A CONSTRUCTIVE THEORY OF APARTNESS ON LATTICES

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Abstract. The theory of apartness spaces is lifted to the more abstract context of complemented frames with habitation, thereby providing another point-free constructive approach to topology.

1 Introduction To date, the constructive[1] theory of apartness has been developed within the context of points and sets, in particular within uniform spaces; see, for example, [12, 9, 10, 13, 14]. The theory has reached a stage where it is possible to see the patterns that might be abstracted to produce a more general theory of apartness on lattices, in the currently popular "point free" spirit [19, 22, 23, 26]. In this paper, which was written partly in response to our being asked (by formal topologists and locale theorists) if we could produce a point-free version of our theory of apartness spaces, we present the foundations of just such a theory. Specifically, after introducing an axiomatic notion of *pre-apartness* on a frame,[2] we show how a pre-apartness is related to a topology-like structure, or *t-structure*, and how such a structure, in turn, gives rise to a frame pre-apartness. We then show how an extra condition of *local decomposability* of the frame leads to the coincidence of its original pre-apartness with that arising from the associated t-structure. In the final part of the paper we introduce, and discuss the connections between, various notions of continuity for maps between frames that carry pre-apartness structures.

We emphasise that the work presented here is just a beginning of the abstraction of the theory of set-set apartness spaces; we are laying the foundations of what may be a substantial edifice, not erecting an entire kit-set building in one go. Further work, dealing with product frames and nearness, will appear in print soon [6]; much, though, remains to be explored.

Since each part of our foundation is based on the model of the theory of (point-set and) set-set apartness, it makes sense for us to begin with the axioms and some fundamental notions of that theory. The basic structure therein is an inhabited set X equipped with a binary relation $\neq$ of inequality satisfying the conditions

$$x \neq y \Rightarrow \neg (x = y),$$
$$x \neq y \Rightarrow y \neq x.$$

A subset S of X has two natural complementary subsets:

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[1]For background on constructive mathematics—by which we mean Bishop-style mathematics, carried out with intuitionistic logic and an appropriate set-theoretic foundation such as Friedman's IZF—we refer the reader to [2, 4, 5, 8, 11, 15, 20, 25]. By using IZF we admit the possibility of impredicativity; however, we believe that large parts of the paper could be formalised in the predicative constructive set theory (CST) of Aczel–Myhill–Rathjen [1, 20].

[2]For a classical theory of nearness in frames see [18]. That theory is quite different to ours in spirit and content.

▷ the **logical complement**

$$\neg S = \{x \in X : \forall_{y \in S} \neg (x = y)\},$$

▷ and the **complement**

$$\sim S = \{x \in X : \forall_{y \in S} \ (x \neq y)\}.$$

It may seem strange that we distinguish between these complements, since classically they would coincide in many, if not most, contexts. Classically, every metric space comes equipped with an inequality relation—namely, the denial inequality; but a metric space also has a natural **metric inequality** $\neq$ defined by

$$x \neq y \Leftrightarrow \rho(x, y) > 0, \tag{1}$$

which cannot be shown to coincide with the denial one unless we use Markov's principle:

> **MP** For each binary sequence $(a_n)_{n \geqslant 1}$ for which it is impossible that all terms are 0, there exists n such that $a_n = 1$.

Being independent of Heyting arithmetic and embodying an unbounded search, **MP** is a principle that constructive mathematicians prefer to do without (except, perhaps, when working with the recursive model of Bishop-style constructive mathematics). For that reason we have both the complements $\sim S$ and $\neg S$ not only in the set–set apartness theory but, later, in our axiomatic theory of apartness on lattices.

We are interested in an inhabited set X that is equipped with a **pre-apartness** relation $\bowtie$ between pairs of subsets of X. Defining the **apartness complement** of a subset S of X by

$$-S = \{x \in X : \{x\} \bowtie S\}, \tag{2}$$

we require that $\bowtie$ satisfy the following four axioms.[3]

B1 $X \bowtie \varnothing$.

B2 $-A \subset \sim A$

B3 $(A_1 \cup A_2) \bowtie (B_1 \cup B_2) \Leftrightarrow \forall_{i,j} (A_i \bowtie B_j)$

B4 $-A \subset \sim B \Rightarrow -A \subset -B$

We call the pair $(X, \bowtie)$, or when no confusion is likely, simply the set X itself, a **pre-apartness space.** Defining

$$x \bowtie A \Leftrightarrow \{x\} \bowtie A, \tag{3}$$

we obtain the so-called "point-set pre-apartness" associated with the given set-set one.

Note that although the apartness complement suffices for many parts of our theory (in particular, much of the work in Section 3 involving the apartness topology, which, in the set-set model, is essentially a point-set notion), in order to abstract to the lattice context the full essence of the set-set apartness theory, we need the relation $\bowtie$.

By a (set-set) **apartness** on X we mean a relation $\bowtie$ between subsets of X that satisfies **B1**–**B3** and

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[3]These axioms differ from the ones in our earlier papers such as [9, 10, 13]. The reason for this is that we believe that they form a minimal set of axioms capturing the essential expected features of a set-set apartness. Note, in particular, that the current axioms impose no requirement of symmetry; this is why the axiom of unions, **B3**, is more complicated than its earlier counterpart.

B5 $x \in -A \Rightarrow \exists_{S \subset X}(x \in -S \wedge X = -A \cup S)$.

It then also satisfies **B4** and so is a pre-apartness. The canonical example of a set-set apartness is the one defined on a uniform space X with uniform structure $\mathcal{U}$ by

$$A \bowtie B \Leftrightarrow \exists_{U \in \mathcal{U}} \left(A \times B \subset {\sim}U\right).$$

This apartness relation is symmetric: for all $A, B \subset X$ we have $A \bowtie B \Leftrightarrow B \bowtie A$.

2 Lattices and a-frames Our aim is to lift the theory of set-set apartness to the context of a certain type of complemented lattice. Before doing so, however, we provide some background information about lattices in the constructive setting.

A lattice is a set $\mathfrak{L}$ together with

▷ two distinguished elements $0, 1$ and

▷ total binary functions[4] $\vee$ ("join") and $\wedge$ ("meet")

that satisfy standard axioms, as found in [3, 16]. Using also the standard notations of lattice theory, we define a partial order $\leqslant$ on the lattice $\mathfrak{L}$ by

$$x \leqslant y \Leftrightarrow x \wedge y = x.$$

This partial order has such properties as the following:

- **consistency:** $x \leqslant y \Leftrightarrow x \vee y = y$;
- $x \wedge y \leqslant x \leqslant x \vee y$;
- $(z \leqslant x \;\&\; z \leqslant y) \Rightarrow z \leqslant x \wedge y$;
- $(x \leqslant z \;\&\; y \leqslant z) \Rightarrow x \vee y \leqslant z$.

We say that the lattice $\mathfrak{L}$ is **distributive** if

$$\forall_{x,y,z \in \mathfrak{L}} \left(x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\right);$$

and **modular** if

$$\forall_{x,y,z} \left(x \leqslant z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z\right).$$

Every distributive lattice is modular. If $\mathfrak{L}$ is distributive, then

$$\forall_{x,y,z \in \mathfrak{L}} \left(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\right).$$

Throughout this paper, the motivating example of a lattice is the lattice of subsets of an inhabited metric space X, with $\varnothing, X,$ union, and intersection playing the roles of $0, 1$, join, and meet, respectively. The presence there of the notion of complement which we mentioned earlier shows that this example is complemented in the sense of our next definition.

If our lattice $\mathfrak{L}$ has a unary function $\sim$ with the following axiomatic properties, then it is called a **complemented lattice**, and ${\sim}x$ is called the **complement** of the element $x \in \mathfrak{L}$.

[4] We use "&" and "or" to denote the logical "and" and "or" respectively, to avoid confusion with the lattice-theoretic symbols "$\wedge$" and "$\vee$".

C1 $x \wedge \sim x = 0$

C2 $x \wedge \sim\sim x = x$

C3 $\sim(x \vee y) = \sim x \wedge \sim y$

In that case,

$$x \leqslant y \Rightarrow \sim y \leqslant \sim x.$$

For if $x \leqslant y$, then $\sim y = \sim(x \vee y) = \sim x \wedge \sim y$, by **C3**. Moreover, $x \leqslant \sim\sim x$, $\sim 1 = 0$, and

$$\sim\sim(x \vee \sim x) = \sim 0 = 1.$$

In view of the metric space model, we would not expect to have either $x \vee \sim x = 1$ or $x = \sim\sim x$. In fact, the reader may prove that in a modular complemented lattice $\mathfrak{L}$,

$$\forall_{x \in \mathfrak{L}} (x = \sim\sim x) \Leftrightarrow \forall_{x \in \mathfrak{L}} (x \vee \sim x = 1).$$

The metric space model also shows us that we cannot prove that

$$\forall_{x,y \in \mathfrak{L}} (\sim(x \wedge y) = \sim x \vee \sim y).$$

However, we have the following replacement (without requiring modularity):

$$\forall_{x,y \in \mathfrak{L}} (\sim x \vee \sim y \leqslant \sim(x \wedge y)).$$

If $\mathfrak{L}$ is a distributive complemented lattice and $a \leqslant \sim a \vee b$, then $a \leqslant b$: for then

$$\begin{aligned} a &= a \wedge (\sim a \vee b) \\ &= (a \wedge \sim a) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b. \end{aligned}$$

We define the **join and meet of an arbitrary family** $(x_i)_{i \in I}$ of elements of a lattice in the standard way:

$$\begin{aligned} x = \bigvee_{i \in I} x_i &\Leftrightarrow \forall_{i \in I} (x_i \leqslant x) \ \&\ \forall_y (\forall_{i \in I} (x_i \leqslant y) \Rightarrow x \leqslant y), \\ x = \bigwedge_{i \in I} x_i &\Leftrightarrow \forall_{i \in I} (x \leqslant x_i) \ \&\ \forall_y (\forall_{i \in I} (y \leqslant x_i) \Rightarrow y \leqslant x). \end{aligned}$$

These elements need not exist when I is an infinite index set. Note that

$$\forall_{i \in I} \left(\bigwedge_{i \in I} x_i \leqslant x_i \leqslant \bigvee_{i \in I} x_i \right)$$

whenever the appropriate elements exist.

We say that our lattice $\mathfrak{L}$ is **habitive** if it has a unary **habitation relation**, denoted by hab, whose axiomatic requirements mirror those of inhabitedness for sets:

H1 $\mathsf{hab}(x) \Rightarrow \neg(x = 0)$.

H2 $(\mathsf{hab}(x) \ \&\ x \leqslant y) \Rightarrow \mathsf{hab}(y)$.

H3 The **join-existential property:** For any family $(x_i)_{i\in I}$ of elements of $\mathfrak{L}$, if $\bigvee_{i\in I} x_i$ exists and $\mathsf{hab}\left(\bigvee_{i\in I} x_i\right)$, then there exists $i\in I$ such that $\mathsf{hab}\,(x_i)$.

H4 $\mathsf{hab}\,(1)$.

If $\mathsf{hab}(x)$, we say that x is an **inhabited** element of $\mathfrak{L}$.

The habitation relation corresponds to the notion of "openness" in a locale (see [19, 26]) and to that of "positivity" in formal topology (see [22, 23]).

In the context of a complemented lattice $\mathfrak{L}$ with a habitation relation, we define an analogue of the metric inequality by

$$x \neq y \Leftrightarrow (\mathsf{hab}(x\wedge {\sim}y) \text{ or } \mathsf{hab}({\sim}x\wedge y)\,.$$

We then have

$$\begin{aligned} x\neq 0 &\Leftrightarrow (\mathsf{hab}(x\wedge{\sim}0) \text{ or } \mathsf{hab}({\sim}x\wedge 0)) \\ &\Leftrightarrow \mathsf{hab}(x) \text{ or } \mathsf{hab}(0) \\ &\Leftrightarrow \mathsf{hab}(x), \end{aligned}$$

in view of **H1**. Clearly,

$$x\neq y \Leftrightarrow y\neq x.$$

Also,

$$\begin{aligned} ((x\neq y)\wedge(x=y)) &\Rightarrow (\mathsf{hab}(x\wedge{\sim}x) \text{ or } \mathsf{hab}({\sim}x\wedge x)) \\ &\Rightarrow \mathsf{hab}(0), \end{aligned}$$

which is absurd, by **H1**; so

$$x\neq y\Rightarrow \neg(x=y)$$

and the relation $\neq$ is a genuine inequality relation. Note that since $1\wedge{\sim}0=1$, it follows from **H4** that $0\neq 1$.

A lattice $\mathfrak{L}$ is said to be **complete** if the join exists for any family of elements of $\mathfrak{L}$; in that case, the meet of any family of elements of $\mathfrak{L}$ also exists (see [26], Proposition 3.5.2).

Each element x of a complete lattice $\mathfrak{L}$ has a **pseudocomplement,** which (in line with the name it is given in the set-set model) we may also call the **logical complement**, defined by

$$\neg x = \bigvee\{y\in\mathfrak{L} : y\wedge x = 0\}\,.$$

The pseudocomplement[5] of x is the unique element z of $\mathfrak{L}$ such that

$$z\wedge x = 0 \;\&\; \forall_{y\in\mathfrak{L}}\,(y\wedge x=0\Rightarrow y\leqslant z)\,.$$

Classically, in a distributive lattice, the pseudocomplement of x is the unique element x' of $\mathfrak{L}$ such that $x\wedge x'=0$ and $x\vee x'=1$. Constructively, in view of the set-set model, we cannot prove that $x\vee\neg x=1$ or that $\neg x={\sim}x$.

We state two facts about pseudocomplements, leaving aside the elementary proofs.

[5]Note that the pseudocomplement satisfies the axioms **C1**–**C4** for a complement on $\mathfrak{L}$. However, to keep our theory in parallel with the set-set model, we normally consider lattices that have a separate relation of complementation satisfying those axioms.

Lemma 1 *If $\mathfrak{L}$ has pseudocomplements and $x \leqslant y$, then $\neg y \leqslant \neg x$. If also $\mathfrak{L}$ is distributive, then $\neg(x \vee y) = \neg x \wedge \neg y$ and $x \vee y \leqslant \neg(\neg x \wedge \neg y)$.*

In connection with the last conclusion of this lemma, note that if the (classically true) proposition

$$\neg(\neg S \cap \neg T) = S \cup T$$

held constructively for all subsets S, T of $\mathbb{N}$, then by taking

$$S = T = \{n \in \mathbb{N} : P\}$$

for any syntactically correct proposition P such that $\neg\neg P$ is provable, we could prove the law of excluded middle in the form $(\neg\neg P \Rightarrow P)$.

By a **frame** we mean a complete, complemented lattice with the property that $\wedge$ is **infinitely distributive** over $\vee$: that is,

$$x \wedge \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \wedge u_i)$$

for all $x \in \mathfrak{L}$ and all families $(u_i)_{i \in I}$ of elements of $\mathfrak{L}$. For arbitrary families $(u_i)_{i \in I}$ and $(v_j)_{j \in J}$ of elements of a frame $\mathfrak{L}$ we have

$$\left(\bigvee_{i \in I} u_i\right) \wedge \left(\bigvee_{j \in J} v_j\right) = \bigvee_{i \in I, j \in J} (u_i \wedge v_j).$$

Acknowledging the importance and success of locale theory and formal topology as constructive foundations for topology, we now introduce something completely new, not found in standard presentations of those theories such as [19, 22, 23, 26].

Let $\mathfrak{L}$ be a habitive frame. Our novelty consists of a binary relation $\bowtie$ on $\mathfrak{L}$ and an associated unary function $-$, where for each $x \in \mathfrak{L}$,

$$-x = \bigvee \{y \in \mathfrak{L} : y \bowtie x\}.$$

For $\bowtie$ to be a **frame pre-apartness** we require that the following axioms, clearly reflecting **B1**–**B3**, be satisfied:

A1 $1 \bowtie 0$

A2 $-x \leqslant {\sim} x$

A3 $(x_1 \vee x_2) \bowtie (y_1 \vee y_2) \Leftrightarrow \forall_{i,j}\,(x_i \bowtie y_j)$

If $x \bowtie y$, we say that x and y are **apart**, and we call $-x$ the **apartness complement** of x. Taken with a frame pre-apartness, $\mathfrak{L}$ becomes an **apartness frame**, or an **a-frame** for short.

Proposition 2 *The following hold in an a-frame $\mathfrak{L}$.*

(i) $-0 = 1$ *and* $-1 = 0$.

(ii) $(x_1 \leqslant y_1 \;\&\; x_2 \leqslant y_2 \;\&\; y_1 \bowtie y_2) \Rightarrow x_1 \bowtie x_2$.

(iii) $-(x \vee y) = -x \wedge -y$.

Proof. It follows from **A1** that $1 \leqslant -0$; whence $1 = -0$; moreover, by **A2,** $-1 \leqslant {\sim}1 = 0$ and therefore $-1 = 0$. Thus (i) holds. To prove (ii) we note that if $x_i \leqslant y_i$, then $y_i = x_i \vee y_i$; so if also $y_1 \bowtie y_2$, then $x_1 \vee y_1 \bowtie x_2 \vee y_2$; whence $x_1 \bowtie x_2$, by **A3**. For (iii), first note that if $t \bowtie x \vee y$, then, by **A3,** $t \leqslant -x$ and $t \leqslant -y$; whence $t \leqslant -x \wedge -y$. Thus $-(x \vee y) \leqslant -x \wedge -y$. On the other hand, if $z \bowtie x$ and $z' \bowtie y$, then by (ii), $z \wedge z' \bowtie x$ and $z \wedge z' \bowtie y$; whence, by **A3**, $z \wedge z' \bowtie x \vee y$. Since

$$\begin{aligned} -x \wedge -y &= -y \wedge \bigvee \{z \in \mathfrak{L} : z \bowtie x\} \\ &= \bigvee \{-y \wedge z : z \bowtie x\} \\ &= \bigvee \{z \wedge -y : z \bowtie x\} \\ &= \bigvee \left\{ z \wedge \bigvee \{z' \in \mathfrak{L} : z' \bowtie y\} : z \bowtie x \right\} \\ &= \bigvee \{z \wedge z' : z \bowtie x \ \&\ z' \bowtie y\}, \end{aligned}$$

it follows that $-x \wedge -y \leqslant -(x \vee y)$. Hence $-x \wedge -y = -(x \vee y)$. ∎

In the set model the analogue of the following **Lodato property** is required (as axiom **B4**) of a pre-apartness between sets.

A4 $-x \leqslant {\sim}y \Rightarrow -x \leqslant -y$.

If this holds, we say that $\bowtie$ is a **Lodato pre-apartness** on $\mathfrak{L}$, and that $\mathfrak{L}$ is a **Lodato a-frame**.

Everything we have done so far with lattices is point-free. However, we can introduce "points"—known as "atoms"—into our theory as follows: an **atom** of a habitive lattice $\mathfrak{L}$ is an element x with the properties

$$x \neq 0 \ \&\ \forall_y \left(0 \neq y \leqslant x \Rightarrow y = x\right).$$

It should be clear that atoms correspond to the singleton subsets of the set-set apartness model.[6] We call $\mathfrak{L}$ an **atomic lattice** if

$$\forall_{x \in \mathfrak{L}} \left(x \neq 0 \Rightarrow x = \bigvee \{y \in \mathfrak{L} : y \text{ is an atom and } y \leqslant x\} \right).$$

For future reference we now prove some elementary properties of atoms.

Lemma 3 *If $\mathfrak{L}$ is a habitive distributive lattice, x is an atom, and $x \leqslant u \vee v$, then $x \leqslant u$ or $x \leqslant v$.*

Proof. Since

$$0 \neq x = x \wedge (u \vee v)) = (x \wedge u) \vee (x \wedge v),$$

it follows from axiom **H3** that either $x \wedge u \neq 0$ or $x \wedge v \neq 0$. In the first case, $0 \neq x \wedge u \leqslant x$, so, as x is an atom, $x \wedge u = x$ and therefore $x \leqslant u$. In the second case, a similar argument gives $x \leqslant v$. ∎

[6] We contemplated writing "$t \in x$" to signify that t is an atom and $t \leqslant x$. We would then have described an element x of our lattice as "set-like" if $0 \neq x = \bigvee_{t \in x} t$.

Lemma 4 *If $\mathfrak{L}$ is a habitive frame, x is an atom, and $x \leqslant \bigvee_{i \in I} u_i$, then there exists i such that $x \leqslant u_i$.*

Proof. Since

$$0 \neq x = x \wedge \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \wedge u_i),$$

axiom **H3** shows that there exists i such that $x \wedge u_i \neq 0$. The same argument as in the conclusion of the proof of Lemma 3 completes the proof in this case also. ∎

Proposition 5 *If $\mathfrak{L}$ is a habitive a-frame, and x is an atom of $\mathfrak{L}$ such that $x \leqslant -y$, then $x \bowtie y$.*

Proof. Since

$$x \leqslant -y = \bigvee \{z \in \mathfrak{L} : z \bowtie y\},$$

it follows from Lemma 4 that there exists $z \in \mathfrak{L}$ such that $x \leqslant z$ and $z \bowtie y$; whence $x \bowtie y$, by Proposition 2(ii). ∎

3 Topology-like structures Referring to the set-set case [9], we define the **nearly open elements** of a frame $\mathfrak{L}$ with a pre-apartness to be those of the form $\bigvee_{i \in I} -u_i$ for some index set I. The join of any family of nearly open elements is then nearly open. Both 0 $(= -1)$ and 1 $(= -0)$ are nearly open. If $a = \bigvee_{i \in I} -u_i$ and $b = \bigvee_{j \in J} -v_j$ are nearly open, then $a \wedge b$ is nearly open, since by the generalised distributive law and Proposition 2(iii),

$$\left(\bigvee_{i \in I} -u_i\right) \wedge \left(\bigvee_{j \in J} -v_j\right) = \bigvee_{i \in I, j \in J} (-u_i \wedge -v_j) = \bigvee_{i \in I, j \in J} -(u_i \vee v_j).$$

The set of nearly open elements of $\mathfrak{L}$ is denoted by $\tau_{\mathfrak{L}}$.

In a uniform apartness space (that is, a uniform space taken with the canonical apartness defined at the end of Section 1), the closure of a subset S relative to the uniform topology is the unique set T with the property[7]

$$\forall_{U \subset X} \left((U \text{ is open and } T \cap U \neq \varnothing) \Rightarrow S \cap U \neq \varnothing \right).$$

Given that uniformly open sets are just unions of apartness complements (see [13]), this suggests the following definition.

The **closure** of an element x of a habitive a-frame $\mathfrak{L}$ is the element

$$\overline{x} = \bigvee \{t \in \mathfrak{L} : \forall_{u \in \mathfrak{L}} (t \wedge -u \neq 0 \Rightarrow x \wedge -u \neq 0)\}.$$

Clearly, $x \leqslant \overline{x}$. Also, for each $y \in \mathfrak{L}$,

$$\overline{x} \leqslant \overline{x \vee y}.$$

[7] We write "$S \neq \varnothing$" to signify that the set S is inhabited, not the weaker property that $\neg(S = \varnothing)$.

For if

$$\forall_{u\in\mathfrak{L}}\,(t\wedge -u\neq 0\Rightarrow x\wedge -u\neq 0)$$

and $t\wedge -u\neq 0$, then

$$0\neq x\wedge -u\leqslant (x\vee y)\wedge -u;$$

so, by **H3,** $(x\vee y)\wedge -u\neq 0$; whence $t\leqslant \overline{x\vee y}$.

Our next proposition is further demonstration that the closure of an element in an a-frame behaves like the closure in a topological space.

Proposition 6 *In a habitive a-frame $\mathfrak{L}$, if $y\leqslant \overline{x}$, then*

$$\forall_{u\in\tau_{\mathfrak{L}}}\,(y\wedge u\neq 0\Rightarrow x\wedge u\neq 0)\,.$$

Proof. Let $u=\bigvee_{i\in I}-a_i\in\tau_{\mathfrak{L}}$ and

$$0\neq y\wedge u=y\wedge\bigvee_{i\in I}-a_i=\bigvee_{i\in I}(y\wedge -a_i)\,.$$

Then by **H3**, there exists i_0 such that $y\wedge -a_{i_0}\neq 0$. Since $y\leqslant\overline{x}$ and $\mathfrak{L}$ is a frame,

$$0\neq\overline{x}\wedge -a_{i_0}=\bigvee\left\{t\wedge -a_{i_0}:\forall_{v\in\mathfrak{L}}\,(t\wedge -v\neq 0\Rightarrow x\wedge -v\neq 0)\right\}.$$

Again applying **H3**, we obtain t such that

$$\forall_{v\in\mathfrak{L}}\,(t\wedge -v\neq 0\Rightarrow x\wedge -v\neq 0)$$

and $t\wedge -a_{i_0}\neq 0$; whence $x\wedge -a_{i_0}\neq 0$. ■

Now consider any frame $\mathfrak{L}$ with a family τ of elements satisfying the following three properties:

TL1 $0\in\tau$ and $1\in\tau$.

TL2 If $(u_i)_{i\in I}$ is a family of elements of τ, then $\bigvee_{i\in I}u_i\in\tau$.

TL3 If $u,v\in\tau$, then $u\wedge v\in\tau$.

We call τ a **topology-like structure**, or a **t-structure**, on $\mathfrak{L}$; the pair $(\mathfrak{L},\tau)$ a **topological frame**; and the elements of τ the corresponding **open elements** of $\mathfrak{L}$. Trivially, the family comprising all elements of $\mathfrak{L}$ is a t-structure on $\mathfrak{L}$. Of more interest and significance is the fact that (as follows from the first paragraph of this section) if $\mathfrak{L}$ is equipped with a pre-apartness, then $\tau_{\mathfrak{L}}$ is a t-structure on $\mathfrak{L}$.

Given a topological frame $(\mathfrak{L},\tau)$, we define a relation $\bowtie_\tau$ on $\mathfrak{L}$ as follows:

$$x\bowtie_\tau y\Leftrightarrow\exists_{u\in\tau}\,(x\leqslant u\leqslant{\sim}y)\,.$$

We also define

$$-_\tau x=\bigvee\left\{z\in\mathfrak{L}:z\bowtie_\tau x\right\}.$$

To show that the relation $\bowtie_\tau$ is a pre-apartness on $\mathfrak{L}$, first observe that, since $1 \leqslant 1 = {\sim}0$ and $0, 1 \in \tau$, we have $1 \bowtie_\tau 0$—that is, **A1**. Next, if $z \bowtie_\tau x$, then there exists $u \in \tau$ such that $z \leqslant u \leqslant {\sim}x$, so $z \leqslant {\sim}x$; whence $-x \leqslant {\sim}x$, and **A2** holds. To deal with **A3,** suppose first that $(x_1 \vee x_2) \bowtie_\tau (y_1 \vee y_2)$. Then there exists $u \in \tau$ such that

$$x_1 \vee x_2 \leqslant u \leqslant {\sim}(y_1 \vee y_2) = {\sim}y_1 \wedge {\sim}y_2.$$

Hence $x_i \leqslant u \leqslant {\sim}y_j$ and therefore $x_i \bowtie_\tau y_j$. Next suppose, conversely, that $x_i \bowtie_\tau y_j$ for $i, j = 1, 2$. For such i, j there exists $u_{ij} \in \tau$ such that $x_i \leqslant u_{ij} \leqslant {\sim}y_j$. Then

$$x_i \leqslant u_{i1} \wedge u_{i2} \leqslant {\sim}y_1 \wedge {\sim}y_2 = {\sim}(y_1 \vee y_2),$$

so

$$x_1 \vee x_2 \leqslant (u_{11} \wedge u_{12}) \vee (u_{21} \wedge u_{22}) \leqslant {\sim}(y_1 \vee y_2).$$

Since

$$(u_{11} \wedge u_{12}) \vee (u_{21} \wedge u_{22}) \in \tau,$$

we now see that $(x_1 \vee x_2) \bowtie_\tau (y_1 \vee y_2)$. This completes the proof that $\bowtie_\tau$ is a pre-apartness, which we call the **topological pre-apartness**, on $\mathfrak{L}$. When we regard a topological frame as an a-frame, it is this topological pre-apartness that we have in mind.

The relation $\bowtie_\tau$ satisfies not only **A4** but even the stronger property

$$\forall_{x,y\in\mathfrak{L}} \left(-_\tau x \leqslant {\sim}y \Rightarrow \forall_{z\in\mathfrak{L}} \left(z \bowtie_\tau x \Rightarrow z \bowtie_\tau y\right)\right). \tag{4}$$

To show this, let $-_\tau x \leqslant {\sim}y$. For each z with $z \bowtie_\tau x$ construct $u \in \tau$ such that $z \leqslant u \leqslant {\sim}x$. Then $u \leqslant u \leqslant {\sim}x$, so $u \bowtie_\tau x$ and therefore $u \leqslant -_\tau x$. Hence $z \leqslant u \leqslant {\sim}y$ and so $z \bowtie_\tau y$.

Our primary motivating example of an a-frame was the power set of a set-set pre-apartness space. Another example is provided by the topology τ on an inhabited set X with an inequality relation. Relative to the operations of union and intersection, τ itself is both a complete, habitive frame and a t-structure on that frame; so we have the corresponding pre-apartness $\bowtie_\tau$ and apartness complement $-_\tau$ on τ, defined by

$$U \bowtie_\tau V \Leftrightarrow \exists_{W\in\tau} \left(U \subset W \subset {\sim}V\right)$$

and

$$-_\tau U \equiv ({\sim}U)^\circ.$$

These turn τ into a Lodato a-frame. Note that the nearly open sets of this a-frame all belong to the original topology τ.

For any element x of a topological frame, the **interior** of x is defined by

$$x^\circ = \bigvee \{u \in \tau : u \leqslant x\}.$$

Then $x^\circ \in \tau$, by **TL2**; and the definition of "join" shows that $x^\circ \leqslant x$. If $x \in \tau$, then since $x \leqslant x$, we also have $x \leqslant x^\circ$ and therefore $x = x^\circ$.

Proposition 7 *Let $(\mathfrak{L}, \tau)$ be a topological frame, and $\bowtie_\tau, -_\tau$ the corresponding topological pre-apartness and apartness complement. Then $-_\tau x = ({\sim}x)^\circ$ for each $x \in \mathfrak{L}$.*

Proof. If $z \bowtie_\tau x$, then there exists $u \in \tau$ such that $z \leqslant u \leqslant \sim x$; since $u \leqslant (\sim x)^\circ$ by definition of "interior", it follows that $z \leqslant (\sim x)^\circ$. Hence $-_\tau x \leqslant (\sim x)^\circ$. Conversely, if $u \in \tau$ and $u \leqslant \sim x$, then, by definition of $\bowtie_\tau$, $u \bowtie_\tau x$ and therefore $u \leqslant -_\tau x$. Hence $(\sim x)^\circ \leqslant -_\tau x$. ■

Starting with an a-frame, we produce the corresponding t-structure, which, in turn, gives rise to a pre-apartness satisfying the (strong Lodato-type) condition (4). How is this pre-apartness related to the original one?

Proposition 8 *Let $\mathfrak{L}$ be a habitive a-frame, τ the corresponding t-structure, and $\bowtie_\tau$ the pre-apartness induced on $\mathfrak{L}$ by τ. For all $x, y \in \mathfrak{L}$, if $x \bowtie y$, then $x \bowtie_\tau y$. Conversely, if $\mathfrak{L}$ also has the Lodato property and x is an atom such that $x \bowtie_\tau y$, then $x \bowtie y$.*

Proof. If $x \bowtie y$, then $x \leqslant -y \leqslant \sim y$, so $x \bowtie_\tau y$. Conversely, suppose that x is an atom satisfying $x \bowtie_\tau y$. Then there exists a family $(a_i)_{i \in I}$ of elements of $\mathfrak{L}$ such that

$$x \leqslant \bigvee_{i \in I} - a_i \leqslant \sim y.$$

By Lemma 4, there exists $i \in I$ such that $x \leqslant -a_i \leqslant \sim y$. It follows from the Lodato property that $-a_i \leqslant -y$; whence $x \leqslant -y$. Since x is an atom, we conclude that $x \bowtie y$. ■

Again, let $(\mathfrak{L}, \tau)$ be a topological frame. Now that we have a lattice pre-apartness $\bowtie_\tau$ and an apartness complement $-_\tau$ on $\mathfrak{L}$, we have the corresponding nearly open elements of $\mathfrak{L}$: namely, the joins of elements of the form $-_\tau x$. Every nearly open element is open: for, by Proposition 7,

$$\bigvee_{i \in I} -_\tau u_i = \bigvee_{i \in I} (\sim u_i)^\circ = \bigvee_{i \in I} \bigvee \{v \in \tau : v \leqslant \sim u_i\},$$

which is a join of open elements. However, as the set model shows, we cannot expect to prove that every open element of $\mathfrak{L}$ is nearly open; see Section 2.2 of [12].[8] If that property does hold, then we call $\mathfrak{L}$ **topologically consistent.** We now consider conditions that guarantee topological consistency.

We say that a frame $\mathfrak{L}$ with a pre-apartness and an apartness complement is **locally decomposable** if

$$\forall_{x \in \mathfrak{L}} (-x = \bigvee \{-y \in \mathfrak{L} : -x \vee y = 1\}.$$

In the context of a set-set pre-apartness space $(X, \bowtie)$ this property becomes

$$\forall_{A \subset X} \left(-A = \bigcup \{-S : X = -A \cup S\}\right) \tag{5}$$

is equivalent to (and a point-free expression of) axiom **B5**, and so turns the pre-apartness into an apartness. For that reason, we say that a locally decomposable pre-apartness on a frame is an **apartness.**

Proposition 9 *Every habitive, locally decomposable a-frame has the Lodato property.*

[8] The paper [17], where a generalisation of point-set apartness spaces is considered, has a particularly illuminating discussion of the constructive plurality of topologies compatible with a given point-set pre-apartness.

Proof. Let $\mathfrak{L}$ be a habitive, locally decomposable a-frame, and let x, y be elements of $\mathfrak{L}$ such that $-x \leqslant \sim y$. For each $z \in \mathfrak{L}$ with $-x \vee z = 1$ we have

$$y = (y \wedge -x) \vee (y \wedge z) = 0 \vee (y \wedge z) = y \wedge z$$

—that is, $y \leqslant z$; whence $-z \leqslant -y$. It follows that

$$-x = \bigvee \{-z : -x \vee z = 1\} \leqslant -y.$$

Thus the Lodato condition holds in $\mathfrak{L}$. ■

Proposition 10 *If $\mathfrak{L}$ is a habitive, locally decomposable a-frame, then for each atom $x \in \mathfrak{L}$ and each $u \in \mathfrak{L}$ with $x \leqslant -u$, there exists $v \in \mathfrak{L}$ such that $x \leqslant -v$ and $-u \vee v = 1$.*

Proof. Let x be an atom with $x \leqslant -u$. Since $\mathfrak{L}$ is locally decomposable,

$$0 \neq x \leqslant \bigvee \{-v : -u \vee v = 1\}.$$

Since x is an atom, it follows from Lemma 4 that there exists $v \in \mathfrak{L}$ such that $-u \vee v = 1$ and $x \leqslant -v$. ■

Lemma 11 *In an a-frame $\mathfrak{L}$, if $-a \leqslant a \vee b$, then $-a \leqslant b$.*

Proof. It is enough to prove that if $z \bowtie a$, then $z \leqslant b$. Our hypotheses ensure that $z \leqslant a \vee b$. Since the lattice is distributive and $z \wedge a \leqslant -a \wedge a = 0$, it follows that

$$z = z \wedge (a \vee b) = (z \wedge a) \vee (z \wedge b) = z \wedge b;$$

whence $z \leqslant b$. ■

Proposition 12 *Let $(\mathfrak{L}, \tau)$ be a topological frame such that*

$$\forall_{u \in \tau} \left(u = \bigvee \{v \in \tau : u \vee \sim v = 1\} \right). \tag{6}$$

Then $(\mathfrak{L}, \bowtie_\tau)$ is locally decomposable.

Proof. Let $x \in \mathfrak{L}$. Then (by Proposition 7) $-_\tau x = (\sim x)^\circ \in \tau$; so, by (6),

$$-_\tau x = \bigvee \{v \in \tau : -_\tau x \vee \sim v = 1\}.$$

Given $v \in \tau$ such that $-_\tau x \vee \sim v = 1$, set $z = \sim v$. Then $v \leqslant \sim\sim v = \sim z$ and $v \in \tau$, so $v \bowtie_\tau z$ and therefore $v \leqslant -_\tau z$. Thus

$$v \leqslant \bigvee \{-_\tau y : -_\tau x \vee y = 1\}$$

and therefore

$$-_\tau x = \bigvee \{v \in \tau : -_\tau x \vee \sim v = 1\} \leqslant \bigvee \{-_\tau y : -_\tau x \vee y = 1\}.$$

Since, by Lemma 11,

$$\bigvee \{-_\tau y : -_\tau x \vee y = 1\} \leqslant -_\tau x,$$

it follows that

$$-_\tau x = \bigvee \{-_\tau y : -_\tau x \vee y = 1\},$$

so $(\mathfrak{L}, \bowtie_\tau)$ is locally decomposable. ■

Proposition 13 *Let* $(\mathfrak{L}, \bowtie)$ *be a frame with a locally decomposable pre-apartness, and let* τ *denote the t-structure comprising the nearly open elements of* $\mathfrak{L}$. *Then* (6) *holds.*

Proof. Let $u \in \tau$. Then there exists a family $(u_i)_{i\in I}$ of elements of $\mathfrak{L}$ with $u = \bigvee\limits_{i\in I} - u_i$. For each $i \in I$, by local decomposability we have

$$-u_i = \bigvee \{-z : -u_i \vee z = 1\}. \tag{7}$$

Let $-u_i \vee z = 1$; then $-z \leqslant -u_i$, by Lemma 11. Moreover,

$$1 = -u_i \vee z \leqslant u \vee {\sim\sim} z \leqslant u \vee \sim - z,$$

so $u \vee \sim - z = 1$. It follows from this, (7), and Lemma 11 that

$$-u_i \leqslant \bigvee \{v \in \tau : u \vee {\sim} v = 1\} \leqslant u.$$

Hence

$$u = \bigvee_{i\in I} - u_i \leqslant \bigvee \{v \in \tau : u \vee {\sim} v = 1\} \leqslant u$$

and therefore

$$u = \bigvee \{v \in \tau : u \vee {\sim} v = 1\},$$

as we required. ■

Proposition 14 *Every topological frame satisfying* (6) *is topologically consistent.*

Proof. Let $\mathfrak{L}$ be a frame with a t-structure τ that satisfies (6), and consider any u, v in τ such that $u \vee {\sim} v = 1$. Since $v \leqslant {\sim\sim} v$ and $v \in \tau$, we have $v \leqslant ({\sim\sim} v)^\circ$ and therefore $v \leqslant -_\tau {\sim} v$. Moreover,

$${\sim\sim} v \leqslant 1 = u \vee {\sim} v,$$

so, by Lemma 11, ${\sim\sim} v \leqslant u$; whence $-_\tau {\sim} v \leqslant u$. It follows that

$$v \leqslant \bigvee \{-x : -x \leqslant u\} \leqslant u.$$

Applying (6), we now obtain

$$u = \bigvee \{v \in \tau : u \vee {\sim} v = 1\} \leqslant \bigvee \{-x : -x \leqslant u\} \leqslant u$$

and therefore

$$u = \bigvee \{-x : -x \leqslant u\},$$

which is a nearly open element of $\mathfrak{L}$. ■

Next, we show that local decomposability implies a lattice version of the point-set property

$$\forall_{x\in X} \forall_{U\subset X} \left(x \in -U \Rightarrow \forall_{y\in X} \left(x \neq y \text{ or } y \in -U\right)\right),$$

which is one of the axioms for a point-set apartness (see [9] or Chapter 2 of [12]) and is easily seen to be equivalent to

$$\forall_{U\subset X} \left(-U = \bigcup \{T \subset X : -U \cup {\sim} T = X\}\right).$$

Proposition 15 *Let $\mathfrak{L}$ be a locally decomposable a-frame. Then*

$$\forall_{x\in\mathfrak{L}}\left(-x=\bigvee\{z\in\mathfrak{L}:-x\vee\sim z=1\}\right). \tag{8}$$

Proof. Fix x in $\mathfrak{L}$. Since $\mathfrak{L}$ is locally decomposable,

$$-x=\bigvee\{-u\in\mathfrak{L}:-x\vee u=1\}.$$

Given $u\in\mathfrak{L}$ with $-x\vee u=1$, we have $-x\vee\sim\sim u=1$, so

$$-u\leqslant\sim u\leqslant\bigvee\{z\in\mathfrak{L}:-x\vee\sim z=1\}.$$

Moreover, if $-x\vee\sim z=1$, then $z\leqslant -x\vee\sim z$, so, by Lemma 11, $z\leqslant -x$. Hence

$$-u\leqslant\bigvee\{z\in\mathfrak{L}:-x\vee\sim z=1\}\leqslant -x.$$

It follows that

$$-x=\bigvee\{-u\in\mathfrak{L}:-x\vee u=1\}\leqslant\bigvee\{z\in\mathfrak{L}:-x\vee\sim z=1\}\leqslant -x$$

from which we deduce (8). ■

4 Join homomorphisms and continuity Consider a mapping $f:X\to Y$ between two sets with inequalities. For any family $(A_i)_{i\in I}$ of subsets of X we have

$$f\left(\bigcup_{i\in I}A_i\right)=\bigcup_{i\in I}f(A_i)$$

and

$$f\left(\bigcap_{i\in I}A_i\right)\subset\bigcap_{i\in I}f(A_i),$$

but generally not

$$\bigcap_{i\in I}f(A_i)\subset f\left(\bigcap_{i\in I}A_i\right).$$

Note also that for any $A\subset X$, $f(A)$ is inhabited if and only if A is inhabited. These observations motivate the following definitions.

Let $\mathfrak{L},\mathfrak{M}$ be habitive frames. We call a mapping $f:\mathfrak{L}\to\mathfrak{M}$ a **join homomorphism** if

▷ for each family $(x_i)_{i\in I}$ of elements of $\mathfrak{L}$,

$$f\left(\bigvee_{i\in I}x_i\right)=\bigvee_{i\in I}f(x_i),$$

and

▷ for all $x\in\mathfrak{L}$, $x\neq 0$ in $\mathfrak{L}$ if and only if $f(x)\neq 0$ in $\mathfrak{M}$.

Such a map is order-preserving: if $a \leqslant b$, then $a \vee b = b$, so $f(a) \vee f(b) = f(a \vee b) = f(b)$ and therefore $f(a) \leqslant f(b)$. It follows that

$$f\left(\bigwedge_{i \in I} x_i\right) \leqslant \bigwedge_{i \in I} f(x_i)$$

for all families $(x_i)_{i \in I}$ of elements of $\mathfrak{L}$.

Comparing our notion of *join homomorphism* with the standard notion of *frame homomorphism* (see page 39 of [19]), we see that the former is, in one respect, more general, as it does not require the mapping to preserve finite meets; but in another, it is less general, as it requires that nonzero elements be mapped to nonzero elements.

For each $v \in \mathfrak{M}$ define

$$f^{-\infty}(v) = \bigvee \{x \in \mathfrak{L} : f(x) \leqslant v\}.$$

In our theory the element $f^{-\infty}(x)$ plays the role of the inverse image of a set in the set model. For example, in that model, topological continuity of a mapping f between set-set apartness spaces X and Y means that $f^{-1}(V)$ is nearly open in X for each nearly open $V \subset Y$. But

$$f^{-1}(V) = \{x \in X : f(x) \in V\} = \bigcup \{U \subset X : f(U) \subset V\},$$

which is precisely the counterpart of the element $f^{-\infty}(v)$ that appears in the definition of *topologically continuous* given below. We have used the special notation $f^{-\infty}(v)$ in order to avoid confusion with

$$f^{-1}(v) = \{x \in \mathfrak{L} : f(x) = v\}.$$

We introduce several types of continuity for a join homomorphism. With the exception of the first, these properties are based on their counterparts in the set-set theory. The first property here is similar to that of *near continuity* in the set-set theory, but a better correspondent to the latter can be introduced once we have a good notion of *nearness* in an a-frame; in turn, that requires an understanding of product apartness structures on a-frames, a topic that will is covered in some detail in [6].

Before defining continuity types, we need one more definition. Let x, y be elements of an a-frame $\mathfrak{L}$. We say that x **approximates** y if

$$\forall_{u \in \tau_{\mathfrak{L}}} (x \leqslant u \Rightarrow y \wedge u \neq 0),$$

and we then write $\mathsf{apr}(x, y)$.

Let $f : \mathfrak{L} \to \mathfrak{M}$ be a join homomorphism between a-frames. We say that f is

- **approximately continuous** if

$$\forall_{x,y \in \mathfrak{L}} (\mathsf{apr}(x, y) \Rightarrow \mathsf{apr}(f(x), f(y)));$$

- **continuous** if

$$\forall_{u,v \in \mathfrak{M}} \left(u \leqslant -v \Rightarrow f^{-\infty}(u) \leqslant -f^{-\infty}(v)\right);$$

- **topologically continuous** if

$$\forall_{v \in \tau_{\mathfrak{M}}} \left(f^{-\infty}(v) \in \tau_{\mathfrak{L}}\right);$$

▶ **strongly continuous** if

$$\forall_{u,v\in\mathfrak{M}}\left(u \bowtie v \Rightarrow f^{-\infty}(u) \bowtie f^{-\infty}(v)\right).$$

The elementary proofs of the next two lemmas are left to the reader.

Lemma 16 *If $\mathfrak{L}, \mathfrak{M}$ are a-frames, $f : \mathfrak{L} \to \mathfrak{M}$ is a join homomorphism, and $b \in \mathfrak{M}$, then $f(f^{-\infty}(b)) \leqslant b$.*

Lemma 17 *Let $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}$ be a-frames, and let $f : \mathfrak{L} \to \mathfrak{M}$ and $g : \mathfrak{M} \to \mathfrak{N}$ be join homomorphisms. Then $(g \circ f)^{-\infty} = f^{-\infty} \circ g^{-\infty}$.*

Using Lemma 17, it is easy to see that the composition of two join homomorphisms of the same continuity type is also of that continuity type.

What relations do we have between our various notions of continuity? To answer this, we first prove

Lemma 18 *If an a-frame $\mathfrak{L}$ satisfies* (8), and *$z \in \mathfrak{L}$ is nearly open, then $\neg z = {\sim}z$.*

Proof. Since ${\sim}z \leqslant \neg z$, it is enough to prove that $\neg z \leqslant {\sim}z$. Pick a family $(a_i)_{i\in I}$ of elements of $\mathfrak{L}$ such that $z = \bigvee\limits_{i\in I} - a_i$. Consider any $y \in \mathfrak{L}$ with $y \wedge z = 0$. For any $i \in I$ and any $t \in \mathfrak{L}$ with $-a_i \vee {\sim}t = 1$ we have

$$y = y \wedge (-a_i \vee {\sim}t) = (y \wedge -a_i) \vee (y \wedge {\sim}t) = y \wedge {\sim}t,$$

so $y \leqslant {\sim}t$. It follows from this and (8) that

$$\begin{aligned} y &\leqslant \bigwedge \{{\sim}t : -a_i \vee {\sim}t = 1\} \\ &\leqslant {\sim}\bigvee \{t : -a_i \vee {\sim}t = 1\} = {\sim} - a_i. \end{aligned}$$

Hence

$$y \leqslant \bigwedge_{i\in I} {\sim} - a_i = {\sim}\bigvee_{i\in I} - a_i = {\sim}z.$$

Thus

$$\neg z = \bigvee \{y : y \wedge z = 0\} \leqslant {\sim}z$$

and the proof is complete. ■

Proposition 19 *Let $\mathfrak{L}, \mathfrak{M}$ be a-frames, and $f : \mathfrak{L} \to \mathfrak{M}$ a topologically continuous join homomorphism. Then f is approximately continuous.*

Proof. Consider elements x, y of $\mathfrak{L}$ such that $\mathsf{apr}(x, y)$. Let v be a nearly open element of $\mathfrak{M}$ such that $f(x) \leqslant v$. By the topological continuity of f, $f^{-\infty}(v)$ is nearly open in $\mathfrak{L}$. Since $x \leqslant f^{-\infty}(v)$ and $\mathsf{apr}(x, y)$, we have $y \wedge f^{-\infty}(v) \neq 0$; so, by the defining properties of a join homomorphism and Lemma 16,

$$0 \neq f(y \wedge f^{-\infty}(v)) \leqslant f(y) \wedge f(f^{-\infty}(v)) \leqslant f(y) \wedge v.$$

Hence $f(y) \wedge v \neq 0$. ■

We say that the inequality $\neq$ on a complemented lattice $\mathfrak{L}$ with a habitation relation is **zero-tight** if

$$\forall_{x\in\mathfrak{L}}\left(\neg\left(x\neq 0\right)\Rightarrow x=0\right).$$

Note that this does not imply that the inequality is **tight** in the sense that

$$\forall_{x,y\in\mathfrak{L}}\left(\neg\left(x\neq y\right)\Rightarrow x=y\right).$$

To see this, consider the lattice $\mathfrak{L}$ of subsets of $X=\{1,2\}$, where the complement of a subset S of X is

$$\sim S=\{x\in X:\forall_{y\in S}\left(|x-y|>0\right)\}$$

and the habitation predicate is the usual one of inhabitedness:

$$\mathsf{hab}(S)\Leftrightarrow\exists_{x}\left(x\in S\right).$$

If $\neg\left(S\neq\varnothing\right)$, then $S=\varnothing$, so the inequality on $\mathfrak{L}$ is zero-tight. Suppose it is tight. Given any syntactically correct statement P, define

$$S=\{1\}\ \text{ and } T=\{x\in X:x=1\text{ and }P\vee\neg P\}.$$

Then $\neg\left(S\neq T\right)$; but if $S=T$, then $P\vee\neg P$.

Proposition 20 *Let $\mathfrak{L}$ be a Lodato a-frame that satisfies* (8) *and has a zero-tight inequality. Then every topologically continuous join homomorphism from $\mathfrak{L}$ to an a-frame is continuous.*

Proof. Let $f:\mathfrak{L}\to\mathfrak{M}$ be a topologically continuous join homomorphism. Consider elements u,v of $\mathfrak{M}$ with $u\leqslant -v$. Let $y=f^{-\infty}(-v)$. By Lemma 16, $f(f^{-\infty}(u))\leqslant u\leqslant -v$, so $f^{-\infty}(u)\leqslant y$. We prove that

$$f^{-\infty}(v)\leqslant\sim y. \tag{9}$$

By Lemma 16,

$$f(f^{-\infty}(v)\wedge y)\leqslant f(f^{-\infty}(v))\wedge f(y)\leqslant v\wedge -v=0;$$

whence $f(f^{-\infty}(v)\wedge y)=0$. It follows from the second defining property of a join homomorphism that $\neg\left(f^{-\infty}(v)\wedge y\neq 0\right)$. Since the inequality on $\mathfrak{L}$ is zero-tight, $f^{-\infty}(v)\wedge y=0$; so $f^{-\infty}(v)\leqslant\neg y$, from which we obtain (9), by Lemma 18.

Now, the topological continuity of f ensures that y is nearly open, so there exists a family $(a_i)_{i\in I}$ of elements of $\mathfrak{L}$ such that $y=\bigvee_{i\in I}-a_i$. For each i, since $-a_i\leqslant y$ we have $f^{-\infty}(v)\leqslant\sim y\leqslant\sim -a_i$ and therefore

$$-a_i\leqslant\sim\sim -a_i\leqslant\sim f^{-\infty}(v).$$

It follows from **A4** that $-a_i\leqslant -f^{-\infty}(v)$. Hence

$$f^{-\infty}(u)\leqslant y=\bigvee_{i\in I}-a_i\leqslant -f^{-\infty}(v)$$

and therefore f is continuous. ■

By a $\mathbf{T}_1$ **frame** we mean an a-frame in which for all x, y with $x \neq y$, either $x \wedge -y \neq 0$ or else $-x \wedge y \neq 0$.

We say that a join homomorphism $f : \mathfrak{L} \to \mathfrak{M}$ between a-frames is **strongly extensional** if

$$\forall_{u,v \in \mathfrak{M}} \left(u \neq v \Rightarrow f^{-\infty}(u) \neq f^{-\infty}(v) \right).$$

Proposition 21 *A continuous join homomorphism of an a-frame into a $\boldsymbol{T}_1$ a-frame is strongly extensional.*

Proof. Let $f : \mathfrak{L} \to \mathfrak{M}$ be a continuous join homomorphism, where $\mathfrak{L}, \mathfrak{M}$ are a-frames and $\mathfrak{M}$ is $\mathbf{T}_1$. Let u, v be elements of $\mathfrak{M}$ such that $u \neq v$. Since $\mathfrak{M}$ is $\mathbf{T}_1$, we may assume that $u \wedge -v \neq 0$. By the continuity of f,

$$f^{-\infty}(u \wedge -v) \leqslant -f^{-\infty}(v) \leqslant \sim f^{-\infty}(v).$$

But, clearly, $f^{-\infty}(u \wedge -v) \leqslant f^{-\infty}(u)$. Since f is a join homomorphism,

$$0 \neq f^{-\infty}(u \wedge -v) \leqslant f^{-\infty}(u) \wedge \sim f^{-\infty}(v).$$

Hence $f^{-\infty}(u) \wedge \sim f^{-\infty}(y) \neq 0$ and therefore $f^{-\infty}(u) \neq f^{-\infty}(y)$. ■

We now have conditions which ensure the equivalence of continuity and topological continuity for a join homomorphism.

Proposition 22 *Let $\mathfrak{L}$ and $\mathfrak{M}$ be a-frames with $\mathfrak{L}$ atomic and $\mathfrak{M}$ locally decomposable. Let $f : \mathfrak{L} \to \mathfrak{M}$ be a continuous join homomorphism which preserves atoms (that is, takes atoms in $\mathfrak{L}$ to atoms in $\mathfrak{M}$). Then f is topologically continuous.*

Proof. Let $v = \bigvee_{i \in I} - s_i$ be a nearly open subset of $\mathfrak{M}$. If $x \in \mathfrak{L}$ is an atom such that $x \leqslant f^{-\infty}(v)$, then $f(x)$ is an atom and $f(x) \leqslant v$. Lemma 4 shows that there exists i such that $f(x) \leqslant -s_i$. By Proposition 10, there exists $t \in \mathfrak{M}$ such that $f(x) \leqslant -t$ and $-s_i \vee t = 1$. Set $y = f^{-\infty}(t)$. Then $f(y) \leqslant t$, so $f(x) \leqslant -t \leqslant -f(y)$; whence, by the continuity of f, $x \leqslant -f^{-\infty}(f(y))$. Since $y \leqslant f^{-\infty}(f(y))$, it follows that $x \leqslant -y$.

Now consider any atom z of $\mathfrak{L}$ with $z \leqslant -y$. Since $\neg(z \leqslant y)$ and $y = f^{-\infty}(t)$, we have $\neg(f(z) \leqslant t)$; but $f(z)$ is an atom in $\mathfrak{M}$ and

$$f(z) \leqslant 1 = -s_i \vee t = 1;$$

so, by Lemma 3, $f(z) \leqslant -s_i$ and therefore $z \leqslant f^{-\infty}(-s_i)$. Since $\mathfrak{L}$ is atomic, it follows that $-y \leqslant f^{-\infty}(-s_i)$. Thus we have proved that for each atom $x \leqslant f^{-\infty}(v)$ there exist $i \in I$ and $y \in \mathfrak{L}$ such that $x \leqslant -y \leqslant f^{-\infty}(-s_i) \leqslant f^{-\infty}(v)$. It follows that

$$\begin{aligned}
f^{-\infty}(v) &= \bigvee \left\{ x \in \mathfrak{L} : x \text{ is an atom and } x \leqslant f^{-\infty}(v) \right\} \\
&\leqslant \bigvee \left\{ -y : y \in \mathfrak{L} \text{ and } \exists_{i \in I} \left(-y \leqslant f^{-\infty}(-s_i) \right) \right\} \\
&= \bigvee_{i \in I} \bigvee \left\{ -y : y \in \mathfrak{L} \text{ and } -y \leqslant f^{-\infty}(-s_i) \right\} \\
&\leqslant f^{-\infty}(v),
\end{aligned}$$

so

$$f^{-\infty}(-v) = \bigvee \left\{ -y : y \in \mathfrak{L} \text{ and } \exists_{i \in I} \left(-y \leqslant f^{-\infty}(-s_i) \right) \right\},$$

which is nearly open in $\mathfrak{L}$. ∎

Finally, we turn to strong continuity.

Proposition 23 *Let $\mathfrak{L}$ and $\mathfrak{M}$ be a-frames with $\mathfrak{L}$ atomic. Let $f : \mathfrak{L} \to \mathfrak{M}$ be a strongly continuous join homomorphism that preserves atoms. Then f is continuous.*

Proof. Let $u \leqslant -v$ in $\mathfrak{M}$, and consider any atom x of $\mathfrak{L}$ such that $x \leqslant f^{-\infty}(u)$. We have

$$\begin{aligned} f(x) \leqslant u &= u \wedge -v \\ &= u \wedge \bigvee \{z \in \mathfrak{M} : z \bowtie v\} \\ &= \bigvee \{u \wedge z : z \in \mathfrak{M},\ z \bowtie v\}. \end{aligned}$$

Since $f(x)$ is an atom, Lemma 4 shows us that there exists $z \in \mathfrak{M}$ with $z \bowtie v$ and $f(x) \leqslant u \wedge z \leqslant z$; whence $f(x) \bowtie v$. The strong continuity of f now yields $f^{-\infty}(f(x)) \bowtie f^{-\infty}(v)$; since $x \leqslant f^{-\infty}(f(x))$, it follows that $x \bowtie f^{-\infty}(v)$ and therefore $x \leqslant -f^{-\infty}(v)$. Thus

$$f^{-\infty}(u) = \bigvee \{x \in \mathfrak{L} : x \text{ is an atom } \& \ f(x) \leqslant u\} \leqslant -f^{-\infty}(v),$$

as we want. ∎

This completes our laying the foundations of a theory of point-free pre-apartness on frames. One advantage of our doing this has been to show exactly where we appear to need points/atoms in the set-set model; for example, axiom **B5** of the latter (dealing with local decomposability) is now seen to be expressible without reference to points, thereby countering a criticism that our set-set theory appeared, in that one axiom, to be point-dependent. On the other hand, it is hard to see how Propositions 22 and 23 could have been proved without the atomic hypotheses.

Further development of the theory can be found in [6] and [7].

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