_{k}\right)=\left(\operatorname{det} A_{1} \cdot \operatorname{det} A_{2} \cdots \operatorname{det} A_{k}\right)^{\frac{1}{k}}
$$

P10 The arithmetic-geometric-harmonic mean inequaility.

$$
\frac{A_{1}+A_{2}+\cdots+A_{k}}{k} \geq G\left(A_{1}, A_{2}, \cdots, A_{k}\right) \geq\left(\frac{A_{1}^{-1}+A_{2}^{-1}+\cdots+A_{k}^{-1}}{k}\right)^{-1}
$$

In this note, we define a geometric mean of $(k+1)$ operators with a parameter $\lambda$ which still satisfies the above properties P1-P10 from a given geometric mean of $k$ operators satisfying all properties. Based on a method in [2], Yamazaki [11] obtained a formula of the geometric mean of $2 \times 2$ matrices under a trace condition. We shall show that the formula does not depend on any special choice of a parameter in the process of construction.

Without occurrence of ambiguity, we shall often abbreviate the letter $\lambda$. All operators (or matrices) are assumed to be positive invertible (or positive definite) if stated otherwise.

## 2. Definition of geometric means of more than two operators

Let $\Omega$ be the set of all (positive invertible) operators on $H$. Then the Thompson metric on $\Omega$ is defined ([10], [3], [4]) by

$$
d(A, B)=\max \{\log M(A / B), \log M(B / A)\} \quad \text { for } A, B \in \Omega
$$

where

$$
M(A / B)=\inf \{\mu>0: A \leq \mu B\}\left(=\left\|B^{-1 / 2} A B^{-1 / 2}\right\|\right)
$$

We remark that $\Omega$ is complete with respect to the Thompson metric topology. As a basic inequality with respect to the metric, the following inequality for a weighted geometric mean of two operators holds [3], [4]:

$$
\begin{align*}
& d\left(A_{1} \sharp_{\alpha} A_{2}, B_{1} \sharp_{\alpha} B_{2}\right) \leq(1-\alpha) d\left(A_{1}, B_{1}\right)+\alpha d\left(A_{2}, B_{2}\right) \\
& \text { for } A_{1}, A_{2}, B_{1}, B_{2} \in \Omega \text { and } \alpha \in(0,1) . \tag{2.1}
\end{align*}
$$

Now in order to define our geometric mean $G_{\lambda}\left(A_{1}, \ldots, A_{k+1}\right)$ of $(k+1)$ operators from a given one of $k(\geq 2)$ operators, we want to assume a useful inequality:

$$
\begin{equation*}
d\left(G\left(A_{1}, \ldots, A_{k}\right), G\left(B_{1}, \ldots, B_{k}\right)\right) \leq \frac{1}{k} \sum_{i=1}^{k} d\left(A_{i}, B_{i}\right) \tag{2.2}
\end{equation*}
$$

for another $k$-tuple of operators $B_{1}, \ldots, B_{k}$.
Theorem 2.1. The geometric mean $G_{\lambda}\left(A_{1}, \ldots, A_{k+1}\right)$ is always defined as the common limit of the following $(k+1)$ sequences $\left\{A_{1}^{(r)}\right\}, \ldots,\left\{A_{k+1}^{(r)}\right\}$ of $(k+1)$ operators $A_{1}, \ldots, A_{k+1}$ :

$$
\begin{align*}
& A_{i}^{(1)}=A_{i} \text { for } i=1, \ldots, k+1, \quad \text { and } \\
& A_{i}^{(r+1)}=A_{i}^{(r)} \sharp_{\lambda} G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right)\left(=A_{i}^{(r)} \sharp_{\lambda} G\left(A_{1}^{(r)}, \ldots, A_{i-1}^{(r)}, A_{i+1}^{(r)}, \ldots, A_{k+1}^{(r)}\right)\right)  \tag{2.3}\\
& \quad \text { for } r \geq 1, \quad i=1, \ldots, k+1 .
\end{align*}
$$

where $\lambda \in(0,1]$ and $G\left(A_{1}, \ldots, A_{k}\right)$ is a geometric mean of $k$ operators satisfying P1-P10 and the inequality (2.2). The geometric mean $G_{\lambda}\left(A_{1}, \ldots, A_{k+1}\right)$ satisfies P1-P10, and furthermore, the following inequality holds:

$$
\begin{equation*}
d\left(G_{\lambda}\left(A_{1}, \ldots, A_{k+1}\right), G_{\lambda}\left(B_{1}, \ldots, B_{k+1}\right)\right) \leq \frac{1}{k+1} \sum_{i=1}^{k+1} d\left(A_{i}, B_{i}\right) \tag{2.4}
\end{equation*}
$$

corresponding to (2.2) for another $(k+1)$-tuple $B_{1}, \ldots, B_{k+1}$ of operators.
Proof. To see that all sequences $\left\{A_{i}^{(r)}\right\}$ are convergent with a common limit we first show that for $i, j=1, \ldots, k+1, i \neq j$

$$
\begin{equation*}
d\left(A_{i}^{(r+1)}, A_{j}^{(r+1)}\right) \leq\left(1-\frac{k-1}{k} \lambda\right)^{r} d\left(A_{i}, A_{j}\right) \tag{2.5}
\end{equation*}
$$

By the definition (2.3) of $A_{i}^{(r)}$ and the inequalities (2.1) and (2.4), we have

$$
\begin{aligned}
d\left(A_{i}^{(r+1)},\right. & \left.A_{j}^{(r+1)}\right)=d\left(A_{i}^{(r)} \sharp_{\lambda} G\left(\left(A_{\ell}^{(r)}\right)_{\ell \neq i}\right), A_{j}^{(r)} \sharp_{\lambda} G\left(\left(A_{\ell}^{(r)}\right)_{\ell \neq j}\right)\right) \\
& \leq(1-\lambda) d\left(A_{i}^{(r)}, A_{j}^{(r)}\right)+\lambda d\left(G\left(\left(A_{\ell}^{(r)}\right)_{\ell \neq i}\right), G\left(\left(A_{\ell}^{(r)}\right)_{\ell \neq j}\right)\right) \\
& \leq(1-\lambda) d\left(A_{i}^{(r)}, A_{j}^{(r)}\right)+\lambda \cdot \frac{1}{k} d\left(A_{i}^{(r)}, A_{j}^{(r)}\right) \\
& =\left(1-\frac{k-1}{k} \lambda\right) d\left(A_{i}^{(r)}, A_{j}^{(r)}\right) .
\end{aligned}
$$

Hence by iteration with respect to $r$ we can obtain the desired inequality. Next we show

$$
\begin{equation*}
d\left(A_{i}^{(r+1)}, A_{i}^{(r)}\right) \leq \frac{\lambda}{k}\left(1-\frac{k-1}{k} \lambda\right)^{r-1} K_{i} \tag{2.6}
\end{equation*}
$$

where $K_{i}=\sum_{\ell=1, \ell \neq i}^{k+1} d\left(A_{i}, A_{\ell}\right)$. Note that

$$
A_{i}^{(r)}=A_{i}^{(r)} \sharp_{\lambda} G(\overbrace{A_{i}^{(r)}, \ldots, A_{i}^{(r)}}^{k}) .
$$

Using (2.2), we have

$$
d\left(A_{i}^{(r+1)}, A_{i}^{(r)}\right) \leq \lambda d(G\left(\left(A_{\ell}^{(r)}\right)_{\ell \neq i}\right), G(\overbrace{A_{i}^{(r)}, \ldots, A_{i}^{(r)}}^{k})) \leq \lambda \cdot \frac{1}{k} \sum_{\ell=1, \ell \neq i}^{k+1} d\left(A_{i}^{(r)}, A_{\ell}^{(r)}\right) .
$$

Hence from (2.5)

$$
d\left(A_{i}^{(r+1)}, A_{i}^{(r)}\right) \leq \frac{\lambda}{k} \cdot \sum_{\ell=1, \ell \neq i}^{k+1}\left(1-\frac{k-1}{k} \lambda\right)^{r-1} d\left(A_{\ell}, A_{i}\right)=\frac{\lambda}{k}\left(1-\frac{k-1}{k} \lambda\right)^{r-1} K_{i}
$$

which is the desired inequality. Now we see that for any $i$, the sequence $\left\{A_{i}^{(r)}\right\}$ is convergent, or a Cauchy sequence. In fact, for $r \leq s$

$$
\begin{aligned}
d\left(A_{i}^{(r+1)}, A_{i}^{(s+1)}\right) & \leq \sum_{\ell=r+1}^{s} d\left(A_{i}^{(\ell)}, A_{i}^{(\ell+1)}\right) \leq \frac{\lambda}{k} K_{i} \sum_{\ell=r+1}^{s}\left(1-\frac{k-1}{k} \lambda\right)^{\ell-1} \\
& \leq \frac{\lambda}{k} K_{i} \cdot\left(1-\frac{k-1}{k} \lambda\right)^{r} /\left(\frac{k-1}{k} \lambda\right)=\frac{K_{i}}{k-1}\left(1-\frac{k-1}{k} \lambda\right)^{r} .
\end{aligned}
$$

Hence $d\left(A_{i}^{(r+1)}, A_{i}^{(s+1)}\right) \rightarrow 0$ as $r(<s) \rightarrow \infty$, so that $\left\{A_{i}^{(r)}\right\}$ is convergent. From (2.5), we easily see that all $\left\{A_{i}^{(r)}\right\}$ have the same limit, which guarantees the desired geometric mean to be defined.

It is not difficult to see that the geometric mean $G_{\lambda}\left(A_{1}, \ldots, A_{k+1}\right)$ satisfies all properties P1-P10. For example, to see P3, let $\pi\left(A_{1}, A_{2}, \ldots, A_{k+1}\right)=\left(A_{\pi(1)}, \ldots, A_{\pi(k+1)}\right)$ be a permutation of $\left(A_{1}, A_{2}, \ldots, A_{k+1}\right)$, and let

$$
\begin{aligned}
B_{i}^{(1)}=A_{\pi(i)}^{(1)}=A_{\pi(i)}, \quad B_{i}^{(r+1)}=B_{i}^{(r)} \not \sharp_{\lambda} G\left(\left(B_{j}\right)_{j \neq i}^{(r)}\right) \\
\quad \text { for } i=1, \ldots, k+1, r \geq 1 .
\end{aligned}
$$

Then we see that $B_{i}^{(r)}=A_{\pi(i)}^{(r)}$. In fact, assuming that $B_{i}^{(r)}=A_{\pi(i)}^{(r)}(i=1, \ldots, k+1)$, we have

$$
B_{i}^{(r+1)}=A_{\pi(i)}^{(r)} \sharp \lambda G\left(\left(A_{\pi(j)}\right)_{j \neq i}\right)=A_{\pi(i)}^{(r+1)} .
$$

Hence $\left\{B_{i}^{(r)}\right\}$ and $\left\{A_{\pi(i)}^{(r)}\right\}$ coincide, so that they converge to the same limit, which is desired.
For the inequality (2.4), let the sequences $\left\{B_{1}^{(r)}\right\}, \ldots,\left\{B_{k+1}^{(r)}\right\}$ be defined corresponding to $B_{1}, \ldots, B_{k+1}$, similarly as (2.3) for $A_{1}, \ldots, A_{k+1}$. Then for each $i$, from (2.1) and the
assumption (2.2), we have

$$
\begin{aligned}
d\left(A_{i}^{(r+1)},\right. & \left.B_{i}^{(r+1)}\right)=d\left(A_{i}^{(r)} \sharp_{\lambda} G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right), B_{i}^{(r)} \sharp_{\lambda} G\left(\left(B_{j}^{(r)}\right)_{j \neq i}\right)\right) \\
& \leq(1-\lambda) d\left(A_{i}^{(r)}, B_{i}^{(r)}\right)+\lambda d\left(G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right), G\left(\left(B_{j}^{(r)}\right)_{j \neq i}\right)\right) \\
& \leq(1-\lambda) d\left(A_{i}^{(r)}, B_{i}^{(r)}\right)+\lambda \cdot \frac{1}{k} \sum_{j=1, j \neq i}^{k+1} d\left(A_{j}^{(r)}, B_{j}^{(r)}\right) \\
& =\left(1-\frac{k+1}{k} \lambda\right) d\left(A_{i}^{(r)}, B_{i}^{(r)}\right)+\frac{\lambda}{k} \sum_{j=1}^{k+1} d\left(A_{j}^{(r)}, B_{j}^{(r)}\right) .
\end{aligned}
$$

Summing up all $d\left(A_{i}^{(r+1)}, B_{i}^{(r+1)}\right)$ with respect to $i$, we have

$$
\begin{aligned}
\sigma_{r+1}:=\sum_{i=1}^{k+1} d\left(A_{i}^{(r+1)}\right. & \left., B_{i}^{(r+1)}\right) \\
& \leq\left(1-\frac{k+1}{k} \lambda\right) \sum_{i=1}^{k+1} d\left(A_{i}^{(r)}, B_{i}^{(r)}\right)+\frac{k+1}{k} \lambda \sum_{j=1}^{k+1} d\left(A_{j}^{(r)}, B_{j}^{(r)}\right) \\
& =\sum_{i=1}^{k+1} d\left(A_{i}^{(r)}, B_{i}^{(r)}\right)\left(=\sigma_{r}\right)
\end{aligned}
$$

Hence $\sigma_{r+1} \leq \sigma_{r} \leq \cdots \leq \sigma_{1}$, that is, $\sigma_{r+1} \leq \sum_{i=1}^{k+1} d\left(A_{i}, B_{i}\right)$. Taking the limit as $r \rightarrow \infty$, we have the desired inequality since $\sigma_{r+1} \rightarrow(k+1) d\left(G_{\lambda}\left(A_{1}, \ldots, A_{k+1}\right), G_{\lambda}\left(B_{1}, \ldots, B_{k+1}\right)\right)$.

Example 2.2 Let

$$
A_{1}=\left[\begin{array}{cc}
10 & 1 \\
1 & 0.2
\end{array}\right], \quad A_{2}=\left[\begin{array}{cc}
4.1 & 4.9 \\
4.9 & 6.1
\end{array}\right] \text { and } A_{3}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

Then by numerical computation we have, (discarded less than $10^{-10}$,)

$$
\begin{aligned}
G_{1 / 3} & =\left[\begin{array}{ll}
1.6472832734 & 0.6138234917 \\
0.6138234917 & 0.8357878097
\end{array}\right]\left(=A_{1}^{(r)}=A_{2}^{(r)}=A_{3}^{(r)} \text { for } r \geq 39\right), \\
G_{1 / 2} & =\left[\begin{array}{ll}
1.6499095763 & 0.6157374707 \\
0.6157374707 & 0.8358837675
\end{array}\right]\left(=A_{1}^{(r)}=A_{2}^{(r)}=A_{3}^{(r)} \text { for } r \geq 20\right), \\
G_{2 / 3} & =\left[\begin{array}{ll}
1.6600838645 & 0.6231334993 \\
0.6231334993 & 0.8362802552
\end{array}\right]\left(=A_{1}^{(r)}=A_{2}^{(r)}=A_{3}^{(r)} \text { for } r \geq 4\right)
\end{aligned}
$$

and

$$
G_{1}=\left[\begin{array}{ll}
1.6970826618 & 0.6497880663 \\
0.6497880663 & 0.8380408114
\end{array}\right]\left(=A_{1}^{(r)}=A_{2}^{(r)}=A_{3}^{(r)} \text { for } r \geq 39\right)
$$

Now for more convenient expression, denote by $(G, \lambda)=(G, \lambda)\left(A_{1}, \ldots, A_{k+1}\right)$ the geometric mean constructed as in Theorem 2.1. Then successively we can define

$$
\left(G, \lambda_{1}, \ldots, \lambda_{\ell}\right)=\left(\left(G, \lambda_{1}, \ldots, \lambda_{\ell-1}\right), \lambda_{\ell}\right)
$$

Let $G=\sharp\left(A_{1}, A_{2}\right)=A_{1} \sharp A_{2}$. Then $(\sharp, \overbrace{1, \ldots, 1}^{k-2})$ is the geometric mean (of $k$ operators) given by Ando-Li-Mathias in [2], and ( $\sharp ; \frac{2}{3}, \ldots, \frac{k-1}{k}$ ) is one given in [9].

Example 2.3. Let

$$
A_{1}=\left[\begin{array}{ll}
2 & 1 \\
1 & 1
\end{array}\right], \quad A_{2}=\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right], \quad A_{3}=\left[\begin{array}{cc}
3 & \sqrt{2} \\
\sqrt{2} & 1
\end{array}\right] \quad \text { and } \quad A_{4}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

Then by numerical computation, we obtain, (discarded less than $10^{-10}$,) for $r \geq 4$, $\left(\sharp ; \frac{2}{3}, \frac{3}{4}\right)\left(A_{1}, A_{2}, A_{3}, A_{4}\right)=\left[\begin{array}{ll}1.4126934750 & 0.7066270669 \\ 0.7066270669 & 1.0331915013\end{array}\right]\left(=A_{1}^{(r)}=A_{2}^{(r)}=A_{3}^{(r)}=A_{4}^{(r)}\right)$.

## 3. Yamazaki's formura for the geometric mean of three matrices

Let $A_{1}, A_{2}$ and $A_{3}$ be $2 \times 2$ (positive definite) matrices. Write

$$
G_{3}=(\sharp, 1)\left(A_{1}, A_{2}, A_{3}\right) \quad\left(G_{2}=A_{1} \sharp A_{2}\right) .
$$

Related to $G_{3}$, refining a result in [2], Yamazaki [11] presented the following formula : If $\operatorname{det} A_{i}=1$ for $i=1,2,3$, then

$$
G_{3}=\frac{A_{1}+A_{2}+A_{3}}{\sqrt{\operatorname{det}\left(A_{1}+A_{2}+A_{3}\right)}}
$$

under the trace condition:

$$
\begin{equation*}
\tau\left(A_{i}^{-1} A_{j}\right)=c(\text { a constant }) \text { for } i, j=1,2,3, i \neq j \tag{3.1}
\end{equation*}
$$

Here $\tau(A)$ is the trace of $A$. (In [11], Yamazaki further presented the similar formula as above for more than three matrices.)

With respect to the above result, we show the following fact, which implies that every parametrized geometric mean $\tilde{G}_{3}=(\sharp, \lambda)\left(A_{1}, A_{2}, A_{3}\right)$ coincides, that is, $\tilde{G}_{3}$ does not depend on $\lambda$ under the condition (3.1):

Theorem 3.1. Let $A_{1}, A_{2}, A_{3}$ be $2 \times 2$ matrices with $\operatorname{det} A_{i}=1(i=1,2,3)$. Then under the condition (3.1)

$$
\begin{equation*}
\tilde{G}_{3}=(\sharp, \lambda)\left(A_{1}, A_{2}, A_{3}\right)=\frac{A_{1}+A_{2}+A_{3}}{\sqrt{\operatorname{det}\left(A_{1}+A_{2}+A_{3}\right)}} \text { for any } \lambda \in(0,1] \text {. } \tag{3.2}
\end{equation*}
$$

Before we prove the theorem, we want to provide a lemma:
Lemma 3.2. Let $A, B$ be $2 \times 2$ matrices such that $\operatorname{det} A=\operatorname{det} B=1$, and let $\lambda \in(0,1]$. Then

$$
\begin{equation*}
A \not{ }_{\lambda} B=\phi_{\lambda}\left(\tau\left(A^{-1} B\right)\right) A+\psi_{\lambda}\left(\tau\left(A^{-1} B\right)\right) B, \tag{3.3}
\end{equation*}
$$

where

$$
\begin{aligned}
& \phi_{\lambda}(t)=\frac{1}{\sqrt{t^{2}-4}}\left\{\left(\frac{t+\sqrt{t^{2}-4}}{2}\right)^{1-\lambda}-\left(\frac{t-\sqrt{t^{2}-4}}{2}\right)^{1-\lambda}\right\} \\
& \psi_{\lambda}(t)=\frac{1}{\sqrt{t^{2}-4}}\left\{\left(\frac{t+\sqrt{t^{2}-4}}{2}\right)^{\lambda}-\left(\frac{t-\sqrt{t^{2}-4}}{2}\right)^{\lambda}\right\} \quad \text { for } t>2
\end{aligned}
$$

and

$$
\phi_{\lambda}(2)=\lim _{t \rightarrow 2} \phi_{\lambda}(t)=1-\lambda, \psi_{\lambda}(2)=\lim _{t \rightarrow 2} \psi_{\lambda}(t)=\lambda
$$

In particular, (for $\lambda=1 / 2$ ) [2, Proposition 2.1]

$$
\begin{equation*}
A \sharp B=\frac{A+B}{\sqrt{\tau\left(A^{-1} B\right)+2}}\left(=\frac{A+B}{\sqrt{\operatorname{det}(A+B)}}\right) . \tag{3.4}
\end{equation*}
$$

Proof. Let $C$ be a $2 \times 2$ matrix such that $C \neq I_{2}$ and $\operatorname{det} C=1$. ( $I_{2}$ is the identity matrix.) Then we can see $\sigma(C)=\left\{p, p^{-1}\right\}$ (the spectrum of $C$ ) with some $p>1$. Hence from [2, p. 310], we have

$$
\begin{equation*}
C^{\lambda}=\frac{p^{1-\lambda}-p^{-(1-\lambda)}}{p-p^{-1}} I_{2}+\frac{p^{\lambda}-p^{-\lambda}}{p-p^{-1}} C . \tag{3.5}
\end{equation*}
$$

It follows from $\tau(C)=p+p^{-1}$ that $p^{ \pm 1}=\frac{\tau(C) \pm \sqrt{\tau(C)^{2}-4}}{2}$. Hence

$$
\begin{equation*}
C^{\lambda}=\phi_{\lambda}(\tau(C)) I_{2}+\psi_{\lambda}(\tau(C)) C \tag{3.6}
\end{equation*}
$$

Now for $A, B$ we may only consider the case $A \neq B$. Then note that $A^{-1 / 2} B A^{-1 / 2} \neq I_{2}$, $\operatorname{det}\left(A^{-1 / 2} B A^{-1 / 2}\right)=\operatorname{det}\left(A^{-1} B\right)=1$, and that $\tau\left(A^{-1 / 2} B A^{-1 / 2}\right)=\tau\left(A^{-1} B\right)$ since $\sigma\left(A^{-1 / 2} B A^{-1 / 2}\right)=\sigma\left(A^{-1} B\right)$. Hence, replacing $C$ by $A^{-1 / 2} B A^{-1 / 2}$ in (3.6), we have

$$
\left(A^{-1 / 2} B A^{-1 / 2}\right)^{\lambda}=\phi_{\lambda}\left(\tau\left(A^{-1} B\right)\right) I_{2}+\psi_{\lambda}\left(\tau\left(A^{-1} B\right)\right) A^{-1 / 2} B A^{-1 / 2} .
$$

Multiplying by $A^{1 / 2}$ both sides of the above identity from the left and the right, we now obtain the identity(3.3). For $\lambda=1 / 2$, we can see that $\phi_{\lambda}(t)=\psi_{\lambda}(t)=\frac{1}{\sqrt{t+2}}$. We can also see that

$$
\operatorname{det}(A+B)=\operatorname{det} A^{-1} \operatorname{det}(A+B)=\operatorname{det}\left(I_{2}+A^{-1} B\right)=\tau\left(A^{-1} B\right)+2
$$

This implies the desired identity (3.4).
Proof of Theorem 3.1. What we have to prove is, for all $r \geq 1$, and $i=1,2,3$

$$
\begin{equation*}
A_{i}^{(r)}=\alpha_{r} A_{i}+\beta_{r} \sum_{j=1, j \neq i}^{3} A_{j} \tag{3.7}
\end{equation*}
$$

holds for some scalars $\alpha_{r}$ and $\beta_{r}$ (each of which does not depend on $i$ ): In fact, if (3.7) holds, then by the similar argument as in [2, p.329-330], or, by direct computation, we have

$$
\lim _{r \rightarrow \infty} \alpha_{r}=\lim _{r \rightarrow \infty} \beta_{r}=\alpha
$$

for some $\alpha$, so that $\tilde{G}_{3}=\alpha\left(A_{1}+A_{2}+A_{3}\right)$, or

$$
\alpha=\frac{1}{\sqrt{\operatorname{det}\left(A_{1}+A_{2}+A_{3}\right)}}
$$

which is desired.
Now our task is to show (3.7) (by induction on $r$ ). For $r=1$, this is clear, since we can put $\alpha_{1}=1$ and $\beta_{1}=0$. Assume that (3.7) holds (for $r$ ). Then we have to show that the identity holds for $r+1$ instead of $r$, that is,

$$
\begin{equation*}
A_{i}^{(r+1)}=\alpha_{r+1} A_{i}+\beta_{r+1} \sum_{j=1, j \neq i}^{3} A_{j} \tag{3.8}
\end{equation*}
$$

for some scalars $\alpha_{r+1}$ and $\beta_{r+1}$. We devide the proof into four steps.
Step 1. $c_{r}:=\tau\left(\left(A_{i}^{(r)}\right)^{-1} A_{j}^{(r)}\right)($ for $i \neq j)$ does not depend on $i, j$, or, more precisely,

$$
\begin{equation*}
c_{r}=(4+2 c) \alpha_{r} \beta_{r}+(2+3 c) \beta_{r}^{2}+c \alpha_{r}^{2} . \tag{3.9}
\end{equation*}
$$

To see this fact, first note that $\operatorname{det} A_{\ell}=1$ (for $\ell=1,2,3$ ) and $\operatorname{det} A_{i}^{(r)}=1$ (by P9), so that the inverse operation on $2 \times 2$ matrices assures

$$
\left(A_{i}^{(r)}\right)^{-1}=\left(\alpha_{r} A_{i}+\beta_{r} \sum_{\ell=1, \ell \neq i}^{3} A_{\ell}\right)^{-1}=\alpha_{r} A_{i}^{-1}+\beta_{r} \sum_{\ell=1, \ell \neq i}^{3} A_{\ell}^{-1}
$$

Hence, (for example,) for $i=1, j=2$,

$$
\begin{aligned}
& \left(A_{1}^{(r)}\right)^{-1} A_{2}^{(r)}=\left(\alpha_{r} A_{1}^{-1}+\beta_{r} A_{2}^{-1}+\beta_{r} A_{3}^{-1}\right)\left(\alpha_{r} A_{2}+\beta_{r} A_{1}+\beta_{r} A_{3}\right) \\
& \quad=\alpha_{r} \beta_{r}\left(2 I_{2}+A_{1}^{-1} A_{3}+A_{3}^{-1} A_{2}\right)+\beta_{r}^{2}\left(I_{2}+A_{2}^{-1} A_{1}+A_{3}^{-1} A_{1}+A_{2}^{-1} A_{3}\right)+\alpha_{r}^{2} A_{1}^{-1} A_{2}
\end{aligned}
$$

¿From this and the trace condition, we have

$$
\tau\left(\left(A_{1}^{(r)}\right)^{-1} A_{2}^{(r)}\right)=(4+2 c) \alpha_{r} \beta_{r}+(2+3 c) \beta_{r}^{2}+c \alpha_{r}^{2}=c_{r}
$$

For other pairs of $i, j$ we can obtain the same value $c_{r}$.

Step 2. Let

$$
d_{r}=\frac{2 \beta_{r}}{\sqrt{2+c_{r}}} \text { and } e_{r}=\frac{\beta_{r}+\alpha_{r}}{\sqrt{2+c_{r}}}
$$

Then from (3.4), (3.7) and Step 1, we have

$$
\begin{aligned}
G\left(\left(A_{j}^{(r)}\right)_{j \neq 3}\right) & =A_{1}^{(r)} \sharp A_{2}^{(r)}=\frac{1}{\sqrt{\tau\left(\left(A_{1}^{(r)}\right)^{-1} A_{2}^{(r)}\right)+2}}\left(A_{1}^{(r)}+A_{2}^{(r)}\right) \\
& =\frac{1}{\sqrt{c_{r}+2}}\left[2 \beta_{r} A_{3}+\left(\beta_{r}+\alpha_{r}\right)\left(A_{1}+A_{2}\right)\right] \\
& =d_{r} A_{3}+e_{r}\left(A_{1}+A_{2}\right) .
\end{aligned}
$$

Similarly, we have $G\left(\left(A_{j}^{(r)}\right)_{j \neq 1}\right)=d_{r} A_{1}+e_{r}\left(A_{2}+A_{3}\right)$ and $G\left(\left(A_{j}^{(r)}\right)_{j \neq 2}\right)=d_{r} A_{2}+e_{r}\left(A_{3}+\right.$ $A_{1}$ ).

Step 3. We have to show that $f_{r}:=\tau\left(\left(A_{i}^{(r)}\right)^{-1} G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right)\right)$ is independent from $i$. For $i=1$, by (3.7) and Step 2, we have

$$
\begin{aligned}
f_{r} & =\tau\left(\left(\alpha_{r} A_{1}+\beta_{r}\left(A_{2}+A_{3}\right)\right)^{-1}\left(d_{r} A_{1}+e_{r}\left(A_{2}+A_{3}\right)\right)\right) \\
= & \tau\left(\left(\alpha_{r} A_{1}^{-1}+\beta_{r}\left(A_{2}^{-1}+A_{3}^{-1}\right)\right)\left(d_{r} A_{1}+e_{r}\left(A_{2}+A_{3}\right)\right)\right) \\
= & \tau\left(\left(\alpha_{r} d_{r}+2 \beta_{r} e_{r}\right) I+\alpha_{r} e_{r}\left(A_{1}^{-1} A_{2}+A_{1}^{-1} A_{3}\right)\right. \\
& \left.\quad+\beta_{r} d_{r}\left(A_{2}^{-1} A_{1}+A_{3}^{-1} A_{1}\right)+\beta_{r} e_{r}\left(A_{2}^{-1} A_{3}+A_{3}^{-1} A_{2}\right)\right) \\
= & 2 \alpha_{r} d_{r}+(4+2 c) \beta_{r} e_{r}+2 c \alpha_{r} e_{r}+2 c \beta_{r} d_{r} .
\end{aligned}
$$

For $i=2$ and 3 , we can obtain the same value $f_{r}$. This is what we want in this step. Now in the final step, we show (3.8):

Step 4. By (3.3), Steps 2 and 3, we have

$$
\begin{aligned}
A_{i}^{(r+1)} & =A_{i}^{(r)} \sharp_{\lambda} G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right) \\
& =\phi_{\lambda}\left(\tau\left(\left(A_{i}^{(r)}\right)^{-1} G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right)\right)\right) A_{i}^{(r)}+\psi_{\lambda}\left(\tau\left(\left(A_{i}^{(r)}\right)^{-1} G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right)\right)\right) G\left(\left(A_{j}^{(r)}\right)_{j \neq i}\right) \\
& =\phi_{\lambda}\left(f_{r}\right)\left(\alpha_{r} A_{i}+\beta_{r} \sum_{j=1, j \neq i}^{3} A_{j}\right)+\psi_{\lambda}\left(f_{r}\right)\left(d_{r} A_{i}+e_{r} \sum_{j=1, j \neq i}^{3} A_{j}\right) \\
& =\left\{\alpha_{r} \phi_{\lambda}\left(f_{r}\right)+d_{r} \psi_{\lambda}\left(f_{r}\right)\right\} A_{i}+\left\{\beta_{r} \phi_{\lambda}\left(f_{r}\right)+e_{r} \psi_{\lambda}\left(f_{r}\right)\right\} \sum_{j=1, j \neq i}^{3} A_{j} .
\end{aligned}
$$

Putting

$$
\alpha_{r+1}=\alpha_{r} \phi_{\lambda}\left(f_{r}\right)+d_{r} \psi_{\lambda}\left(f_{r}\right) \text { and } \beta_{r+1}=\beta_{r} \phi_{\lambda}\left(f_{r}\right)+e_{r} \psi_{\lambda}\left(f_{r}\right)
$$

we obtain the desired (3.8).
Remark 3.4. A nontrivial example of a triple of $2 \times 2$ matrices with their determinants 1 , satisfying the trace condition (3.1) is given as follows: Let $c>2$, and let $a, b$ be positive numbers such that $a(c-a)>1, b(c-b)>1$, respectively, and further the identity

$$
\begin{equation*}
(c+2)\left(a^{2}+b^{2}+1\right)-2 c(a b+a+b)=0 \tag{3.10}
\end{equation*}
$$

holds (In fact, the parameters $a=1, b=2$ and $c=3$ satisfy these conditions). In case $(a+b-1) c-2 a b \geq 0$, we put

$$
A_{1}=\left[\begin{array}{cc}
a & \sqrt{a(c-a)-1} \\
\sqrt{a(c-a)-1} & c-a
\end{array}\right], \quad A_{2}=\left[\begin{array}{cc}
b & \sqrt{b(c-b)-1} \\
\sqrt{b(c-b)-1} & c-b
\end{array}\right]
$$

and $A_{3}=I_{2}$. Then it is easy to see that $\operatorname{det} A_{i}=1$ for all $i$. For the trace condition, it is also easy to see $\tau\left(A_{1}^{-1} A_{3}\right)=\tau\left(A_{2}^{-1} A_{3}\right)=c$. So it suffices to show that $\tau\left(A_{1}^{-1} A_{2}\right)=c$. Since

$$
\tau\left(A_{1}^{-1} A_{2}\right)=(a+b) c-2 a b-2 \sqrt{a(c-a)-1} \sqrt{b(c-b)-1}
$$

we have to show that

$$
(a+b) c-2 a b-2 \sqrt{a(c-a)-1} \sqrt{b(c-b)-1}-c\left(=\tau\left(A_{1}^{-1} A_{2}\right)-c\right)=0
$$

or, equivalently,

$$
\{(a+b-1) c-2 a b\}^{2}-(2 \sqrt{a(c-a)-1} \sqrt{b(c-b)-1})^{2}=0
$$

Write $Q$ the left side of the above identity. Then

$$
\begin{aligned}
Q & =\left\{(a+b-1)^{2}-4 a b\right\} c^{2}+4(a b+a+b) c-4\left(a^{2}+b^{2}+1\right) \\
& =(c-2)\left\{(c+2)\left(a^{2}+b^{2}+1\right)-2 c(a b+a+b)\right\}=0
\end{aligned}
$$

by the assumption (3.10). Hence $A_{1}, A_{2}$ and $A_{3}$ are desired matrices.
In case $(a+b-1) c-2 a b \leq 0$, we define $A_{1}^{\prime}$ by

$$
A_{1}^{\prime}=\left[\begin{array}{cc}
a & -\sqrt{a(c-a)-1} \\
-\sqrt{a(c-a)-1} & c-a
\end{array}\right]
$$

Then $A_{1}^{\prime}, A_{2}$ and $A_{3}$ become a desired triple of matrices.
Remark 3.5. By the similar argument as in Theorem 3.1, we can show the similar fact as (3.2) of the geometric mean for more than three $(2 \times 2)$ matrices, though the trace condition then becomes very restrictive.

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