A FAST ALGORITHM FOR COMPUTING JONES POLYNOMIALS OF MONTESINOS LINKS

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Abstract. We propose a fast algorithm for computing Jones polynomials of Montesinos links. The Jones polynomial is a useful invariant and Montesinos links are one of the fundamental classes in knot theory. Given the Tait graph of a Montesinos diagram with n edges, our algorithm runs with $\mathcal{O}(n)$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, namely in $\mathcal{O}(n^2 \log n)$ time.

1 Introduction Knot theory is a subfield of topology. A knot is a simple (non-self-intersecting) closed curve embedded in $\mathbb{R}^3$. More generally, one may study links. A link is a finite collection of disjointly embedded knots. Works on knot theory have led to many important advances in other areas, biology, chemistry, physics and so on [1].

For classifying and characterizing links, various invariants have been defined and profoundly studied in knot theory. The Jones polynomial [4] is a useful invariant. L.H. Kauffman [5] gave a combinatorial method for calculating the Jones polynomial by means of the Kauffman bracket polynomial. By using Kauffman's method, the Jones polynomial is computable with $\mathcal{O}(2^{\mathcal{O}(n)})$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, where n is the number of the edges in the input Tait graph. F. Jaeger, D.L. Vertigan and D.J.A. Welsh showed that computing the Jones polynomial is generally #**P**–hard [3, 12]. It is expected to require exponential time in the worst case.

Recently, it has been recognized that it is important to compute Jones polynomials for links with reasonable restrictions. J.A. Makowsky [6, 7] showed that Jones polynomials are computable in polynomial time if treewidths of input Tait graphs are bounded by a constant. J. Mighton [8] showed that Jones polynomials are computable with $\mathcal{O}(n^4)$ operations in polynomials of degree $\mathcal{O}(n)$ if treewidths of input Tait graphs are at most two, where n is the number of the edges in the input Tait graph. M. Hara, S. Tani and M. Yamamoto [2] showed that Jones polynomials of arborescent links are computable with $\mathcal{O}(n^3)$ operations in polynomials of degree $\mathcal{O}(n)$, where n is the number of the edges in the input Tait graph. T. Utsumi and K. Imai [11] showed that Jones polynomials of pretzel links are computable in $\mathcal{O}(n^2)$ time, where n is the number of the edges in the input Tait graph. M. Murakami, M. Hara, M. Yamamoto and S. Tani [10] showed that Jones polynomials of 2–bridge links and closed 3–braid links are computable with $\mathcal{O}(n)$ operations in polynomials of degree $\mathcal{O}(n)$, where n is the number of the edges in the input Tait graph.

In this paper, we propose a fast algorithm for computing Jones polynomials of Montesinos links. Montesinos links, introduced by J.M. Montesinos [9], are one of the fundamental classes in knot theory and a generalization of pretzel links and 2–bridge links. Montesinos diagrams, defined below, are standard link diagrams representing Montesinos links and represented by sequences of integer sequences because every Montesinos diagram consists of rational tangles and every rational tangle is represented by an integer sequence. Given the Tait graph of a Montesinos diagram, our algorithm recognizes the rational tangles

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of the diagram, constructs the integer sequences representing the tangles and computes the Kauffman bracket polynomial of the diagram. The Jones polynomial is directly computable from the Kauffman bracket polynomial.

We characterize Tait graphs of Montesinos diagrams in order to recognize rational tangles of Montesinos diagrams in linear time. The recognition algorithm can be modified to a linear time algorithm for constructing sequences of integer sequences representing Montesinos diagrams. We show a recurrence formula of Kauffman bracket polynomials of Montesinos diagrams represented by sequences of integer sequences. By using the formula, Kauffman bracket polynomials of Montesinos diagrams are computable with $\mathcal{O}(n)$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, where n is the number of the edges in the input Tait graph. Therefore, Jones polynomials of Montesinos links are computable with $\mathcal{O}(n)$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, namely in $\mathcal{O}(n^2 \log n)$ time. Although treewidths of Tait graphs of Montesinos diagrams are two, our algorithm is faster than Mighton's algorithm.

This paper is organized in the following way. Section 2 contains some basic notations and definitions of knot theory. In section 3, we characterize Tait graphs of Montesinos diagram. Section 4 deals with the algorithm for recognizing rational tangles of Montesinos diagrams from its Tait graphs and the algorithm for computing sequences of integer sequence of Montesinos diagrams. In Section 5, we provide the algorithm for computing Kauffman bracket polynomials of Montesinos diagrams from its sequences of integer sequences.

2 Preliminaries In this section, we give some basic notations and definitions of knot theory. For details, see C.C. Adams [1].

A *link* of n components is n mutually disjoint simple closed curves embedded in $\mathbb{R}^3$. A link of one component is a *knot*. A *link diagram* is an image of a link by an orthogonal projection from $\mathbb{R}^3$ to a plane which has finitely many traverse double points, called *crossings*, and no other multiple points, together with information specifying which arc is on top at each crossing (see Figure 1).

Given any link diagram, we can color the faces black and white in such a way that no two faces with a common arc are the same color. We color the unique unbounded face white. Such a coloring is called the *Tait coloring* of the diagram (see Figure 2). We can get the edge-labeled planar graph. Its vertices are the black faces of the Tait coloring and two vertices are joined by a labeled edge if they share a crossing. The label of the edge is $+1$ or -1 according to the (conventional) rule. We may call the label the *sign*. We call the graph the *Tait graph* of the diagram. Note that the number of the edges in the graph is equal to the number of the crossings of the diagram. A Tait graph G is *isomorphic* to a Tait graph G' if there exists a bijection f from the vertex set of G to the vertex set of G' satisfying that the number of the edges labeled "+1" joining u and v in G is equal to the number of the edges labeled "+1" joining $f(u)$ and $f(v)$ in G' and the number of the edges labeled "-1" joining u and v in G is equal to the number of the edges labeled "-1" joining $f(u)$ and $f(v)$ in G' for any pair of vertices u and v in G. Such a function f is called an *isomorphism* from G to G'.

Let $G = (V, E, s)$ be a Tait graph, where V is the vertex set of G, E is the edge set of G and s is the edge-labeling function from E to $\{-1, +1\}$. $\deg_G(v)$ denotes the degree of v in G and $N_G(v)$ denotes the set of the neighbors of v in G for any vertex $v \in V$. For any subset V' of V, the *induced subgraph* $G[V']$ is the subgraph of G consisting of V' and the edges in E whose endvertices are both in V'. A *Hamilton cycle* of G is an alternating sequence $v_1 e_1 v_2 \cdots v_n e_n v_1$ of all vertices in V and edges in E satisfying that $v_1, \ldots, v_n$ are distinct, e_n is incident to v_1 and v_n, and e_i is incident to v_i and v_{i+1} for $i = 1, \ldots, n-1$. We define a function $\mathrm{edge\_sum}_G$ from $V \times V$ to $\mathbb{Z}$ satisfying that $\mathrm{edge\_sum}_G(u, v)$ is the

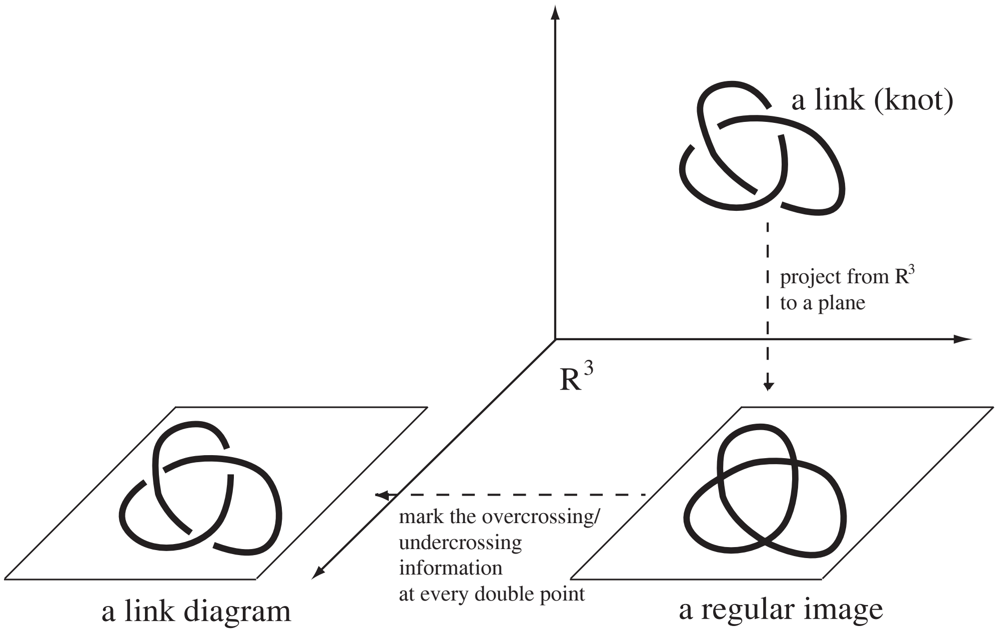


Figure 1: A link and a link diagram.

sum of the signs of all edges in G whose endvertices are u and v for any adjacent pair of vertices u and v in G. We denote the size of S by $|S|$ for a set S.

A continuous bijection f from $\mathbb{R}^3$ to $\mathbb{R}^3$ is called *homeomorphism* if f has a continuous inverse mapping. A link L is *ambient isotopic* or *equivalent*[1] to a link L' if there exists a homeomorphism h_t from $\mathbb{R}^3$ to $\mathbb{R}^3$ for any real number t in the closed interval $[0, 1]$ satisfying the following:

(i) h_0 is the identity.

(ii) $h_1(L) = L'$.

(iii) The mapping f_p from the open interval $(0, 1)$ to $\mathbb{R}^3$ satisfying $f_p(t) = h_t(p)$ is continuous for any point $p \in \mathbb{R}^3$.

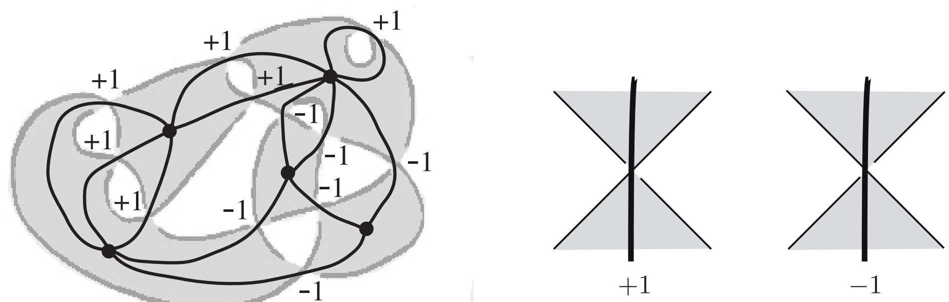
Definition 2.1 The *Kauffman bracket polynomial* is a function from link diagrams to the Laurent polynomial ring with integer coefficients in an indeterminate A. It maps a link diagram $\widetilde{L}$ to $\langle \widetilde{L} \rangle \in \mathbb{Z}[A^{\pm 1}]$ and is characterized by

(i) $\langle \bigcirc \rangle = 1$,

(ii) $\langle \widetilde{L} \sqcup \bigcirc \rangle = (-A^{-2} - A^2)\langle \widetilde{L} \rangle$ and

(iii) $\langle \diagup\!\!\!\diagdown \rangle = A\langle\ )(\ \rangle + A^{-1}\langle \asymp \rangle$.

[1]Intuitively, a link L is equivalent to a link L' if L can be continuously deformed to L' without ever having any one of the loops intersects itself or any of the other loops in the process. Therefore, we can regard a link as "rubber bands" and deform it topologically.



The Tait coloring of a link diagram and its Tait graph

Signs of edges in Tait graphs

Figure 2: A Tait coloring, a Tait graph and signs of edges in Tait graphs.

Here, $\bigcirc$ is the knot diagram without a crossing and $\widetilde{L} \sqcup \bigcirc$ is the disjoint sum of $\widetilde{L}$ and $\bigcirc$. In (iii), the formula refers to three link diagrams that are exactly the same except near a point where they differ in the way indicated.

Note that the degree of the Kauffman bracket polynomial of the diagram is $\mathcal{O}(n)$ and the coefficients of the Kauffman bracket polynomial of the diagram are $\mathcal{O}(2^n)$ for any link diagram with n crossings. A link is *oriented* if each of its components is given an orientation. The *writhe* $w(\widetilde{L})$ of an oriented link diagram $\widetilde{L}$ is the sum of the signs of the crossings of $\widetilde{L}$, where each crossing has sign $+1$ or -1 as defined (by convention) in Figure 3.

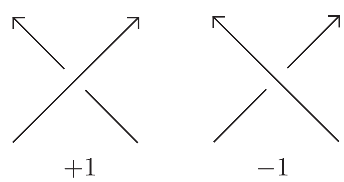


Figure 3: Signs of crossings.

The *Jones polynomial* $V(L)$ of an oriented link L is defined by

$$V(L) = \left((-A)^{-3w(\widetilde{L})}\langle \widetilde{L} \rangle\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{\pm 1/2}],$$

where $\widetilde{L}$ is a link diagram of L. It is known that $V(L)$ is well-defined.

A *tangle* is a portion of a link diagram from which there emerge just four arcs pointing in the compass directions NW, NE, SW, SE (see Figure 4). The tangle consisting of two vertical strings without a crossing is called 0-*tangle*. The 0-tangle twisted k times is called k-*tangle* and denoted by I_k. They are called *integer tangles* (see Figure 4). The tangle consisting of integer tangles $I_{a_1}, \ldots, I_{a_m}$ as shown in Figure 4 is called a *rational tangle*, where $a_1, \ldots, a_m$ are integers. Let $a_{11}, \ldots, a_{1m_1}, \ldots, a_{l1}, \ldots, a_{lm_l}$ and a be integers. We denote the link diagram consisting of integer tangles $I_{a_{11}}, \ldots, I_{a_{1m_1}}, \ldots, I_{a_{l1}}, \ldots, I_{a_{lm_l}}$ (l rational tangles) and I_a as shown in Figure 5 by $\widetilde{M}(a_{11}, \ldots, a_{1m_1}| \cdots |a_{l1}, \ldots, a_{lm_l}||a)$.

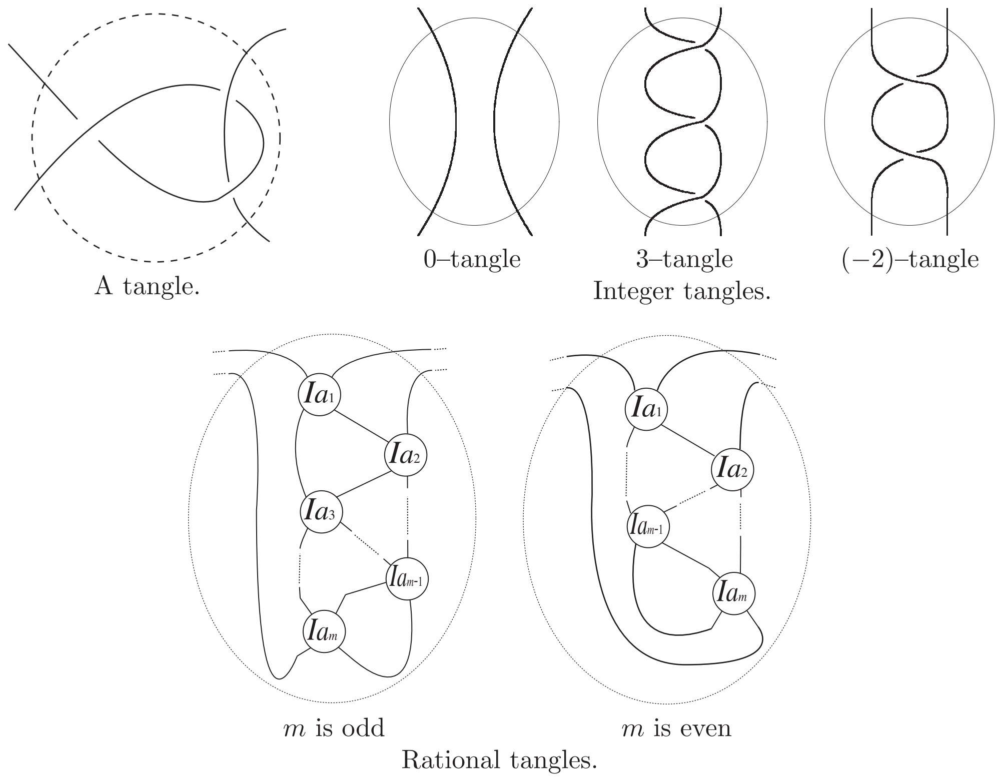


Figure 4: A tangle, integer tangles and rational tangles.

Definition 2.2 We call a link diagram a *Montesinos diagram* if the diagram can be represented by $\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$, where $l \geq 3$, $m_i \geq 3$, m_i is an odd number and $a_{ij} \neq 0$ for $i=1,\ldots,l$ and $j=1,\ldots,m_i$ (see Figure 6). We call $(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$ a *normal representation* of the diagram.

A link is called a *Montesinos link* if there exists a Montesinos diagram representing the link. Montesinos links are a generalization of pretzel links and 2–bridge links. Pretzel links are Montesinos links represented by Montesinos diagrams all of whose rational tangles are integer tangles. 2–bridge links are Montesinos links represented by Montesinos diagrams consisting of at most two rational tangles.

Let $(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$ be a normal representation of a Montesinos diagram $\widetilde{L}$ and $G=(V,E,s)$ the Tait graph of the diagram. For each $i=1,\ldots,l$, we call the induced subgraph $G[V_i]$ a *rational subgraph* of G where V_i is the set of the endvertices of the edges in E corresponding to the crossings of $I_{a_{i1}},\ldots,I_{a_{im_i}}$ in $\widetilde{L}$. If $a \neq 0$, then we call the induced path $G[V_0]$ the *top path* in G where V_0 is the set of the endvertices of the edges in E corresponding to the crossings of I_a in $\widetilde{L}$. Let V_{ij} be the set of the endvertices of the edges in E corresponding to the crossings of $I_{a_{ij}}$ in $\widetilde{L}$ for $i=1,\ldots,l$ and $j=1,\ldots,m_i$. Note that $|\bigcap_{j=1}^{\frac{m_i+1}{2}} V_{i,2j-1}|=1$ and $|V_{i1}|=2$ for $i=1,\ldots,l$. Let u_{01} be the vertex in $\bigcap_{j=1}^{\frac{m_1+1}{2}} V_{1,2j-1}$ and u_{i1} the vertex in $V_{i1}-\{u_{i-1,1}\}$ for $i=1,\ldots,l$. Let $n_i=|V_i|-1$ and u_{ij} the j-th visited vertex where we go along the path $G[V_i-\{u_{i-1,1}\}]$ from u_{i1} to the other endvertex of $G[V_i-\{u_{i-1,1}\}]$ for $i=1,\ldots,l$ and $j=1,\ldots,n_i$. Note that $G[\{u_{i-1,1}\}\cup\bigcup_{j=1}^{n_i}\{u_{ij}\}]$

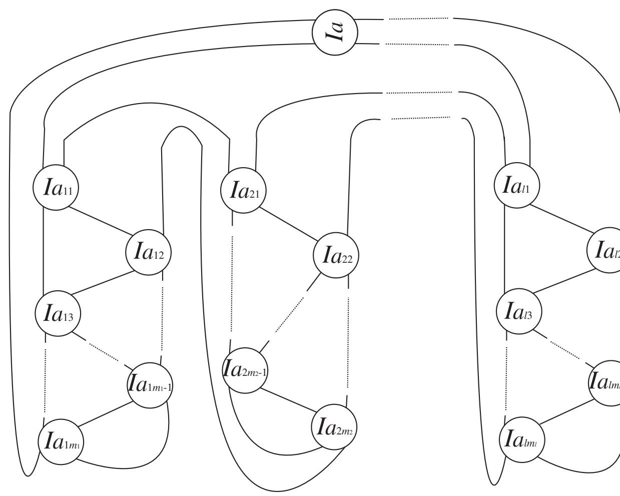


Figure 5: $\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$.

is the rational subgraph $G[V_i]$ for $i=1,\ldots,l$. Let $n_0=|a|+1$. If $a\neq 0$, then let u_{0j} be the j-th visited vertex where we go along the top path path $G[V_0]$ from u_{01} to u_{l1} for $j=1,\ldots,n_0$. We call the label $u_{01},\ldots,u_{0n_0},u_{11},\ldots,u_{1n_1},\ldots,u_{l1},\ldots,u_{ln_l}$ ($u_{l1}=u_{0n_0}$) the *normal label* of V for $(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$ (see Figure 6).

3 Characteristic of Tait graphs of Montesinos diagrams In this section, we characterize Tait graphs of Montesinos diagrams.

Proposition 3.1 *Let $G=(V,E,s)$ be the Tait graph of a Montesinos diagram and V' a subset of V. The induced subgraph $G[V']$ is a rational subgraph of G if and only if V' is a maximal subset of V such that there exists a vertex $v_1\in V'$ satisfying the following conditions:*

(i) $|N_G(v_1)|\geq 3$ *and* $G[V'-\{v_1\}]$ *is a path with endvertices* v_2 *and* v_3 *in* G.

(ii) $|N_G(v_2)|\geq 3$ *and* $v_2\in N_G(v_1)$.

(iii) $|N_G(v_3)|=2$ *and* $v_3\in N_G(v_1)$.

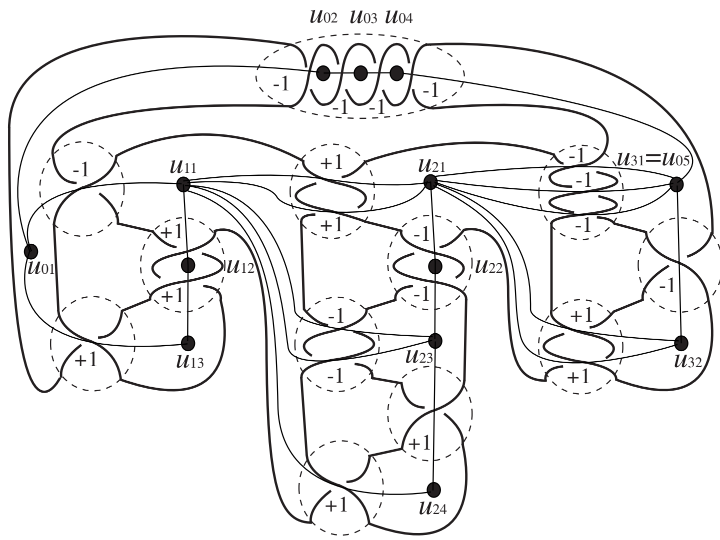
(iv) $|N_G(v_1)|\geq 4$ *or* $|N_G(v_2)|\geq 4$.

(v) *For any vertex* $v\in V'-\{v_1,v_2,v_3\}$, $2\leq|N_G(v)|\leq 3$. *For any vertex* $v\in V'-\{v_1,v_2,v_3\}$, *if* $|N_G(v)|=3$, *then* $v\in N_G(v_1)$.

(vi) *All edges in* E *whose endvertices are* v *and* v_1' *have the same sign for any vertex* $v\in N_{G[V']}(v_1)$.

(vii) *The two edges incident to* u *have the same sign for any vertex* $u\in\{v\in V'-\{v_1,v_2,v_3\}:|N_G(v)|=2\}$.

Proof. Let $u_{01},\ldots,u_{0n_0},u_{11},\ldots,u_{1n_1},\ldots,u_{l1},\ldots,u_{ln_l}$ ($u_{l1}=u_{0n_0}$) the normal label of V for a normal representation of the Montesinos diagram.
($\Rightarrow$) Because $G[V']$ is a rational subgraph of G, $V'=\{u_{i-1,1}\}\cup\bigcup_{j=1}^{n_i}\{u_{ij}\}$ for an integer $i\in\{1,\ldots,l\}$. Take $u_{i-1,1}$ as v_1, and the conditions (i), (ii), (iii), (iv), (v), (vi) and (vii)



A normal representation $\widetilde{M}(1, 2, -1| - 2, -2, 2, 1, -1|3, -1, -2|| - 4)$

Figure 6: A Montesinos diagram and the normal label for a normal representation.

are satisfied. We show the maximality of V' by contradiction. We assume that there exist a subset V'' of V and a vertex $v'_1 \in V''$ satisfying $V'' \supsetneq V'$ and the following conditions:

(i′) $|N_G(v'_1)| \geq 3$ and $G[V'' - \{v'_1\}]$ is a path with endvertices v'_2 and v'_3 in G.

(ii′) $|N_G(v'_2)| \geq 3$ and $v'_2 \in N_G(v'_1)$.

(iii′) $|N_G(v'_3)| = 2$ and $v'_3 \in N_G(v'_1)$.

(iv′) $|N_G(v'_1)| \geq 4$ or $|N_G(v'_2)| \geq 4$.

(v′) For any vertex $v \in V'' - \{v'_1, v'_2, v'_3\}$, $2 \leq |N_G(v)| \leq 3$. For any vertex $v \in V'' - \{v'_1, v'_2, v'_3\}$, if $|N_G(v)| = 3$, then $v \in N_G(v'_1)$.

(vi′) All edges in E whose endvertices are v and v'_1 have the same sign for any vertex $v \in N_{G[V'']}(v'_1)$.

(vii′) The two edges incident to u have the same sign for any vertex $u \in \{v \in V'' - \{v'_1, v'_2, v'_3\} : |N_G(v)| = 2\}$.

Since $V'' \supset V'$ and $G[V']$ contains a Hamilton cycle, v'_1 should be in V' by (i′) and $G[V' - \{v'_1\}]$ should be a path. Note that $\{v'_3\} = \{v \in N_{G[V'']}(v'_1) : |N_G(v)| = 2\}$ by (i′), (ii′) and (iii′). If v'_1 is $u_{i-1,1}$, then v'_3 should be u_{in_i}. If v'_1 is u_{i1}, then v'_3 should be u_{i2}. It holds that one of the following three cases Case 1: $|N_G(u_{i-1,1})| \geq 4$ and $|N_G(u_{i1})| \geq 4$, Case 2: $|N_G(u_{i-1,1})| = 3$ and $|N_G(u_{i1})| \geq 4$ or Case 3: $|N_G(u_{i-1,1})| \geq 4$ and $|N_G(u_{i1})| = 3$.

Case 1: $|N_G(u_{i-1,1})| \geq 4$ and $|N_G(u_{i1})| \geq 4$.

v'_1 is $u_{i-1,1}$ and v'_2 is u_{i1} or v'_1 is u_{i1} and v'_2 is $u_{i-1,1}$ by (i′), (ii′), (iii′) and (v′). If v'_1 is $u_{i-1,1}$ and v'_2 is u_{i1}, then v'_3 should be u_{in_i} and V'' should be $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i} \{u_{ij}\} = V'$. This contradicts $V'' \supsetneq V'$. If v'_1 is u_{i1} and v'_2 is $u_{i-1,1}$, then v'_3 should be u_{i2} and V'' should be $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i} \{u_{ij}\} = V'$. This contradicts $V'' \supsetneq V'$.

Case 2: $|N_G(u_{i-1,1})| = 3$ and $|N_G(u_{i1})| \geq 4$.
i should be 1, $n_0 > 1$ and $|N_{G[V']}(u_{01})|$ should be 2. Because $v'_1 \in V'$ and $|N_G(v)| = 2$ for any vertex $v \in V' - \{u_{01}, u_{11}\}$, it holds that either Case 2.1: v'_1 is u_{01} and v'_2 is u_{11} or Case 2.2: v'_1 is u_{11} and v'_2 is u_{01} by (i′), (ii′), (iii′) and (v′).
Case 2.1: v'_1 is u_{01} and v'_2 is u_{11}.
v'_3 should be u_{1n_1} and V'' should be $\{u_{01}\} \cup \bigcup_{j=1}^{n_1}\{u_{1j}\} = V'$. This contradicts $V'' \supsetneq V'$.
Case 2.2: v'_1 is u_{11} and v'_2 is u_{01}.
v'_3 should be u_{12} and V'' should be $\{u_{01}\} \cup \bigcup_{j=1}^{n_1}\{u_{1j}\} = V'$. This contradicts $V'' \supsetneq V'$.
Case 3: $|N_G(u_{i-1,1})| \geq 4$ and $|N_G(u_{i1})| = 3$.
i should be l and $n_0 > 1$. Recall v'_1 should be in $V' = \{u_{l-1,1}\} \cup \bigcup_{j=1}^{n_l}\{u_{lj}\}$.
Case 3.1: v'_1 is $u_{l-1,1}$.
v'_3 should be u_{ln_l}. v'_2 should be in $\{u_{l-2,1}, u_{l-1,2}\} \cup \bigcup_{j=1}^{n_l-1}\{u_{lj}\}$ by (ii′). If v'_2 is $u_{l-2,1}$ or $u_{l-1,2}$, then u_{01} should be contained in $G[V'' - \{v'_1\}]$. This contradicts (v′). If $v'_2 \in \bigcup_{j=1}^{n_l-1}\{u_{lj}\}$, then v'_2 should be u_{l1} by the maximality of V'' and V'' should be $\{u_{l-1,1}\} \cup \bigcup_{j=1}^{n_l}\{u_{lj}\} = V'$. This contradicts $V'' \supsetneq V'$.
Case 3.2: $v'_1 \in \bigcup_{j=1}^{n_l}\{u_{lj}\}$.
v'_2 should be $u_{l-1,1}$ by (ii′), (iii′) and (v′). If v'_1 is u_{l1}, then v'_3 should be u_{l2} and V'' should be $\{u_{l-1,1}\} \cup \bigcup_{j=1}^{n_l}\{u_{lj}\} = V'$. This contradicts $V'' \supsetneq V'$. Hence, $v'_1 \in \bigcup_{j=2}^{n_l}\{u_{lj}\}$. Since $|N_G(u_{l1})| = 3$ and $N_G(u_{l1}) \cap \bigcup_{j=2}^{n_l}\{u_{ij}\} = \{u_{l2}\}$, v'_1 should be u_{l2} by (v′). Because $N_G(v'_1) = \{v'_2, u_{l1}, u_{l3}\}$, v'_3 should be u_{l3} by (iii′). Therefore, u_{l1} should not be contained in $G[V'' - \{v'_1\}]$. This contradicts $u_{l1} \in V' \subset V''$.
(⇐) We show $V' = \{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$ for an integer $i \in \{1, \ldots, l\}$. It holds that one of the following three cases Case 1: $|N_G(v_1)| \geq 4$ and $|N_G(v_2)| \geq 4$, Case 2: $|N_G(v_1)| \geq 4$ and $|N_G(v_2)| = 3$ or Case 3: $|N_G(v_1)| = 3$ and $|N_G(v_2)| \geq 4$ by (i), (ii) and (iv).
Case 1: $|N_G(v_1)| \geq 4$ and $|N_G(v_2)| \geq 4$.
For an integer $i \in \{1, \ldots, l\}$, it holds that either Case 1.1: v_1 is $u_{i-1,1}$ and v_2 is u_{i1} or Case 1.2: v_1 is u_{i1} and v_2 is $u_{i-1,1}$ by (ii).
Case 1.1: v_1 is $u_{i-1,1}$ and v_2 is u_{i1}.
Case 1.1.1: $i = 1$.
v_3 should be u_{02}, u_{l2} or u_{1n_1} by (iii). If v_3 is u_{02} or u_{l2}, then $u_{l-1,1}$ should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_3 should be u_{1n_1} and V' should be $\{u_{01}\} \cup \bigcup_{j=1}^{n_1}\{u_{1j}\}$.
Case 1.1.2: $1 < i < l$.
v_3 should be $u_{i-1,2}$ or u_{in_i} by (iii). If v_3 is $u_{i-1,2}$, then u_{l1} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_3 should be u_{in_i} and V' should be $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$.
Case 1.1.3: $i = l$.
v_3 should be $u_{l-1,2}$ or u_{ln_l} by (iii). If v_3 is $u_{l-1,2}$, then $u_{l-2,1}$ should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_3 should be u_{ln_l} and V' should be $\{u_{l-1,1}\} \cup \bigcup_{j=1}^{n_l}\{u_{lj}\}$.
Case 1.2: v_1 is u_{i1} and v_2 is $u_{i-1,1}$.
Case 1.2.1: $i = 1$.
v_3 should be u_{12} or u_{2n_2} by (iii). If v_3 is u_{2n_2}, then u_{21} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_3 should be u_{12} and V' should be $\{u_{01}\} \cup \bigcup_{j=1}^{n_1}\{u_{1j}\}$ by the maximality of V'.
Case 1.2.2: $1 < i < l$.
v_3 should be u_{i2} or $u_{i+1,n_{i+1}}$ by (iii). If v_3 is $u_{i+1,n_{i+1}}$, then $u_{i-2,1}$ should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_3 should be u_{i2} and V' should be $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$ by the maximality of V'.
Case 1.2.3: $i = l$.
v_3 should be u_{l2} or u_{1n_1} by (iii). If v_3 is u_{1n_1}, then u_{11} should be contained in $G[V' - \{v_1\}]$.

This contradicts (v). Hence, v_3 should be u_{l2} and V' should be $\{u_{l-1,1}\}\cup\bigcup_{j=1}^{n_l}\{u_{lj}\}$ by the maximality of V'.

Case 2: $|N_G(v_1)| \geq 4$ and $|N_G(v_2)| = 3$.

v_1 should be u_{i1} for an integer $i \in \{0, 1, \ldots, l-1\}$.

Case 2.1: $i = 0$.

v_2 should be in $\{u_{l1}, u_{l2}\} \cup \bigcup_{j=2}^{n_1-1}\{u_{1j}\}$ by (ii). v_3 should be u_{l2}, u_{02} or u_{1n_1} by (iii). If v_2 is u_{l1} or u_{l2}, then v_3 should be u_{1n_1} and u_{11} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, $v_2 \in \bigcup_{j=2}^{n_1-1}\{u_{1j}\}$. If v_3 is u_{l2} or u_{02}, then u_{11} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). If v_3 is u_{1n_1}, then V' should be a proper subset of $\{u_{01}\} \cup \bigcup_{j=1}^{n_1}\{u_{1j}\}$. This contradicts the maximality of V'.

Case 2.2: $i = 1$.

v_2 should be in $\{u_{01}, u_{12}\} \cup \bigcup_{j=2}^{n_2-1}\{u_{2j}\}$ by (ii). v_3 should be u_{12} or u_{2n_2} by (iii). If $v_2 \in \{u_{01}, u_{12}\}$ and v_3 is u_{2n_2} or $v_2 \in \bigcup_{j=2}^{n_2-1}\{u_{2j}\}$ and v_3 is u_{12}, then u_{21} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). If $v_2 \in \bigcup_{j=2}^{n_2-1}\{u_{2j}\}$ and v_3 is u_{2n_2}, then V' should be a proper subset of $\{u_{11}\} \cup \bigcup_{j=1}^{n_2}\{u_{2j}\}$. This contradicts the maximality of V'. Hence, v_2 should be u_{01}, v_3 should be u_{12} and V' should be $\{u_{01}\} \cup \bigcup_{j=1}^{n_1}\{u_{1j}\}$.

Case 2.3: $1 < i < l-1$.

v_2 should be in $\{u_{i2}\}\cup\bigcup_{j=2}^{n_{i+1}-1}\{u_{i+1,j}\}$ by (ii). v_3 should be u_{i2} or $u_{i+1,n_{i+1}}$ by (iii). If v_2 is u_{i2} and v_3 is $u_{i+1,n_{i+1}}$ or $v_2 \in \bigcup_{j=2}^{n_{i+1}-1}\{u_{i+1,j}\}$ and v_3 is u_{i2} then $u_{i-1,1}$ should be contained in $G[V' - \{v_1\}]$. This contradicts (v). If $v_2 \in \bigcup_{j=2}^{n_{i+1}-1}\{u_{i+1,j}\}$ and v_3 is $u_{i+1,n_{i+1}}$, then V' should be a proper subset of $\{u_{i1}\} \cup \bigcup_{j=1}^{n_{i+1}}\{u_{i+1,j}\}$. This contradicts the maximality of V'.

Case 2.4: $i = l-1$.

v_2 should be in $\{u_{l-1,2}\} \cup \bigcup_{j=1}^{n_l-1}\{u_{lj}\}$ by (ii). v_3 should be $u_{l-1,2}$ or u_{ln_l} by (iii). If v_2 is $u_{l-1,2}$ and v_3 is u_{ln_l} or $v_2 \in \bigcup_{j=1}^{n_l-1}\{u_{lj}\}$ and v_3 is $u_{l-1,2}$, then $u_{l-2,1}$ should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_3 should be u_{ln_l}, v_2 should be u_{l1} by the maximality of V' and V' should be $\{u_{l-1,1}\} \cup \bigcup_{j=1}^{n_l}\{u_{lj}\}$.

Case 3: $|N_G(v_1)| = 3$ and $|N_G(v_2)| \geq 4$.

v_1 should be in $\{u_{01}, u_{l1}\} \cup \bigcup_{i=1}^{l}\bigcup_{j=2}^{n_i-1}\{u_{ij}\}$.

Case 3.1: v_1 is u_{01}.

v_2 should be u_{11} by (ii). v_3 should be u_{02} or u_{1n_1} by (iii). If v_3 is u_{02}, then u_{l1} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_3 should be u_{1n_1} and V' should be $\{u_{01}\} \cup \bigcup_{j=1}^{n_1}\{u_{1j}\}$.

Case 3.2: v_1 is u_{l1}.

v_2 should be $u_{l-1,1}$ or u_{01} by (ii). v_3 should be u_{l2} or u_{0,n_0-1} by (iii). If v_2 is $u_{l-1,1}$ and v_3 is u_{0,n_0-1}, then u_{01} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). If v_2 is u_{01}, then v_3 should be u_{l2} and $u_{l-1,1}$ should be contained in $G[V' - \{v_1\}]$. This contradicts (v). Hence, v_2 should be $u_{l-1,1}$, v_3 should be u_{l2} and V' should be $\{u_{l-1,1}\} \cup \bigcup_{j=1}^{n_l}\{u_{lj}\}$ by the maximality of V'.

Case 3.3: v_1 is u_{ij} for integers $i \in \{1, \ldots, l\}$ and $j \in \{2, 3, \ldots, n_i - 1\}$.

v_2 should be $u_{i-1,1}$ or u_{i1} by (ii). v_3 should be $u_{i,j-1}$ or $u_{i,j+1}$ by (iii). If v_2 is $u_{i-1,1}$ and v_3 is $u_{i,j-1}$, then u_{i1} should be contained in $G[V' - \{v_1\}]$. This contradicts (v). If v_2 is $u_{i-1,1}$ and v_3 is $u_{i,j+1}$, then V' should be $\{u_{i-1,1}\} \cup \bigcup_{k=j}^{n_i}\{u_{ik}\}$ by the maximality of V'. This contradicts the maximality of V'. If v_2 is u_{i1}, then v_1 should be u_{i2} by (ii), v_3 should be u_{i3} by (iii) and $u_{i-1,1}$ should be contained in $G[V' - \{v_1\}]$. This contradicts (v). □

Definition 3.2 Let $G = (V, E, s)$ be a Tait graph. A subset V' of V has *rational subset property* if there exists a vertex $v_1 \in V'$ satisfying the following conditions:

(i) $|N_G(v_1)| \geq 3$ and $G[V' - \{v_1\}]$ is a path with endvertices v_2 and v_3 in G.

(ii) $|N_G(v_2)| \geq 3$ and $v_2 \in N_G(v_1)$.

(iii) $|N_G(v_3)| = 2$ and $v_3 \in N_G(v_1)$.

(iv) $|N_G(v_1)| \geq 4$ or $|N_G(v_2)| \geq 4$.

(v) For any vertex $v \in V' - \{v_1, v_2, v_3\}$, $2 \leq |N_G(v)| \leq 3$. For any vertex $v \in V' - \{v_1, v_2, v_3\}$, if $|N_G(v)| = 3$, then $v \in N_G(v_1)$.

(vi) All edges in E whose endvertices are v and v_1' have the same sign for any vertex $v \in N_{G[V']}(v_1)$.

(vii) The two edges incident to u have the same sign for any vertex $u \in \{v \in V' - \{v_1, v_2, v_3\} : |N_G(v)| = 2\}$.

In this case, we call v_1 a *head vertex* of V' and v_2 a *tail vertex* of V'. V' has *maximal rational subset property* if V' is a maximal subset of V with rational subset property.

The definition of top paths of Tait graphs of Montesinos diagrams and Proposition 3.1 imply the following proposition.

Proposition 3.3 *Let $G = (V, E, s)$ be the Tait graph of a Montesinos diagram and V' a subset of V. The induced subgraph $G[V']$ is the top path of G if and only if V' satisfies the following:*

(i) *$G[V']$ is a path with endvertices v_1 and v_2.*

(ii) *v_1 is a tail vertex of one of subsets of V with maximal rational subset property, v_2 is a head vertex of another subset of V with maximal rational subset property, and no subset of V with maximal rational subset property contains both v_1 and v_2.*

(iii) *For any vertex $v \in V' - \{v_1, v_2\}$, $\deg_G(v) = 2$.*

(iv) *All edges in $G[V']$ have the same sign.*

Definition 3.4 Let $G = (V, E, s)$ be a Tait graph. A subset V' of V has *top subset property* if V' satisfies the following:

(i) $G[V']$ is a path with endvertices v_1 and v_2.

(ii) v_1 is a tail vertex of one of subsets of V with maximal rational subset property, v_2 is a head vertex of another subset of V with maximal rational subset property, and no subset of V with maximal rational subset property contains both v_1 and v_2.

(iii) For any vertex $v \in V' - \{v_1, v_2\}$, $\deg_G(v) = 2$.

(iv) All edges in $G[V']$ have the same sign.

The definition of normal labels of Tait graphs of Montesinos diagrams, Propositions 3.1 and 3.3 imply the following lemma.

Lemma 3.5 *Let G be the Tait graph of a Montesinos diagram and $u_{01}, \ldots, u_{0n_0}, u_{11}, \ldots, u_{1n_1}, u_{21}, \ldots, u_{2n_2}, \ldots, u_{l1}, \ldots, u_{ln_l}$ ($u_{l1} = u_{0n_0}$) the normal label of G for a normal representation of the diagram. Then, the following hold.*

(i) *$l \geq 3$.*

(ii) $u_{01}, \ldots, u_{0,n_0-1}, u_{11}, \ldots, u_{1n_1}, \ldots, u_{l1}, \ldots, u_{ln_l}$ *are distinct vertices and for* $i = 0, 1, \ldots, l$ *and* $j = 1, \ldots, n_i$, u_{ij} *and* $u_{i,j+1}$ *are adjacent.*

(iii) *For every* $i \in \{1, \ldots, l\}$, $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$ *has maximal rational subset property,* $u_{i-1,1}$ *is a head vertex of* $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$ *and* $u_{i-1,1}$ *is a tail vertex of* $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$.

(iv) $\bigcup_{j=1}^{n_0}\{u_{0j}\}$ *has top subset property or* $n_0 = 1$.

(v) *Every edge in* G *is contained in either* $G[\bigcup_{j=1}^{n_0}\{u_{0j}\}]$ *or* $G[\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}]$ *for an integer* $i \in \{1, \ldots, l\}$.

Definition 3.6 Let G be a Tait graph. G has *Montesinos diagram property* if there exists a label $u_{01}, \ldots, u_{0n_0}, u_{11}, \ldots, u_{1n_1}, u_{21}, \ldots, u_{2n_2}, \ldots, u_{l1}, \ldots, u_{ln_l}$ ($u_{l1} = u_{0n_0}$) of all vertices in G satisfying the following:

(i) $l \geq 3$.

(ii) $u_{01}, \ldots, u_{0,n_0-1}, u_{11}, \ldots, u_{1n_1}, \ldots, u_{l1}, \ldots, u_{ln_l}$ are distinct vertices and for $i = 0, 1, \ldots, l$ and $j = 1, \ldots, n_i$, u_{ij} and $u_{i,j+1}$ are adjacent.

(iii) For every $i \in \{1, \ldots, l\}$, $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$ has maximal rational subset property, $u_{i-1,1}$ is a head vertex of $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$ and $u_{i-1,1}$ is a tail vertex of $\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}$.

(iv) $\bigcup_{j=1}^{n_0}\{u_{0j}\}$ has top subset property or $n_0 = 1$.

(v) Every edge in G is contained in either $G[\bigcup_{j=1}^{n_0}\{u_{0j}\}]$ or $G[\{u_{i-1,1}\} \cup \bigcup_{j=1}^{n_i}\{u_{ij}\}]$ for an integer $i \in \{1, \ldots, l\}$.

We say that $u_{01}, \ldots, u_{0n_0}, u_{11}, \ldots, u_{1n_1}, \ldots, u_{l1}, \ldots, u_{ln_l}$ ($u_{l1} = u_{0n_0}$) has *normal label property.*

Lemma 3.7 *Let* G *be a Tait graph with Montesinos diagram property and* $u_{01}, \ldots, u_{0n_0}, u_{11}, \ldots, u_{1n_1}, u_{21}, \ldots, u_{2n_2}, \ldots, u_{l1}, \ldots, u_{ln_l}$ ($u_{l1} = u_{0n_0}$) *a label of all vertices in* G *with normal label property. Then,* G *is isomorphic to the Tait graph of the Montesinos diagram* $\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}||a)$ *where* $a_{11}, \ldots, a_{1m_1}, \ldots, a_{l1}, \ldots, a_{lm_l}$ *and* a *are constructed by the following way:*

(i) *For every* $i \in \{1, \ldots, l\}$, *set* $m_i = 2|N_G(u_{i-1,1}) \cap \bigcup_{j=1}^{n_i}\{u_{ij}\}| - 1$.

(ii) *For* $i = 1, \ldots, l$ *and* $j = 1, \ldots, \frac{m_i+1}{2}$, *let* v_{01} *be* u_{01} *and* v_{ij} *the* j*-th visited neighbor of* $u_{i-1,1}$ *where we go along* $G[\bigcup_{j'=1}^{n_i}\{u_{ij'}\}]$ *from* u_{i1} *to* u_{in_i} *(see Figure 7).*

(iii) *For* $i = 1, \ldots, l$ *and* $j = 1, \ldots, \frac{m_i+1}{2}$, *set* $a_{i,2j-1} = -\mathrm{edge\_sum}_G(v_{i-1,1}, v_{ij})$.

(iv) *For* $i = 1, \ldots, l$ *and* $j = 1, \ldots, \frac{m_i-1}{2}$, *Assign the sum of the sign of the edges in the path connecting* v_{ij} *and* $v_{i,j+1}$ *and not containing* $v_{i-1,1}$ *to* $a_{i,2j}$.

(v) *If* $G[\bigcup_{j=1}^{n_0}\{u_{0j}\}]$ *has top path property, then assign the sum of the sign of the edges in* $G[\bigcup_{j=1}^{n_0}\{u_{0j}\}]$ *to* a, *otherwise set* $a = 0$.

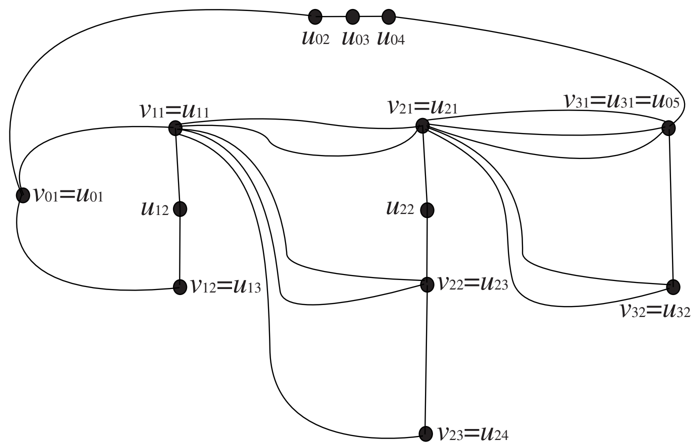


Figure 7: Indices of G.

Proof. It is clear that $\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$ is a Montesinos diagram because $l\geq 3$, $m_i\geq 3$, m_i is an odd number and $a_{ij}\neq 0$ for $i=1,\ldots,l$ and $j=1,\ldots,m_i$. We show that there exists an isomorphism from the Tait graph of the diagram to G.

Let $G=(V,E,s)$, $G'=(V',E',s')$ the Tait graph of the Montesinos diagram $\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$ and $u'_{01},\ldots,u'_{0n'_0},u'_{11},\ldots,u'_{1n'_1},\ldots,u'_{l1},\ldots,u'_{ln'_l}$ ($u'_{l1}=u'_{0n'_0}$) a normal label of G' for $\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$. Note that $n_0=1+|a|=n'_0$ and $n_i=1+\sum_{k=1}^{\frac{m_i-1}{2}}|a_{i,2k}|=n'_i$ for $i=1,\ldots,l$. Let v'_{01} be u'_{01} and v'_{ij} the j-th visited neighbor of $u'_{i-1,1}$ where we go along $G'[\bigcup_{j'=1}^{n'_i}\{u'_{ij'}\}]$ from u'_{i1} to $u'_{in'_i}$ for $i=1,\ldots,l$ and $j=1,\ldots,\frac{m_i+1}{2}$.

Let f be a bijection from V' to V satisfying $f(u'_{ij})=u_{ij}$ for $i=1,\ldots,l$ and $j=1,\ldots,n_i$. Note that $f(v'_{ij})=v_{ij}$ since $v_{ij}=u_{1+\sum_{k=1}^{j-1}|a_{i,2k}|}$ and $v'_{ij}=u'_{1+\sum_{k=1}^{j-1}|a_{i,2k}|}$ for $i=1,\ldots,l$ and $j=1,\ldots,\frac{m_i+1}{2}$. Every pair of adjacent vertices in G consists of $v_{i-1,1}$ and v_{ij} or $u_{i'j'}$ and $u_{i',j'+1}$ for a pair of integers $i\in\{1,\ldots,l\}$ and $j\in\{1,\ldots,\frac{m_i+1}{2}\}$ or a pair of integers $i'\in\{0,1,\ldots,l\}$ and $j'\in\{1,\ldots,n_{i'}-1\}$. Every pair of adjacent vertices in G' consists of $v'_{i-1,1}$ and v'_{ij} or $u'_{i'j'}$ and $u'_{i',j'+1}$ for a pair of integers $i\in\{1,\ldots,l\}$ and $j\in\{1,\ldots,\frac{m_i+1}{2}\}$ or a pair of integers $i'\in\{0,1,\ldots,l\}$ and $j'\in\{1,\ldots,n_{i'}-1\}$. Therefore, two vertices u and v in V' are adjacent in G' if and only if the two vertices $f(u)$ and $f(v)$ in V are adjacent in G.

We show that f is an isomorphism from G' to G. The number of all edges whose endvertices are $v_{i-1,j}$ and v_{ij} is $|a_{i,2j-1}|$, the number of all edges whose endvertices are $v'_{i-1,j}$ and v'_{ij} is $|a_{i,2j-1}|$, and these edges have the same sign $-\frac{a_{i,2j-1}}{|a_{i,2j-1}|}$ for $i=1,\ldots,l$ and $j=1,\ldots,\frac{m_i+1}{2}$. The number of all edges whose endvertices are $u_{i'j'}$ and $u_{i',j'+1}$ are one, the number of all edges whose endvertices are $u'_{i'j'}$ and $u'_{i',j'+1}$ are one, and these edges have the same sign $\frac{a_{i',2k'}}{|a_{i',2k'}|}$ where k' satisfies $1+\sum_{k=1}^{k'-1}|a_{i,2k}|\leq j'<1+\sum_{k=1}^{k'}|a_{i,2k}|$ for $i'=0,\ldots,l$ and $j'=1,\ldots,n_{i'}-1$ (Recall $v_{ij}=u_{1+\sum_{k=1}^{j-1}|a_{i,2k}|}$ and $v'_{ij}=u'_{1+\sum_{k=1}^{j-1}|a_{i,2k}|}$ for $i=1,\ldots,l$ and $j=1,\ldots,\frac{m_i+1}{2}$). $\square$

Lemmas 3.5 and 3.7 imply the following theorem.

Theorem 3.8 *A Tait graph G is isomorphic to the Tait graph of a Montesinos diagram if and only if G has Montesinos diagram property.*

4 Construction of normal representations Taking the advantage of the characterization of Tait graphs of Montesinos diagram shown in section 3, we give a linear time algorithm for constructing normal representations of Montesinos diagrams from its Tait graphs in this section.

Given a Tait graph $G = (V, E, s)$, Procedure montesinos_diagram_property determines whether G has Montesinos diagram property (Definition 3.6) or not. If G has Montesinos diagram property, then the procedure also constructs a label of all vertices in V with normal label property.

Recall that if G has Montesinos diagram property, then there exist at least two vertices which have at least four neighbors and every such vertex is a head vertex of a subset of V with maximal rational subset property. At first, the procedure chooses a vertex $v \in V$ satisfying $|N_G(v)| \geq 4$, and tries to find a subset of V with maximal rational subset property whose head vertex is v by using Procedure maximal_rational_subset_property as a subprocedure. If there exists such a subset V', then the subprocedure finds V' by traversing all vertices in $V' - \{v\}$ along the path $G[V' - \{v\}]$ from the endvertex of the path which has at most two neighbors to the other endvertex which is a tail vertex of V'. If the subprocedure finds V' and u is the tail vertex, then the procedure tries to find another subset of V with maximal subset property satisfying that u is a head vertex of the subset. The procedure iterates the above. During the iteration, the subprocedure may find the subset of V with top subset property. When the subset of V with top subset property is found, the vertices of the subset are labeled by Procedure top_labeler. If G having Montesinos diagram property is realized, then Procedure normal_labeler relabel all vertices in V so that the label of all vertices in V has normal label property.

The detail of the procedure and the subprocedure are described in the following:

Procedure montesinos_diagram_property
  Input: A Tait graph $G = (V, E, s)$.
  Output: A label of all vertices in V with normal label property if G has Montesinos diagram property, otherwise "Failure".
  { Preprocess }
  Construct $V_{\geq 4} = \{u \in V : |N_G(u)| \geq 4\}$;
  Construct $N_G(v)$ for each vertex $v \in V - V_{\geq 4}$;
  `if` $|V_{\geq 4}| < 2$ `then output` "Failure";
  `if` there exists a vertex $v \in V_{\geq 4}$ satisfying $|\{u \in N_G(v) : |N_G(u)| = 2\}| = 0$,
  $|\{u \in N_G(v) : |N_G(u)| = 2\}| \geq 3$, $|\{u \in N_G(v) : u \in V_{\geq 4}\}| = 0$ or
  $|\{u \in N_G(v) : u \in V_{\geq 4}\}| \geq 3$ `then output` "Failure"
  Construct $N_{=2}(v) = \{u \in N_G(v) : |N_G(u)| = 2\}$ and $N_{\geq 4}(v) = \{u \in N_G(v) : u \in V_{\geq 4}\}$ for every $v \in V$;
  Compute $\text{edge\_sum}_G(u, v)$ for each pair of vertices u and v in V;
  `if` there exists a pair of vertices in V satisfying that the multiple edges incident to the two vertices have the different signs `then output` "Failure";
  Label $v \in V_{\geq 4}$ as u'_{01};
  `if` $|N_{=2}(u'_{01})| = 1$
    `then` set w_1 as the vertex in $N_{=2}(u'_{01})$
    `else` set w_1 and w_2 as the two vertices in $N_{=2}(u'_{01})$;
  { Finding subsets of V with maximal rational subset property }
  Initialize k as 1;

```
while k ≤ |N_{=2}(u'_{01})| do begin
  { Note that |N_{=2}(u'_{01})| = 1 or 2}
  V' := maximal_rational_subset_property(G, u'_{01}, w_k);
  if V' = ∅ then break;
  { V' has maximal rational subset property }
  Set n_1 = |V'| − 1;
  Label all vertices in V' − {u'_{01}} in the way which w_k is u'_{1n_1} and u'_{1j} and u'_{1,j+1} are
  adjacent for j = 1, ..., n_1 − 1;
  if |N_G(u'_{11})| ≥ 4
    then begin Initialize i as 2, top as 0 and start as 0; end;
    else begin
      { The top path is found }
      Initialize top as 1, start as 2 and v_{tmp} as the vertex in N_G(u'_{11}) − {u'_{01}, u'_{12}};
      top_labeler(G, u'_{11}, v_{tmp}, 2);
      Initialize i as 3;
    end;
  while do begin
    if |N_{=2}(u'_{i−1,1}) − {u'_{i−1,2}}| ≠ 1 then break;
    Set v_3 as the vertex in N_{=2}(u_{i−1,1}) − {u_{i−1,2}};
    V' := maximal_rational_subset_property(G, u'_{i−1,1}, v_3);
    if V' = ∅ then break;
    { V' has maximal rational subset property }
    Set n_i = |V'| − 1;
    Label all vertices in V' − {u'_{i−1,1}} in the way which v_3 is u'_{in_i} and u'_{ij} and u'_{i,j+1}
    are adjacent for j = 1, ..., n_i − 1;
    if |N_G(u'_{i1})| ≥ 4
      then Increment i
      else begin
        if top = 1 then output "Failure";
        { The top subset property is found }
        Set top as 1, start as i + 1 and v_{tmp} as the vertex in N_G(u'_{i1}) − {u'_{i−1,1}, u'_{i2}};
        top_labeler(G, u'_{i1}, v_{tmp}, i + 1);
        i := i + 2;
      end;
    if u_{i−1,1} = u_{01}
      then if ( i − top > 3 ) ∧ ( all vertices in V are labeled ) ∧ ( every edge in G is
      contained in either G[⋃_{j=1}^{n_{start}} {u'_{start,j}}] or G[{u_{i'−1,1}} ∪ ⋃_{j=1}^{n_{i'}} {u_{i'j}}] for an integer
      i' ∈ {1, ..., start − 1, start + 1, ..., i} )
        then output normal_labeler(G, (u'_{11}, ..., u'_{1n_1}, ..., u'_{i1} ..., u'_{in_i}), start)
        else output "Failure";
    if there exists a pair of integers i' ∈ {1, ..., i} and j' ∈ {1, ..., n_{i'}} such that
    u_{i−1,1} = u_{i'j'} then break;
  end;
  Increment k;
end;
output "Failure";
```

Procedure maximal_rational_subset_property
Input: A Tait graph $G = (V, E, s)$, a vertex $v_1 \in V$ and a vertex $v_3 \in \{v \in N_G(v_1) : |N_G(v) = 2|\}$, where v_1 satisfies that $|N_G(v_1)| \geq 4$ or $|N_G(v_1)| = 3$ and there exists a path whose endvertices are v_1 and a tail vertex of a maximal rational subgraph and whose innervertices have at most two neighbors.
Output: The subset V' of V with maximal rational subset property satisfying that v_1 is a head vertex of V' and v_3 is an endvertex of $G[V']$ if there exists the subset, otherwise the empty set $\emptyset$.
Initialize v_{prv} as v_3, v_{crr} as the vertex in $N_G(v_3) - \{v_1\}$, and thr as 0;
`while do begin`
  `if` there exist multiple edges incident to v_{crr} and v_{prv}, $v_1 \notin N_G(v_{prv})$ and the two edges incident to v_{prv} have different signs `then output` $\emptyset$;
  `if` $v_1 \in N_G(v_{crr})$
    `then if` $|N_G(v_{crr})| \geq 3$
      `then if` $|N_G(v_{crr})| \geq 4$
        `then output` the subset of V consisting of v_1, v_3 and all vertices substituted for v_{crr};
        `else begin` thr `:=` 1; v_{pr3} `:=` v_{crr}; `end`
    `else output` $\emptyset$;
  `else if` $|N_G(v_{crr})| \geq 3$
    `then if` $thr = 1$
      `then output` the subset of V consisting of v_1, v_3 and all vertices substituted for v_{crr} from the vertex in $N_G(v_3) - \{v_1\}$ to v_{pr3}; { The top path is found }
      `else output` $\emptyset$;
    `else begin` v_{tmp} `:=` v_{crr}; v_{crr} `:=` $v \in N_G(v_{crr}) - \{u_{i1}, v_{prv}\}$; v_{prv} `:=` v_{tmp}; `end;`
`end;`

Procedure top_labeler
Input: A Tait graph $G = (V, E, s)$, an endvertex v_1 of the top path in G, a vertex $v_3 \in N_G(v_1)$ in the top path in G and an integer i.
{ Label all vertices in the top path in G satisfying that u'_{i1} and $u_{in_i} = v_1$ are the endvertices of the top path and u'_{ij} and $u'_{i,j+1}$ are adjacent for $j = 1, \ldots, n_i - 1$ where n_i is the number of the vertices in the top path. }
Initialize v_{prv} as v_1 and v_{crr} as v_3;
`while` $|N_G(v_{cur})| = 2$
  `do begin` v_{tmp} `:=` v_{crr}; v_{crr} `:=` $v \in N_G(v_{crr}) - \{v_{prv}\}$; v_{prv} `:=` v_{tmp}; `end;`
Label v_{crr} as u'_{i1} and v_{prv} as u'_{i2};
Set v_{prv} as u'_{i1}, v_{crr} as u'_{i2} and n_i as 2;
`while` $v_{cur} \neq v_1$ `do begin`
  v_{tmp} `:=` v_{crr};
  v_{crr} `:=` $v \in N_G(v_{crr}) - \{v_{prv}\}$;
  v_{prv} `:=` v_{tmp};
  Increment n_i;
  Label v_{crr} as u'_{in_i};
`end;`

Procedure normal_labeler

Input: A Tait graph G with Montesinos diagram property, a label $u'_{01}, u'_{11}, \dots, u'_{1n_1}, \dots, u'_{i1}, \dots, u'_{in_i}$ ($u_{i1} = u_{01}$) of all vertices in G and integer $start \in \{0, 1, \dots, i\}$ satisfying that

(i) $u'_{11}, \dots, u'_{1n_1}, \dots, u'_{i1}, \dots, u'_{in_i}$ are distinct vertices and for $i' = 1, \dots, i$ and $j' = 1, \dots, n_{i'}$, $u'_{i'j'}$ and $u'_{i',j'+1}$ are adjacent.

(ii) For every $i' \in \{1, \dots, start-1, start+1, \dots, i\}$, $\{u'_{i'-1,1}\} \cup \bigcup_{j'=1}^{n_{i'}} \{u'_{i'j'}\}$ has maximal rational subset property, $u'_{i-1,1}$ is a head vertex of $\{u_{i'-1,1}\} \cup \bigcup_{j'=1}^{n_{i'}} \{u_{i'j'}\}$ and $u_{i'-1,1}$ is a tail vertex of $\{u_{i'-1,1}\} \cup \bigcup_{j'=1}^{n_{i'}} \{u_{i'j'}\}$.

(iii) $\bigcup_{j'=1}^{n_{start}} \{u_{start,j'}\}$ has top subset property or $n_{start} = 1$.

Output: A label $u_{01}, \dots, u_{0n_0}, u_{11}, \dots, u_{1n_1}, \dots, u_{l1}, \dots, u_{ln_l}$ ($u_{l1} = u_{0n_0}$) of all vertices in the Tait graph with normal label property.

Label all vertices in $\bigcup_{j=1}^{n_{start}} \{u'_{start,j}\}$ in the way which u_{0j} is $u_{start,j}$ for $j = 1, \dots, n_{start}$;
Label all vertices in $\bigcup_{j=1}^{n_{i'+start}} \{u'_{i'+start,j}\}$ in the way which $u_{i'j'}$ is $u'_{i'+start,j'}$ for $i' = 1, \dots, i - start$ and $j' = 1, \dots, n_{i'}$;
Label all vertices in $\bigcup_{j=1}^{n_{i'-i+start}} \{u'_{i'-i+start,j}\}$ in the way which $u_{i'j'}$ is $u'_{i'-i+start,j'}$ for $i' = i - start + 1, i - start + 2, \dots, i - 1$ and $j' = 1, \dots, n_{i'}$;

By Definition 3.6 and the construction of Procedure montesinos_diagram_property, it is easy to certify whether the input graph has montesinos diagram property. In the rest of this section, we analyse the time complexity of the procedure.

Lemma 4.1 *The preprocess of* Procedure montesinos_diagram_property *finishes in* $\mathcal{O}(|E|)$ *time.*

Proof. Let $V = \{v'_1, \dots, v'_{|V|}\}$. We construct the following array of the structures:

`V[1]`		`V[`i`]`		`V[`$\lvert V\rvert$`]`
N`[1]`		N`[1]`		N`[1]`
N`[2]`		N`[2]`		N`[2]`
N`[3]`		N`[3]`		N`[3]`
$N_{\geq 4}$`[1]`		$N_{\geq 4}$`[1]`		$N_{\geq 4}$`[1]`
$N_{\geq 4}$`[2]`		$N_{\geq 4}$`[2]`		$N_{\geq 4}$`[2]`
$N_{=2}$`[1]`	$\cdots$	$N_{=2}$`[1]`	$\cdots$	$N_{=2}$`[1]`
$N_{=2}$`[2]`		$N_{=2}$`[2]`		$N_{=2}$`[2]`
$\#N$		$\#N$		$\#N$
edge_sum(N`[1]`)		edge_sum(N`[1]`)		edge_sum(N`[1]`)
edge_sum(N`[2]`)		edge_sum(N`[2]`)		edge_sum(N`[2]`)
edge_sum(N`[3]`)		edge_sum(N`[3]`)		edge_sum(N`[3]`)
edge_sum($N_{\geq 4}$`[1]`)		edge_sum($N_{\geq 4}$`[1]`)		edge_sum($N_{\geq 4}$`[1]`)
edge_sum($N_{\geq 4}$`[2]`)		edge_sum($N_{\geq 4}$`[2]`)		edge_sum($N_{\geq 4}$`[2]`)

We denote a member m of an object o of our structure type by $o \to m$. Each of the first seven members of the structure stores a vertex and each of the rest six members stores an integer. For each $i \in \{1, \dots, |V|\}$, the three members `V[`i`]` $\to N$`[1]`, `V[`i`]` $\to N$`[2]` and

$\mathtt{V[}i\mathtt{]} \to N\mathtt{[3]}$ store distinct neighbors of v'_i. For each $i \in \{1, \ldots, |V|\}$, the two members $\mathtt{V[}i\mathtt{]} \to N_{\geq 4}\mathtt{[1]}$ and $\mathtt{V[}i\mathtt{]} \to N_{\geq 4}\mathtt{[2]}$ store distinct neighbors of v'_i which have at least four neighbors and the two members $\mathtt{V[}i\mathtt{]} \to N_{=2}\mathtt{[1]}$ and $\mathtt{V[}i\mathtt{]} \to N_{=2}\mathtt{[2]}$ store distinct neighbors of v'_i which have at most two neighbors. For each $i \in \{1, \ldots, |V|\}$, the member $\mathtt{V[}i\mathtt{]} \to \#N$ stores $|N_G(v'_i)|$ if $|N_G(v'_i)| < 4$, otherwise 4. For each $i \in \{1, \ldots, |V|\}$ and $j \in \{1, 2, 3\}$, the member $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N\mathtt{[}j\mathtt{]})$ stores $\text{edge\_sum}_G(v'_i, \mathtt{V[}i\mathtt{]} \to N\mathtt{[}j\mathtt{]})$. For each $i \in \{1, \ldots, |V|\}$ and $j \in \{1, 2\}$, the member $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N_{\geq 4}\mathtt{[}j\mathtt{]})$ stores $\text{edge\_sum}_G(v'_i, \mathtt{V[}i\mathtt{]} \to N_{\geq 4}\mathtt{[}j\mathtt{]})$.

The array of the structures is constructible in $\mathcal{O}(|E|)$ time by the following way. Initialize all members of the structures of the array as 0. First, for each edge in E where its endvertices are v'_i and v'_j, we do the following:

(i) If there exists a natural number $k \in \{1, 2, 3\}$ satisfying that $\mathtt{V[}i\mathtt{]} \to N\mathtt{[}k\mathtt{]} = v'_j$, then check whether the sign of the edge is equal to the sign of $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N\mathtt{[}k\mathtt{]})$ and add the sign of the edge to $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N\mathtt{[}k\mathtt{]})$.

(ii) If there exists no natural number $k \in \{1, 2, 3\}$ satisfying that $\mathtt{V[}i\mathtt{]} \to N\mathtt{[}k\mathtt{]} = v'_j$ and $\mathtt{V[}i\mathtt{]} \to \#N < 3$, then increment $\mathtt{V[}i\mathtt{]} \to \#N$, assign v'_j to $\mathtt{V[}i\mathtt{]} \to N\mathtt{[V[}i\mathtt{]} \to \#N\mathtt{]}$ and assign the sign of the edge to $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N\mathtt{[V[}i\mathtt{]} \to \#N\mathtt{]})$.

(iii) If there exists no natural number $k \in \{1, 2, 3\}$ satisfying that $\mathtt{V[}i\mathtt{]} \to N\mathtt{[}k\mathtt{]} = v'_j$ and $\mathtt{V[}i\mathtt{]} \to \#N \geq 3$, then assign 4 to $\mathtt{V[}i\mathtt{]} \to \#N$.

(iv) Do for $\mathtt{V[}j\mathtt{]}$ in a similar way to (i), (ii) and (iii).

Next, for each edge in E where its endvertices are v'_i and v'_j, if $|N_G(v'_j)| \geq 4$, then we do the following:

(i) If there exists a natural number $k \in \{1, 2\}$ satisfying that $\mathtt{V[}i\mathtt{]} \to N_{\geq 4}\mathtt{[}k\mathtt{]} = v'_j$, then check whether the sign of the edge is equal to the sign of $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N_{\geq 4}\mathtt{[}k\mathtt{]})$ and add the sign of the edge to $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N_{\geq 4}\mathtt{[}k\mathtt{]})$.

(ii) If there exists no natural number $k \in \{1, 2\}$ satisfying that $\mathtt{V[}i\mathtt{]} \to N_{\geq 4}\mathtt{[}k\mathtt{]} = v'_j$ and there exists a natural number $k' \in \{1, 2\}$ satisfying that $\mathtt{V[}i\mathtt{]} \to N_{\geq 4}\mathtt{[}k'\mathtt{]} = 0$, then assign v'_j to $\mathtt{V[}i\mathtt{]} \to N_{\geq 4}\mathtt{[}k'\mathtt{]}$ and assign the sign of the edge to $\mathtt{V[}i\mathtt{]} \to \text{edge\_sum}(N_{\geq 4}\mathtt{[}k'\mathtt{]})$.

If $|N_G(v'_i)| \geq 4$, then do for $\mathtt{V[}j\mathtt{]}$ as same as the case where $|N_G(v'_j)| \geq 4$.

Finally, for each edge in E where its endvertices are v'_i and v'_j, if $|N_G(v'_j)| = 2$, then we do the following:

(i) If there exists no natural number $k \in \{1, 2\}$ satisfying that $\mathtt{V[}i\mathtt{]} \to N_{=2}\mathtt{[}k\mathtt{]} = v'_j$ and there exists a natural number $k' \in \{1, 2\}$ satisfying that $\mathtt{V[}i] \to N_{=2}[k'] = 0$, then assign v'_j to $\mathtt{V[}i] \to N_{=2}[k']$.

If $|N_G(v'_i)| = 2$, then do for $\mathtt{V[}j\mathtt{]}$ as same as the case where $|N_G(v'_j)| = 2$.

Note that whether there exists a vertex $v \in V_{\geq 4}$ satisfying $|\{u \in N_G(v) : |N_G(u)| = 2\}| = 0$, $|\{u \in N_G(v) : |N_G(u)| = 2\}| \geq 3$, $|\{u \in N_G(v) : u \in V_{\geq 4}\}| = 0$ or $|\{u \in N_G(v) : u \in V_{\geq 4}\}| \geq 3$ is determined in the construction of the array. Whether there exists a pair of vertices in V satisfying that the multiple edges incident to the two vertices have the different signs is also determined in the construction of the array. Therefore, the preprocess of Procedure montesinos_diagram_property finishes in $\mathcal{O}(|E|)$ time. □

Lemma 4.2 *If* Procedure maximal_rational_subset_property *outputs a nonempty subset V' when $v_1 \in N_G(v_{crr})$, then the subprocedure finishes in $\mathcal{O}(|V'|)$ time. If* Procedure maximal_rational_subset_property *outputs a nonempty subset V' when $v_1 \notin N_G(v_{crr})$, then the subprocedure finds a subset V'' of V with top subgraph property and finishes in $\mathcal{O}(|V'|+|V''|)$ time. If* Procedure maximal_rational_subset_property *outputs the empty set $\emptyset$, then the subprocedure finishes in $\mathcal{O}(|V|)$ time.*

Proof. By looking up $\text{edge\_sum}_G(v_{crr}, v_{prv})$, whether there exist multiple edges incident to v_{crr} and v_{prv} is determinable in constant time. For each vertex $v \in V$, whether $v_1 \in N_G(v)$ or not is determinable in constant time by looking up $N_{\geq 4}(v_1)$ if $v_1 \in V_{\geq 4}$ and $v \in V_{\geq 4}$, otherwise by looking up $N_G(u)$ where $u \in \{v_1, v\} - V_{\geq 4}$. If $v_1 \notin N_G(v_{prv})$ and there exist no multiple edges incident to v_{prv} and v for each vertex $v \in N_G(v_{prv})$, then whether the two edges incident to v_{prv} have different signs is determinable in constant time by looking up $\text{edge\_sum}_G(v_{prv}, v)$ for each vertex $v \in N_G(v_{prv})$. It is clear that whether $|N_G(v_{crr})| \geq 3$ and whether $|N_G(v_{crr})| \geq 4$ are determinable in constant time. It is also clear that the subset of V consisting of v_1, v_3 and all vertices substituted for v_{crr} is outputed in $\mathcal{O}(n)$ time where n is the number of the vertices substituted for v_{crr}. Because every vertex is substituted for v_{crr} at most once, the `while` loop iterates n times where n is the number of the vertices substituted for v_{crr}. □

Lemma 4.3 Procedure montesinos_diagram_property *runs in $\mathcal{O}(|E|)$ time.*

Proof. The preprocess finishes in $\mathcal{O}(|E|)$ time by Lemma 4.1. Because $|N_{=2}(u'_{01})|$ is at most two, the outer `while` loop of the procedure iterates at most twice. Therefore, it is sufficient to show that each iteration of the outer `while` loop of the procedure finishes in $\mathcal{O}(|E|)$ time.

If maximal_rational_subset_property outputs a nonempty subset V' when $v_1 \in N_G(v_{crr})$, then the subprocedure finishes in $\mathcal{O}(|V'|)$ time by Lemma 4.2. If maximal_rational_subset_property outputs a nonempty subset V' when $v_1 \notin N_G(v_{crr})$, then the subprocedure finds a subset V'' of V with top subgraph property and finishes in $\mathcal{O}(|V'| + |V''|)$ time by Lemma 4.2. If maximal_rational_subset_property outputs the empty set $\emptyset$, then the subprocedure finishes in $\mathcal{O}(|V|)$ time by Lemma 4.2. It is clear that for each V' outputed by maximal_rational_subset_property, all vertices in V' are labeled in $\mathcal{O}(|V'|)$. It is also clear that Procedure top_labeler finishes in $\mathcal{O}(|V''|)$ time where V'' found by maximal_rational_subset_property and has top subgraph property. Whether $|N_{=2}(u'_{i-1,1}) - \{u'_{i-1,2}\}| \neq 1$ is determinable in constant time for each i because $|N_{=2}(u'_{i-1,1})|$ is at most two. Whether every edge in G is contained in either $G[\bigcup_{j=1}^{n_{start}} \{u'_{start,j}\}]$ or $G[\{u_{i'-1,1}\} \cup \bigcup_{j=1}^{n_{i'}} \{u_{i'j}\}]$ for an integer $i' \in \{1, \ldots, start-1, start+1, \ldots, i\}$ is determinable in constant time by looking up the labels of both the endvertices of the edge. It is clear that Procedure normal_labeler finishes in $\mathcal{O}(|V|)$ time.

When a vertex except $u'_{start,n_{start}}$ is labeled twice or maximal_rational_subset_property outputs the empty set $\emptyset$, the inner `while` loop of the procedure finishes. Therefore, each iteration of the outer `while` loop of the procedure finishes in $\mathcal{O}(|E|)$ time. □

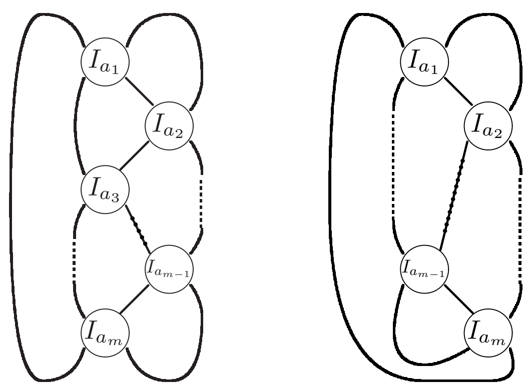
Theorem 4.4 Procedure montesinos_diagram_property *determines whether G has Montesinos diagram property or not in $\mathcal{O}(|E|)$ time. Furthermore, the procedure constructs a label of all vertices in G with normal label property if G has Montesinos diagram property.*

Theorem 4.4 and Lemma 3.7 imply the following corollary.

Corollary 4.5 *Normal representations of Montesinos diagrams are constructible in $\mathcal{O}(n)$ time from its Tait graphs, where n is the number of the edges in the input Tait graph.*

5 Computation of Kauffman bracket polynomials In this section, we show a recurrence formula of Kauffman bracket polynomials of Montesinos diagrams represented by normal representations. By using the formula, Kauffman bracket polynomials of Montesinos diagrams are computable with $\mathcal{O}(n)$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$ from normal representations of Montesinos diagrams, where n is the number of the crossings of the Montesinos diagram represented by the input normal representation.

We denote the link diagram $\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}||0)$ by $\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})$, the link diagram consisting of integer tangles $I_{a_1},\dots,I_{a_m}$ as shown in Figure 8 by $\widetilde{R}(a_1,\dots,a_m)$ and the link diagram consisting of integer tangles $I_{a_{11}},\dots,I_{a_{1m_1}},\dots,I_{a_{l1}},\dots,I_{a_{lm_l}}$ (l rational tangles) as shown in Figure 9 by $\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})$.



m is an odd number. m is an even number.

Figure 8: $\widetilde{R}(a_1,\dots,a_m)$.

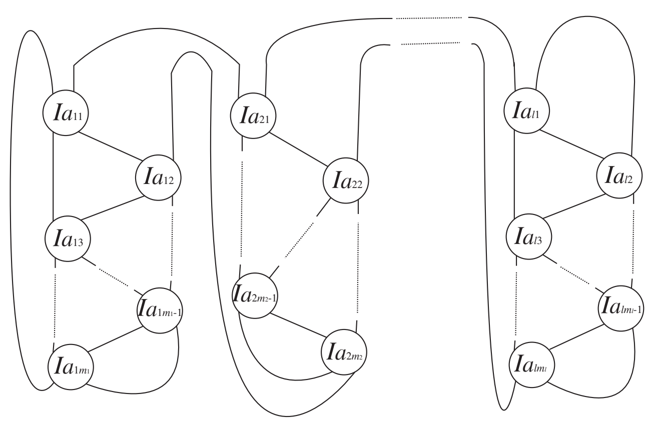
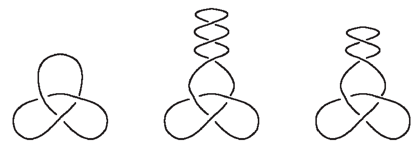


Figure 9: $\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})$.

For a link diagram $\widetilde{L}$, a link diagram $\widetilde{L}\#(k)$ is shorthand for a link diagram twisted it k times as shown in Figure 10. For convenience, $\widetilde{L}\#(0)$ denotes $\widetilde{L}$ itself.



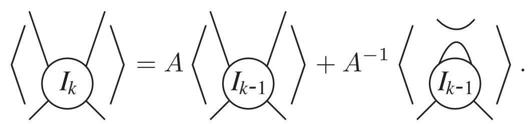
$\widetilde{L}\#(0) = \widetilde{L}$ $\quad$ $\widetilde{L}\#(3)$ $\quad$ $\widetilde{L}\#(-2)$

Figure 10: $\widetilde{L}\#(k)$.

For any integer n, we set

$$Q_n = \frac{1-(-A^4)^n}{1-(-A^4)} = \begin{cases} 1 + (-A^4) + \cdots + (-A^4)^{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -(-A^4)^{-1} - (-A^4)^{-2} - \cdots - (-A^4)^{n} & \text{if } n < 0. \end{cases}$$

Proposition 5.1 [10] *For any integer k,*



Proposition 5.2 [10] *For any link diagram $\widetilde{L}$ and any integer k,*

$$\langle \widetilde{L}\#(k) \rangle = (-A^{-3})^k \langle \widetilde{L} \rangle.$$

Proposition 5.3 [10] *Let $\widetilde{L}$ be a link diagram and $\{b_n\}_{n\in\mathbb{Z}}$ a sequence of polynomials in $\mathbb{Z}[A^{\pm 1}]$. Suppose that for any integer n,*

$$b_n = Ab_{n-1} + A^{-1}\langle \widetilde{L}\#(n-1) \rangle.$$

Then,

$$b_n = A^n b_0 - (-A)^{-3n+2} Q_n \langle \widetilde{L} \rangle.$$

Lemma 5.4 *For any pair of a sequence of integer sequence $(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l})$ and an integer a,*

$$\begin{aligned} & \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l} || a) \rangle \\ = \; & A^a \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l}) \rangle \\ & -(-A)^{-3a+2} Q_a \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l}) \rangle. \end{aligned}$$

Proof. We have

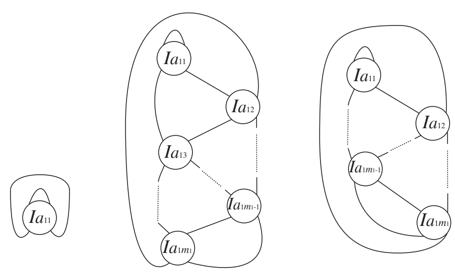
$$\begin{aligned} & \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l} || a) \rangle \\ = \; & A \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l} || a-1) \rangle \\ & + A^{-1} \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l}) \#(a-1) \rangle \end{aligned}$$

by applying Proposition 5.1 to I_a of $\widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l} || a)$. Then, the lemma is implied by Proposition 5.3. □

Lemma 5.5 *For any sequence of integer sequences* $(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})$, *the following recurrence formula holds.*

$$
\begin{aligned}
&\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})\rangle\\
=&\begin{cases}
(-A^{-3})^{a_{11}} & \textit{if } l=1 \textit{ and } m_1=1,\\
(-A^{-3})^{a_{11}}\langle\widetilde{R}(a_{12},\dots,a_{1m_1})\rangle & \textit{if } l=1 \textit{ and } m_1\geq 2,\\
A^{a_{l1}}\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\rangle & \\
\quad -(-A)^{-3a_{l1}+2}Q_{a_{l1}} & \\
\qquad \times\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\rangle & \textit{if } l\geq 2 \textit{ and } m_l=1,\\
(-1)^{a_{l1}}A^{-3a_{l1}+a_{l2}} & \\
\quad\times\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\rangle & \\
\qquad -(-A)^{-3a_{l2}+2}Q_{a_{l2}} & \\
\qquad\quad \times\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1})\rangle & \textit{if } l\geq 2 \textit{ and } m_l=2,\\
(-1)^{a_{lm_l-1}}A^{-3a_{lm_l-1}+a_{lm_l}} & \\
\quad\times\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l-2})\rangle & \\
\qquad -(-A)^{-3a_{lm_l}+2}Q_{a_{lm_l}} & \\
\qquad\quad \times\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l-1})\rangle & \textit{if } l\geq 2 \textit{ and } m_l\geq 3.
\end{cases}
\end{aligned}
$$

Proof. The case where $l=1$ and $m_1=1$ is implied by $\widetilde{M}(a_{11})=\bigcirc\#(a_{11})$, Proposition 5.2 and Definition 2.1(i). The case where $l=1$ and $m_1\geq 2$ is implied by $\widetilde{M}(a_{11},\dots,a_{m_1})=\widetilde{R}(a_{12},\dots,a_{1m_1})\#(a_{11})$ and Proposition 5.2 (see Figure 11).



m_1 is an odd number $\quad$ m_1 is an even number

Figure 11: The case where $l=1$.

We consider the case where $l\geq 2$ and $m_l=1$. We have

$$
\begin{aligned}
(1)\qquad & \langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1})\rangle\\
= \;& A\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1}-1)\rangle\\
& +A^{-1}\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\#(a_{l1}-1)\rangle
\end{aligned}
$$

by applying Proposition 5.1 to $I_{a_{l1}}$ of $\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1})$. We also have

$$
\begin{aligned}
(2)\qquad & \langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|0)\rangle\\
= \;& \langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\rangle
\end{aligned}
$$

by $\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|0)=\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})$ (see Figure 12). Hence, the equations (1) and (2) imply the case where $l\geq 2$ and $m_l=1$ by Proposition 5.3.

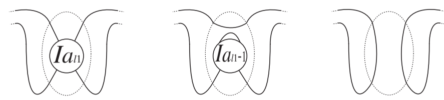


Figure 12: The case where $l \geq 2$ and $m_l = 1$.

We consider the case where $l \geq 2$ and $m_l = 2$. We have

$$\begin{aligned} (3) \qquad & \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}, a_{l2}) \rangle \\ = \; & A \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}, a_{l2} - 1) \rangle \\ & + A^{-1} \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}) \# (a_{l2} - 1) \rangle \end{aligned}$$

by applying Proposition 5.1 to $I_{a_{l2}}$ of $\widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}, a_{l2})$. We also have

$$\begin{aligned} (4) \qquad & \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}, 0) \rangle \\ = \; & (-A^{-3})^{a_{l1}} \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}}) \rangle \end{aligned}$$

by

$$\begin{aligned} & \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}, 0) \\ = \; & \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}}) \# (a_{l1}) \end{aligned}$$

and Proposition 5.2 (see Figure 13). Hence, the equations (3) and (4) imply the case where $l \geq 2$ and $m_l = 2$ by Proposition 5.3.

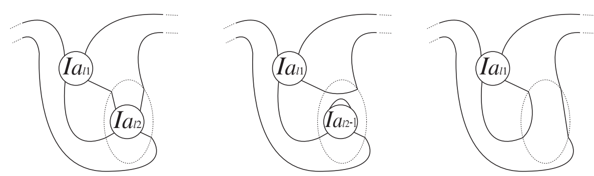


Figure 13: The case where $l \geq 2$ and $m_l = 2$.

We consider the case where $l \geq 2$ and $m_l \geq 3$. We have

$$\begin{aligned} (5) \qquad & \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l}) \rangle \\ = \; & A \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l} - 1) \rangle \\ & + A^{-1} \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l - 1}) \# (a_{lm_l} - 1) \rangle \end{aligned}$$

by applying Proposition 5.1 to $I_{a_{lm_l}}$ of $\widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l})$. We also have

$$\begin{aligned} (6) \qquad & \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l - 1}, 0) \rangle \\ = \; & (-A^{-3})^{a_{lm_l - 1}} \langle \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l - 2}) \rangle \end{aligned}$$

by $\widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l - 1}, 0) = \widetilde{M}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l - 2}) \# (a_{lm_l - 1})$ and Proposition 5.2 (see Figures 14 and 15). Hence, the equations (5) and (6) imply the case where $l \geq 2$ and $m_l \geq 3$ by Proposition 5.3.

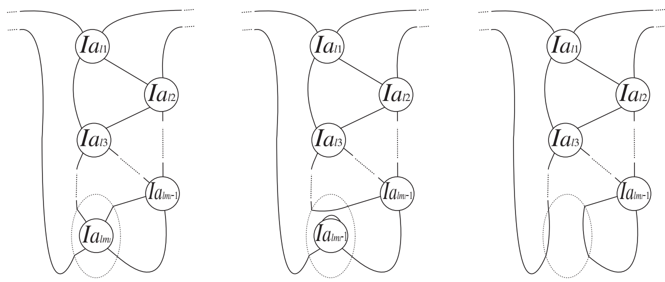


Figure 14: The case where $l \geq 2$, $m_l \geq 3$ and m_l is an odd number.

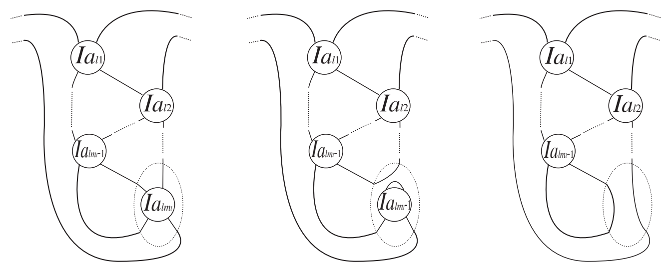


Figure 15: The case where $l \geq 2$, $m_l \geq 3$ and m_l is an even number.

$\square$

Lemma 5.6 *For any sequence of integer sequences* $(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l})$*, the following recurrence formula holds.*

$$
\begin{aligned}
&\langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l}) \rangle \\
= &\begin{cases}
\langle \widetilde{R}(a_{11}, \ldots, a_{1m_1}) \rangle & \textit{if } l = 1, \\
\left(A^{a_{l1}}(-A^{-2} - A^{2}) - (-A)^{-3a_{l1}+2} Q_{a_{l1}}\right) & \\
\quad \times \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}}) \rangle & \textit{if } l \geq 2 \textit{ and } m_l = 1, \\
(-1)^{a_{l1}} A^{-3a_{l1}+a_{l2}} & \\
\quad \times \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}}) \rangle & \\
\qquad -(-A)^{-3a_{l2}+2} Q_{a_{l2}} & \\
\qquad\quad \times \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}) \rangle & \textit{if } l \geq 2 \textit{ and } m_l = 2, \\
(-1)^{a_{lm_l-1}} A^{-3a_{lm_l-1}+a_{lm_l}} & \\
\quad \times \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l-2}) \rangle & \\
\qquad -(-A)^{-3a_{lm_l}+2} Q_{a_{lm_l}} & \\
\qquad\quad \times \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l-1}) \rangle & \textit{if } l \geq 2 \textit{ and } m_l \geq 3.
\end{cases}
\end{aligned}
$$

Proof. The case where $l = 1$ is implied by $\widetilde{N}(a_{11}, \ldots, a_{1m_1}) = \widetilde{R}(a_{11}, \ldots, a_{1m_1})$.

We consider the case where $l \geq 2$ and $m_l = 1$. We have

$$
\begin{aligned}
(7) \qquad & \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1}) \rangle \\
= \; & A \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}} | a_{l1} - 1) \rangle \\
& + A^{-1} \langle \widetilde{N}(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l-11}, \ldots, a_{l-1m_{l-1}}) \# (a_{l1} - 1) \rangle
\end{aligned}
$$

by applying Proposition 5.1 to $I_{a_{l1}}$ of $\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1})$. We also have

$$\begin{aligned}(8) \qquad & \langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|0)\rangle \\ = \;& (-A^{-2}-A^{2})\langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\rangle\end{aligned}$$

by $\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|0)=\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\sqcup\bigcirc$ and Definition 2.1(ii) (see Figure 12). Hence, the equations (7) and (8) imply the case where $l\geq 2$ and $m_l=1$ by Proposition 5.3.

We consider the case where $l\geq 2$ and $m_l=2$. We have

$$\begin{aligned}(9) \qquad & \langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1},a_{l2})\rangle \\ = \;& A\langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1},a_{l2}-1)\rangle \\ & +A^{-1}\langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1})\#(a_{l2}-1)\rangle\end{aligned}$$

by applying Proposition 5.1 to $I_{a_{l2}}$ of $\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1},a_{l2})$. We also have

$$\begin{aligned}(10) \qquad & \langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1},0)\rangle \\ = \;& (-A^{-3})^{a_{l1}}\langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\rangle\end{aligned}$$

by

$$\begin{aligned}& \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}}|a_{l1},0) \\ = \;& \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l-11},\dots,a_{l-1m_{l-1}})\#(a_{l1})\end{aligned}$$

and Proposition 5.2 (see Figure 13). Hence, the equations (9) and (10) imply the case where $l\geq 2$ and $m_l=2$ by Proposition 5.3.

We consider the case where $l\geq 2$ and $m_l\geq 3$. We have

$$\begin{aligned}(11) \qquad & \langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})\rangle \\ = \;& A\langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}-1)\rangle \\ & +A^{-1}\langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l-1})\#(a_{lm_l}-1)\rangle\end{aligned}$$

by applying Proposition 5.1 to $I_{a_{lm_l}}$ of $\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})$. We also have

$$\begin{aligned}(12) \qquad & \langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l-1},0)\rangle \\ = \;& (-A^{-3})^{a_{lm_l-1}}\langle \widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l-2})\rangle\end{aligned}$$

by $\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l-1},0)=\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l-2})\#(a_{lm_l-1})$ and Proposition 5.2 (see Figures 14 and 15). Hence, the equations (11) and (12) imply the case where $l\geq 2$ and $m_l\geq 3$ by Proposition 5.3. □

Lemma 5.7 [10] *The Kauffman bracket polynomial $\langle\widetilde{R}(a_1,\dots,a_m)\rangle$ is computable with $\mathcal{O}(n)$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, where n is the number of the crossings of $\widetilde{R}(a_1,\dots,a_m)$.*

Given normal representations of Montesinos diagrams, Procedure bracket_montesinos computes Kauffman bracket polynomial of Montesinos diagrams, by using the recurrence formulas in Lemmas 5.4, 5.5 and 5.6. While the procedure is running, every Kauffman bracket polynomial is computed at most once.

Procedure bracket_montesinos
 Input: A sequence of integer sequences $(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})$ and an integer a.
 Output: The Kauffman bracket polynomial $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}||a)\rangle$.
 Compute $\langle\widetilde{M}(a_{11},\dots,a_{1m_1})\rangle$ and $\langle\widetilde{N}(a_{11},\dots,a_{1m_1})\rangle$;
 `for` i `:=` 2 `to` l `do begin`
  `for` j `:=` 1 `to` m_i `do begin`
   Compute $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij})\rangle$ and $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij})\rangle$;
  `end;`
 `end;`
 Compute $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}||a)\rangle$;

Theorem 5.8 Procedure bracket_montesinos *computes the Kauffman bracket polynomial* $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}||a)\rangle$ *with* $\mathcal{O}(n)$ *additions and multiplications in polynomials of degree* $\mathcal{O}(n)$*, where* n *is the number of the crossings of* $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}||a)\rangle$.

Proof. Q_k is computable in $\mathcal{O}(|k|)$ time for any integer k. $\langle\widetilde{M}(a_{11},\dots,a_{1m_1})\rangle$ and $\langle\widetilde{N}(a_{11},\dots,a_{1m_1})\rangle$ are computable with $\mathcal{O}(n)$ operations in polynomials of degree $\mathcal{O}(n)$ by Lemmas 5.5, 5.6 and 5.7. We consider the case where $i=2,\dots,l$.

$\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}}|a_{i1})\rangle$ and $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}}|a_{i1})\rangle$ are computable with $\mathcal{O}(1)$ operations in polynomials of degree $\mathcal{O}(n)$ from $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}})\rangle$, $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}})\rangle$, and $Q_{a_{i1}}$ by Lemmas 5.5 and 5.6.

$\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}}|a_{i1},a_{i2})\rangle$ and $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}}|a_{i1},a_{i2})\rangle$ are computable with $\mathcal{O}(1)$ operations in polynomials of degree $\mathcal{O}(n)$ from $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}})\rangle$, $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}}|a_{i1})\rangle$, $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}})\rangle$, $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i-11},\dots,a_{i-1m_{i-1}}|a_{i1})\rangle$ and $Q_{a_{i2}}$ by Lemmas 5.5 and 5.6.

$\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij})\rangle$ and $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij})\rangle$ are computable with $\mathcal{O}(1)$ operations in polynomials of degree $\mathcal{O}(n)$ from $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij-2})\rangle$, $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij-1})\rangle$, $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij-2})\rangle$, $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{i1},\dots,a_{ij-1})\rangle$ and $Q_{a_{ij}}$ for $j=3,\dots,m_i$ by Lemmas 5.5 and 5.6.

$\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}||a)\rangle$ is computable with $\mathcal{O}(1)$ operations in polynomials of degree $\mathcal{O}(n)$ from $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})\rangle$ and $\langle\widetilde{N}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l})\rangle$ and Q_a by Lemmas 5.4. Therefore, the procedure computes $\langle\widetilde{M}(a_{11},\dots,a_{1m_1}|\cdots|a_{l1},\dots,a_{lm_l}||a)\rangle$ with $\mathcal{O}(n)$ operations of polynomials in degree $\mathcal{O}(n)$. □

Corollary 5.9 *Jones polynomials of Montesinos links are computable with* $\mathcal{O}(n)$ *additions and multiplications in polynomials of degree* $\mathcal{O}(n)$ *from Tait graphs of Montesinos diagrams, namely in* $\mathcal{O}(n^2\log n)$ *time, where* n *is the number of the edges in the input Tait graph.*

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