CONTRACTION SEMIGROUPS ON HYPERGROUP ALGEBRAS

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ABSTRACT. In this paper we develop potential theory on hypergroups and study contraction semigroups on hypergroups, we state and prove a necessary and sufficient condition that the domain of zero-resolvent or the domain of potential operator of $\{P_t\}_{t>0}$ is dense in $L^2(K)$.

1. INTRODUCTION AND NOTATION

Let K be a commutative locally compact hypergroup. We denote by $M(K)$ the space of all regular complex-valued Borel measures on K, by $M^+(K)$ the subset of positive measures in $M(K)$, and by δ_x the Dirac measure at the point x. The closure of any $A \subseteq K$ is denoted by A^c.

First we recall the definition and basic properties of a hypergroup. The main references are [2] and [6]. You can see also [5] and [8] for more properties of hypergroups.

Definition 1.1. Let K be a locally compact Hausdorff space. The space K is a hypergroup if there exists a binary mapping $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $M^+(K)$ satisfying the following conditions,
(1) The mapping $(\delta_x, \delta_y) \mapsto \delta_x * \delta_y$ extends to a bilinear associative operator $*$ from $M(K) \times M(K)$ into $M(K)$ such that

$$\int_K f d(\mu * \nu) = \int_K \int_K \int_K f d(\delta_x * \delta_y) d\mu(x) d\nu(y)$$

for all continuous functions f on K vanishing at infinity.
(2) For each $x, y \in K$ the measure $\delta_x * \delta_y$ is a probability measure with compact support.
(3) The mapping $(\mu, \nu) \mapsto \mu * \nu$ is continuous from $M^+(K) \times M^+(K)$ into $M^+(K)$; the topology on $M^+(K)$ being the cone topology.
(4) There exists $e \in K$ such that $\delta_e * \delta_x = \delta_x = \delta_x * \delta_e$ for all $x \in K$.
(5) There exists a homeomorphism involution $x \mapsto x^-$ from K onto K such that, for all $x, y \in K$, we have $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ where for $\mu \in M(K)$, μ^- is defined by

$$\int_K f(t) d\mu^-(t) = \int_K f(t^-) d\mu(t),$$

and also,

$$e \in \operatorname{supp}(\delta_x * \delta_y) \text{ if and only if } y = x^-$$

where $\operatorname{supp}(\delta_x * \delta_y)$ is the support of the measure $\delta_x * \delta_y$.
(6) The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is continuous from $K \times K$ into the space $\mathbf{C}(K)$ of

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compact subsets of K, where $\mathbf{C}(K)$ is given the topology whose subbasis is given by all

$$\mathbf{C}_{U,V} = \{A \in \mathbf{C}(K) : A \cap U \neq \emptyset \text{ and } A \subseteq V\}$$

where U , V are open subsets of K.

Note that $\delta_x * \delta_y$ is not necessarily a Dirac measure. A hypergroup K is commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all x, y in K. Let 's recall some properties of locally compact commutative hypergroups. Such a hypergroup K carries a Haar measure m such that $\delta_x * m = m$ for all $x \in K$ [11]. In any commutative hypergroup K we have $m = m^-$ ([6], 5.3). Let f, g be Borel functions on K and $\mu \in M(K)$. For any $x, y \in K$ we denote $\bar{f}(x) = \overline{f(x)}$, $f^-(x) = f(x^-)$, $\tilde{f}(x) = \overline{f(x^-)}$ and $pos(f) = \{x \in K : f(x) > 0\}$, and define $f_x(y) = f(x * y) := \int_K f d(\delta_x * \delta_y)$. Also we define

$$(\mu * f)(x) := \int_K f(y^- * x)\, d\mu(y) \;\text{ and }\; (f * g)(x) := \int_K f(x * y) g(y^-)\, dm(y),$$

where $x \in K$. For any subsets A, B of K, we denote $A*B = \bigcup\{\text{supp}(\delta_x*\delta_y) : x \in A,\, y \in B\}$ and $A^- = \{x^- : x \in K\}$. A non-empty closed subset E of K is a subhypergroup of K if $E^- = E$ and $E * E \subseteq E$.

A complex-valued continuous function ξ on K is said to be multiplicative if $\xi(x*y) = \xi(x)\xi(y)$ holds for all $x, y \in K$. The space of all multiplicative functions on K is denoted by $X_b(K)$. A nonzero multiplicative function ξ on K is called a character if $\xi(x^-) = \overline{\xi(x)}$ for all x in K. The dual $\hat{K}$ of K is the locally compact Hausdorff space of all characters with the topology of uniform convergence on compacta. In general $\hat{K}$ is not necessarily a hypergroup. A hypergroup K is called *strong* if its dual $\hat{K}$ is also a hypergroup with complex conjugation as involution, pointwise product as convolution, that is

$$\eta(x)\chi(x) = \int_{\hat{K}} \xi(x)\, d\delta_\eta * \delta_\chi(\xi)$$

for all $\eta, \chi \in \hat{K}$ and $x \in K$, and the constant function $\mathbf{1}$ as the identity element.

We denote $L^1(K) = L^1(K, m)$ and $L^1(\hat{K}) = L^1(\hat{K}, \pi)$ where π is the Plancherel measure on $\hat{K}$ associated with m.

For any $f \in L^1(K)$ and $\mu \in M(K)$, the Fourier-Stieltjes transform $\hat{\mu}$ of μ and the Fourier transform $\hat{f}$ of f are defined by

$$\hat{\mu}(\xi) = \int_K \overline{\xi(t)} d\mu(t) \;\text{ and }\; \hat{f}(\xi) = \int_K \overline{\xi(t)} f(t) dm(t),$$

where $\xi \in \hat{K}$.

For any $k \in L^1(\hat{K})$ and $\sigma \in M(\hat{K})$, the inverse Fourier transform $\check{k}$ and $\check{\sigma}$ of k and σ, respectively, are defined by

$$\check{k}(x) = \int_{\hat{K}} \xi(x) k(\xi)\, d\pi(\xi) \;\text{ and }\; \check{\sigma}(x) = \int_{\hat{K}} \xi(x)\, d\sigma(\xi),$$

where $x \in K$.

2. The Periodicity on Hypergroups

Definition 2.1. A measure μ on K is called periodic with period $p \in K$ if $\delta_p * \mu = \mu$. The set of all periods of μ is denoted by $Per(\mu)$. We shall show that $Per(\mu)$ is a closed subhypergroup of K.

Let f be a continuous function on K. An element $p \in K$ is called a period of f if $\delta_p * f = f$. Similarly the set of all periods of ϕ is denoted by $Per(\phi)$

We shall see that for every continuous bounded positive definite function ϕ on K and any $\mu \in M(K)$, $Per(\phi)$ and $Per(\mu)$ are closed subhypergroups of K.

Definition 2.2. Let $A \subseteq K$, $B \subseteq \hat{K}$. We denote $A^\perp = \{\xi \in \hat{K} : \xi(x) = 1 \text{ for all } x \in A\}$, $B^\perp = \{x \in K : \xi(x) = 1 \text{ for all } \xi \in B\}$.

$B^\perp$ is a subhypergroup of K and if K is a strong hypergroup, then $A^\perp$ is a subhypergroup of $\hat{K}$ [9].

Definition 2.3. A measure $\mu \in M(K)$ is called shift-bounded if $\mu * f \in C_b(K)$ for all $f \in C_c(K)$, and is called weakly shift-bounded if $\mu * f * \tilde{f} \in C_b(K)$ for all $f \in C_c(K)$.

Remark. Clearly if μ is shift-bounded, then μ is weakly shift-bounded but the converse is not true in general ([2], 1.2.32). If K is a locally compact group, then μ is a positive definite measure (i.e. $\int_K f * \tilde{f} d\mu \geq 0$ for all $f \in C_c(K)$) if and only if $\mu * f * \tilde{f}$ is a continuous positive definite function ([1], proposition 4.4). This result is also true for a hypergroup K. As a consequence, in the group case every positive definite measure is weakly shift-bounded ([1], 4.4) and this is not the case for hypergroups. For instance $(\mathbb{Z}_+, *(Q_n))$ is a polynomial hypergroup with semicharacters $\chi_x(n) = Q_n(x)$ ([2], Theorem 3.2.12), and $\chi_x m$ is positive definite for all $x \in \mathbb{R}$, but for $x > x_0$, $\chi_x m$ is not shift-bounded.

For every weakly shift-bounded positive definite measure μ on $M(K)$ there is a unique positive measure σ in $M(\hat{K})$ such that

$$\int_K f * \tilde{f} d\mu = \int_{\hat{K}} |(f^-)\hat{}|^2 \, d\sigma \quad (f \in C_c(K)).$$

The measure σ has the following properties

(i) $\int |\hat{f}|^2 d\sigma < \infty$

(ii) $(\mu * f * \tilde{f})(x) = \int_{\hat{K}} \xi(x) |\hat{f}(\xi)|^2 d\sigma(\xi)$

(iii) $(\mu * f * \tilde{g})(x) = \int_{\hat{K}} \xi(x) \hat{f}(\xi) \hat{g}(\xi) d\sigma(\xi)$,

where $f, g \in C_c(K)$ and $x \in K$. The measure σ is called the associated measure of μ [2].

Lemma 2.4. Let U be a compact neighborhood of identity e in K and $\{V_i\}$ be a neighborhood base at e included in U. There is a bounded approximate identity $\{k_i\}$ in $L^1(K)$ satisfying $k_i \in C_c^+(K)$, $\|k_i\|_1 = 1$, $\text{supp}(k_i) \subseteq V_i$, $\widehat{k_i} \in L^1_+(\hat{K})$ and $\lim_i \widehat{k_i} = \mathbf{1}$ uniformly on compact subsets of $\hat{K}$.

Proof. Refer to Theorem 2.2.28 in [2] page 88. $\square$

Theorem 2.5. Let K be a commutative strong hypergroup and μ be a weakly shift-bounded and positive definite measure on K with associated measure σ. For every γ in $\hat{K}$ the measure $\gamma\mu$ is also a positive definite measure with associated measure $\delta_\gamma * \sigma$.

Note that the proof of the theorem in the group case is based on $(fg)_x = f_x g_x$ which does not hold for hypergroups. So our proof is completely different (c.f. [1], Proposition 4.10).

Proof. Let g be in $C_c^+(K)$ and put $h^- := g * \tilde{g}$. For each $f \in C_c(K)$ we have

$$
\begin{aligned}
&\textstyle\int_K [\gamma(f * \tilde{f}) * h(x)] d\mu(x) \\
&\quad = \int_K \int_K \gamma(y)(f * \tilde{f})(y) h(x * y^-)\, dm(y) d\mu(x) \\
&\quad = \int_K \gamma(y)(f * \tilde{f})(y) \int_K h(x * y^-)\, d\mu(x) dm(y) \\
&\quad = \int_K \gamma(y)(f * \tilde{f})(y)(\mu * h^-)(y)\, dm(y) \\
&\quad = \int_K \gamma(y)(f * \tilde{f})(y)(\mu * g * \tilde{g})(y)\, dm(y) \\
&\quad = \int_K \int_{\hat{K}} \gamma(y)(f * \tilde{f})(y)\xi(y)|\hat{g}(\xi)|^2\, d\sigma(\xi) dm(y) \\
&\quad = \int_{\hat{K}} \int_{\hat{K}} |\hat{g}(\xi)|^2 \int_K (f * \tilde{f})(y)\eta(y)\, dm(y) d\delta_\gamma * \delta_\xi(\eta) d\sigma(\xi) \\
&\quad = \int_{\hat{K}} \int_{\hat{K}} |\hat{g}(\xi)|^2 \int_K f(y)(\eta * (\tilde{f})^-)(y)\, dm(y) d\delta_\gamma * \delta_\xi(\eta) d\sigma(\xi) \quad ([6], 5.5O) \\
&\quad = \int_{\hat{K}} \int_{\hat{K}} |\hat{g}(\xi)|^2 \int_K f(y)(\eta * \bar{f})(y)\, dm(y) d\delta_\gamma * \delta_\xi(\eta) d\sigma(\xi) \\
&\quad = \int_{\hat{K}} \int_{\hat{K}} |\hat{g}(\xi)|^2 \int_K \int_K (\eta f)(y)\overline{(\eta f)(x^-)}\, dm(x) dm(y) d\delta_\gamma * \delta_\xi(\eta) d\sigma(\xi) \\
&\quad = \int_{\hat{K}} \int_{\hat{K}} |\hat{g}(\xi)|^2 (\eta f)\hat{}(\mathbf{1})\overline{(\eta f)\hat{}(\mathbf{1})}\, d\delta_\gamma * \delta_\xi(\eta) d\sigma(\xi) \\
&\quad = \int_{\hat{K}} \int_{\hat{K}} |\hat{g}(\xi)|^2 |(\eta f)\hat{}(\mathbf{1})|^2\, d\delta_\gamma * \delta_\xi(\eta) d\sigma(\xi).
\end{aligned}
$$

Now if we put $j(\eta) = |(\eta f)\hat{}(\mathbf{1})|^2$ then since $j(\gamma * \xi) = \int_{\hat{K}} j(\eta)\, d(\delta_\gamma * \delta_\xi)(\eta)$ the last integral is equal to

$$\int_{\hat{K}} \int_{\hat{K}} |\hat{g}(\xi)|^2 j(\eta)\, d\sigma(\xi) = \int_{\hat{K}} |\hat{g}(\xi)|^2 j(\gamma * \xi)\, d\sigma(\xi) = \int_{\hat{K}} |\hat{g}(\xi)|^2 j_\gamma(\xi)\, d\sigma(\xi).$$

Now we replace g by k_i in the above relations (the net $\{k_i\}$ has been introduced in Lemma 2.4). Consider $h \in C_c^+(K)$ such that $h \equiv \|f\|_\infty$ on the compact set $\mathrm{supp}(f) * U$, where U is a compact neighborhood of the identity e as in Lemma 2.4. By ([6], 6.2E) we have $\|\,|f| * k_i\|_\infty \leq \|f\|_\infty \|k_i\|_1 = \|f\|_\infty$. Then

$$|\gamma(f * \tilde{f}) * h| = |\gamma(f * \tilde{f}) * k_i * k_i^-| \leq (|f| * k_i) * (|f| * k_i)^- \leq h * h^- \in L^1(K, \mu).$$

On the other hand for any $\xi \in \hat{K}$,

$$|\widehat{k_i}(\xi)| \leq \int_K |\overline{\xi(x)}|\, k_i(x)\, dm(x) \leq \|k_i\|_1 = 1.$$

Then for any $\xi \in \hat{K}, |\,|\hat{k_i}(\xi)|^2 j_\gamma(\xi)| \leq j_\gamma(\xi)$. Similar to the above relations we also have

$$
\begin{aligned}
\int_{\hat{K}} j_\gamma(\xi)\, d\sigma(\xi) &= \int_K \int_{\hat{K}} \gamma(y)(f * \tilde{f})(y)\xi(y)\, d\sigma(\xi) dm(y) \\
&= \int_K \gamma(y)(f * \tilde{f})(y)\check{\sigma}(y)\, dm(y) < \infty,
\end{aligned}
$$

because $\check{\sigma} \in C(K)$ and so that $\gamma(f * \tilde{f})\check{\sigma} \in C_c(K)$. So $j_\gamma \in L^1(\hat{K}, \sigma)$.

Therefore we can apply the dominated convergence theorem on two sides of the equality

$$\int_K \gamma(f * \tilde{f}) * (k_i * \tilde{k_i}) d\mu = \int_{\hat{K}} |\hat{k_i}|^2 j_\gamma \, d\sigma,$$

and so by limiting,

$$\begin{aligned}\int_K \gamma(f * \tilde{f}) d\mu &= \int_{\hat{K}} j_\gamma \, d\sigma \\ &= \int_{\hat{K}} (\delta_{\gamma^-} * j) \, d\sigma \\ &= \int_{\hat{K}} j \;\; d(\delta_\gamma * \sigma) \quad ([6], \text{Theorem 4.2H}).\end{aligned}$$

But since $(\eta f\hat{)}(\mathbf{1}) = \int_K f(x)\eta(x)\, dm(x) = \int_K f(x^-)\eta(x^-)\, dm(x) = (f^-\hat{)}(\eta)$, we have

$$j(\eta) = |(\eta f\hat{)}(\mathbf{1})|^2 = |(f^-\hat{)}(\eta)|^2.$$

Thus

$$\int_K f * \tilde{f} \, d(\gamma\mu) = \int_{\hat{K}} |(f^-\hat{)}|^2 \, d(\delta_\gamma * \sigma).$$

This completes the proof. □

Corollary 2.6. Let μ be a weakly shift-bounded positive definite measure on K with associated measure σ. Then $Per(\sigma) = \operatorname{supp}(\mu)^\perp$.

Proof. Let $\gamma \in \hat{K}$. By Theorem 2.5 the measure associated with $\gamma\mu$ is $\delta_\gamma * \sigma$. Then $\gamma \in Per(\sigma)$ if and only if the measure associated with $\gamma\mu$ is σ. But the mapping taking a weakly shift-bounded positive definite measure into its associated measure is injective ([2], Corollary 4.3.11), so $\gamma \in Per(\sigma)$ if and only if $\gamma\mu = \mu$, and this is equivalent with γ being 1 on the support of μ. □

A matrix $A = (a_{ij})$ of complex numbers is called *positive hermitian* if

$$\sum_{i,j=1}^{n} a_{ij} c_i \overline{c_j} \geq 0$$

for each complex numbers $c_1, c_2, \ldots, c_n$.

A continuous function $\phi : K \to \mathbb{C}$ is called

(i) *positive definite* if the matrix $(\phi(x_i * x_j^-))$ is positive hermitian for every $x_1, x_2, \ldots, x_n$ in K; and

(ii) *negative definite* if the matrix $(\phi(x_i) + \overline{\phi(x_j)} - \phi(x_i * x_j^-))$ is positive hermitian for every $x_1, x_2, \ldots, x_n$ in K.

By the Bochner theorem every bounded positive definite function ϕ on K is associated with a measure $\sigma \in M^+(K)$ such that $\phi = \check{\sigma}$.

The following theorem has been established in ([2], Proposition 5.2.40) under the condition that K is strong. Here we prove it with a completely different technique for commutative hypergroups.

Theorem 2.7. Let $\sigma \in M^+(K)$ be the measure associated with a bounded positive definite function ϕ on K. Then

$$Per(\phi) = \operatorname{supp}(\sigma)^\perp = \{p \in K : \phi(p) = \phi(e)\}.$$

Proof. Since $\phi = \check{\sigma}$, $p \in Per(\phi)$ if and only if $\check{\sigma} = \delta_p * \check{\sigma}$. For any $x \in K$,

$$(\delta_p * \check{\sigma})(x) = (\widehat{\delta_p}\sigma\check{)}(x) = \int_{\hat{K}} \xi(x)\widehat{\delta_p}(\xi)\, d\sigma(\xi) = \overline{\xi(p)} \int_{\hat{K}} \xi(x)\, d\sigma(\xi) = \overline{\xi(p)}\check{\sigma}(x).$$

By injectivity of the inverse Fourier transform, this implies that $p \in Per(\phi)$ if and only if $\xi(p) = 1$ for each $\xi \in \mathrm{supp}(\sigma)$. Therefore $Per(\phi) = \mathrm{supp}(\sigma)^{\perp}$.

To prove the second equality we proceed as follows:

If $\phi(e) = 0$, since for a bounded positive definite function ϕ we have $\|\phi\|_\infty = \phi(e)$ ([6], Theorem 11.1E), $\phi \equiv 0$ and the second equality trivially holds. Let $\phi(e) \neq 0$ (and so $\phi(e) > 0$). If $p \in Per(\phi)$ evidently we have $\phi(p) = \overline{\phi(p^- * e)} = \overline{\phi(e)} = \phi(e)$, since positive definity of ϕ implies that $\phi(p^-) = \overline{\phi(p)}$. Conversely let $\phi(p) = \phi(e)$. By inequality $2|\phi(x)| \leq \phi(e) + \phi(x * x^-)$ $(x \in K)$ ([2], Lemma 4.1.3(f)) we have $\phi(e) \leq \phi(p * p^-)$, and by $\|\phi\|_\infty = \phi(e)$ we have $\phi(p * p^-) \leq \phi(e)$. Thus $\phi(p * p^-) = \phi(e)$. Let $y \in K$ and put $k = \phi(e)$, $l = \overline{\phi(y)}$, $m = \phi(p * y^-)$, $n = \phi(y * y^-)$. Since ϕ is positive definite, the matrix

$$\begin{pmatrix} k & k & l \\ k & k & m \\ \bar{l} & \bar{m} & n \end{pmatrix}$$

is positive hermitian, and so its determinant, that is $-k\,|m - l|^2$, is non-negative. Then since $k > 0$, $m = l$. Hence for any $y \in K$, $\phi(p^- * y) = \phi(y)$, i.e. $p \in Per(\phi)$. □

Proposition 2.8. For every bounded measure μ of K we have
(i) $Per(\mu) = \mathrm{supp}(\hat{\mu})^{\perp}$,
(ii) $Per(\hat{\mu}) = \mathrm{supp}(\mu)^{\perp}$.

Proof. Refer to ([2], Theorem 5.2.40). □

Corollary 2.9. For every bounded measure μ on K, $Per(\mu)$ is a closed subhypergroup of K.

Proof. Recall that $Per(\mu) = \mathrm{supp}(\hat{\mu})^{\perp}$ and for any subset A of $\hat{K}$, $A^{\perp}$ is a closed subhypergroup of K. □

Proposition 2.10. Let $\psi : \hat{K} \to \mathbb{C}$ be a negative definite function such that $Re(\psi) \geq 0$ and $\psi(\mathbf{1}) = 0$. Then $Per(\psi) = \{\xi \in \hat{K} : \psi(\xi) = 0\}$.

Proof. $\xi \in Per(\psi)$ obviously implies that $\psi(\xi) = \psi(\mathbf{1}) = 0$. Conversely let $\psi(\xi) = 0$. By inequality in ([2], Proposition 4.4.3(c)), $\psi(\xi * \xi^-) + \psi(\mathbf{1}) \leq 2\psi(\mathbf{1})$ and so $\psi(\xi * \xi^-) \leq 0$. Also since $\psi(\xi * \xi^-) \in \mathbb{R}$, $\psi(\xi * \xi^-) = Re(\psi)(\xi * \xi^-) \geq 0$. Thus $\psi(\xi * \xi^-) = 0$. Now by the inequality in ([2], Proposition 4.4.3(e)), $\delta_\xi * \psi = \psi$. □

3. Translation Invariant Contraction Semigroups on hypergroups

Throughout this section K is a commutative hypergroup.

If f is a continuous function with compact support on K, then the right translations of f, $\delta_a * f$ ($a \in K$), are also continuous with compact support (see 3.1B and 4.2F of [6] or Proposition 1.2.16(iii) of [2]).

Definition 3.1. A positive linear mapping $T : C_c(K) \to C(K)$ is called *translation invariant* if for any $a \in K$ and $f \in C_c(K)$, $T(\delta_a * f) = \delta_a * Tf$.

Proposition 3.2. A mapping $T : C_c(K) \to C(K)$ is translation invariant if and only if there exists a unique positive measure $\mu \in M(K)$ such that $Tf = \mu * f$ for any f in $C_c(K)$.

Proof. Suppose that there exists a unique positive measure $\mu \in M(K)$ such that $Tf = \mu * f$ for all $f \in C_c(K)$. Then T is obviously linear and positive by $pos(Tf) = \text{supp}(\mu) * pos(f)$ ([6], 4.2D). Also we have

$$T(\delta_a * f) = \mu * (\delta_a * f) = \delta_a * (\mu * f) = \delta_a * Tf,$$

where $a \in K$, i.e. T is translation invariant.

Conversely let T be a translation invariant mapping. Since the mapping $f \mapsto T(f^-)(e)$ is linear and positive, by Rietz representation theorem there is a measure $\mu \in M(K)$ such that

$$T(f^-)(e) = \int f(x) d\mu(x),$$

where $f \in C_c(K)$. Also for any $f \in C_c(K), x \in K$ we have

$$\begin{aligned} Tf(x) = Tf(x * e) &= (\delta_{x^-} * Tf)(e) \\ &= (T(\delta_{x^-} * f))(e) \\ &= \int (\delta_{x^-} * f)(t^-) d\mu(t) \\ &= \int f(x * t^-) d\mu(t) = (\mu * f)(x). \end{aligned}$$

□

The following definition is similar to definition 3.1.

Definition 3.3. Let $1 \leq p < \infty$ and $T : L^p(K) \to L^p(K)$ be a bounded operator on $L^p(K)$.
(i) T is called translation invariant if for any $a \in K$ and any $f \in L^p(K)$ we have $\delta_a * Tf = T(\delta_a * f)$.
(ii) T is called submarkovian if for any $f \in L^p(K)$ with $0 \leq f \leq 1$ a.e. we have $0 \leq Tf \leq 1$ a.e..

Proposition 3.4. Let $T : L^p(K) \to L^p(K)$ be a bounded operator on $L^p(K)$ $(1 \leq p < \infty)$. Then T is submarkovian translation invariant if and only if there exists a positive bounded measure $\mu \in M(K)$ such that $\mu(K) \leq 1$ and $Tf = \mu * f$ for any $f \in L^p(K)$.

Proof. First, suppose that for a positive bounded measure $\mu \in M(K)$ with $\mu(K) \leq 1$ we have $Tf = \mu * f$ $(f \in L^p(K))$. Then for every $a \in K$, $\delta_a * Tf = \delta_a * (\mu * f) = \mu * (\delta_a * f) = T(\delta_a * f)$. Also if $f \in L^p(K)$ and $0 \leq f \leq 1$ a.e., then since

$$Tf(x) = \mu * f(x) = \int_K f(y^- * x) d\mu(y) = \int_K \int_K f(t) d(\delta_{y^-} * \delta_x)(t) d\mu(y),$$

for almost every x in K, we have

$$0 \leq Tf(x) \leq \int_K \int_K d(\delta_{y^-} * \delta_x)(t) d\mu(y) = \int_K d\mu(y) = \mu(K) \leq 1.$$

Conversely let T be a submarkovian translation invariant operator.

First, assume that for every $f \in C_c(K)$ there exists a h_f in $C(K)$ such that $Tf = h_f$. Then the restriction $T|_{C_c(K)} : C_c(K) \to C(K)$ defined by $f \mapsto h_f$ is obviously translation invariant. By Proposition 2.2 there exists a positive measure $\mu \in M(K)$ such that $Tf = \mu * f$ for all f in $C_c(K)$. For this μ and any $x_0 \in K$ we have

$$\mu(K) = \int_K d\mu(x) = \int_K \mathbf{1}(x^- * x_0)d\mu(x) = (\mu * \mathbf{1})(x_0) = T\mathbf{1}(x_0).$$

Since T is submarkovian, this shows that $0 \leq \mu(K) \leq 1$. In addition since $C_c(K)$ is dense in $L^p(K)$, the restricted T has an extension $Tf = \mu * f$ to $L^p(K)$.

In general case, by Lemma 2.4, for a neighborhood base $\{V_i\}$ at e there is a net $\{k_i\} \subseteq C_c^+(K)$ with $\text{supp}(k_i) \subseteq V_i$ and $\|k_i\|_1 = 1$ such that $\{k_i\}$ is a bounded approximate identity of $L^p(K)$. For every k_i we define $T_i : L^p(K) \to L^p(K)$ by $T_i f = Tf * k_i$, $f \in L^p(K)$. Every T_i is clearly linear and since for any $f \in L^p(K)$, $\|Tf * k_i\|_p \leq \|Tf\|_p \|k_i\|_1 = \|Tf\|_p$ ([6], 5.5Q), every T_i is also bounded. For any $a \in K$ and $f \in L^p(K)$,

$$\delta_a * T_i f = \delta_a * (Tf * k_i) = T(\delta_a * f) * k_i = T_i(\delta_a * f).$$

Then every T_i is translation invariant. If $f \in L^p(K)$ and $0 \leq f \leq 1$ a.e., we have

$$T_i f(x) = \int_K Tf(x * y)k_i(y^-)dm(y) = \int_K \int_K Tf(t)k_i(y^-)d(\delta_x * \delta_y)(t)dm(y).$$

Since T is submarkovian, for almost every x in K,

$$0 \leq T_i f(x) \leq \int_K k_i(y^-)dm(y) = \|k_i\|_1 = 1.$$

Then any T_i is submarkovian too. By ([6], 5.5D) for any $f \in L^p(K)$, $Tf * k_i$ is continuous. Then by case 1 for every T_i there is a positive bounded measure $\mu_i \in M(K)$ with $\mu_i(K) \leq 1$ such that $T_i f = \mu_i * f$. By Banach-Alouglu theorem there is a positive bounded measure $\mu \in M(K)$ with $\mu(K) \leq 1$ such that $\mu_i \to \mu$ in $\sigma(M(K), C_c(K))$. Now for every f in $C_c(K)$ we have

$$Tf = \lim_i Tf * k_i = \lim_i \mu_i * f = \mu * f.$$

Since $C_c(K)$ is dense in $L^p(K)$, we can extend this function to $L^p(K)$.

Uniqueness of the measure μ is obvious. □

Definition 3.5. Let $1 \leq p < \infty$. A family $\{P_t\}_{t>0}$ of bounded operators on $L^p(K)$ is called *strongly continuous contraction semigroup* if
(i) $\|P_t\| \leq 1$ for all $t > 0$;
(ii) $P_t P_s = P_{t+s}$ for all $t, s > 0$;
(iii) $\|P_t f - f\|_p \to 0$ as $t \to 0^+$, for all $f \in L^p(K)$.

Definition 3.6. A family $\{\mu_t\}_{t>0}$ of positive bounded measures in $M(K)$ is called a *convolution semigroup* on K if
(i) $\mu_t(K) \leq 1$ for all $t > 0$;
(ii) $\mu_t * \mu_s = \mu_{t+s}$ for all $t, s > 0$;
(iii) $\mu_t \to \delta_e$ as $t \to 0^+$ in $\sigma(M(K), C_c(K))$.

For a strong hypergroup K, if $\{\mu_t\}_{t>0}$ is a convolution semigroup on K, then there exists a unique negative definite function ψ on $\hat{K}$ such that $Re(\psi) \geq 0$ and $\widehat{\mu_t} = e^{-t\psi}$ for every $t > 0$. The function ψ is called the negative definite function associated with $\{\mu_t\}_{t>0}$. We refer to [7] or [2] for basic properties of convolution semigroups.

Theorem 3.7. Let $1 \leq p < \infty$. There is a one to one correspondence between convolution semigroups on K and strongly continuous contraction semigroups of translation invariant and submarkovian operators on $L^p(K)$.

Proof. Suppose that $\{\mu_t\}_{t>0}$ is a convolution semigroup on K. We define $P_t : L^p(K) \to L^p(K)$ by $P_t f = \mu_t * f$, $f \in L^p(K)$. By Proposition 3.4 every P_t is a translation invariant and submarkovian bounded operator. In addition since $\|\mu_t * f\|_p \leq \|\mu_t\| \, \|f\|_p$ ([6], 5.4G), $\|P_t\| = sup_{\|f\|_p=1}\|P_t f\| \leq \|\mu_t\| = \mu_t(K) \leq 1$ and $P_t P_s f = P_t(\mu_s * f) = \mu_t * \mu_s * f = \mu_{t+s} * f = P_{t+s} f$. By taking $\alpha = \mathbf{1}$ in Theorem 2.2 of [7] we have $\mu_t(K) > 0$ for any $t > 0$, so for every $f \in L^p(K)$,

$$\begin{aligned}
\|P_t f - f\|_p^p &= \int_K |\mu_t * f(x) - f(x)|^p \, dm(x) \\
&= \int_K |\int_K f(x * y^-) - \frac{f(x)}{\mu_t(K)} \, d\mu_t(y)|^p \, dm(x) \\
&\leq \int_K \int_K |f(x * y^-) - \frac{f(x)}{\mu_t(K)}|^p \, d\mu_t(y) \, dm(x) \\
&= \int_K \int_K |\delta_y * f - \frac{f}{\mu_t(K)}|^p \, dm \, d\mu_t(y) \\
&= \int_K g_t(y) \, d\mu_t(y),
\end{aligned}$$

where $g_t(y) = \|\delta_y * f - \frac{f}{\mu_t(K)}\|_p^p$. We put $g(y) = \|\delta_y * f - f\|_p^p$. Then g, g_t are bounded and continuous. Consider $h \in C_c^+(K)$ such that $0 \leq h \leq 1$ and $h(e) = 1$. Since $\mu_t(K) = \hat{\mu_t}(\mathbf{1}) = e^{-t\psi(\mathbf{1})}$, $\mu_t(K) \to 1$ as $t \to 0^+$. We have

$$\int_K g_t d\mu_t = \int_K hg d\mu_t - \int_K h(g - g_t) d\mu_t + \int_K (1-h) g_t d\mu_t,$$

and so

$$0 \leq \limsup_t \int_K g_t(y) d\mu_t(y) \leq 2^p \|f\|_p^p \int_K (1-h) d\delta_e = 0.$$

Therefore $\{P_t\}_{t>0}$ is a strongly continuous contraction semigroup.

Conversely let $\{P_t\}_{t>0}$ be a strongly continuous contraction semigroup of translation invariant and submarkovian operators on $L^p(K)$. By Proposition 3.4 for every $t > 0$ there exists a unique positive and bounded measure $\mu_t \in M(K)$ such that $\mu_t(K) \leq 1$ and $P_t f = \mu_t * f$, for all $f \in L^p(K)$. For any $f \in C_c(K)$ and any $t, s > 0$,

$$\begin{aligned}
\int_K f(x) d\mu_{t+s}(x) &= \int_K f^-(e * x^-) d\mu_{t+s}(x) = (\mu_{t+s} * f^-)(e) \\
&= P_{t+s} f^-(e) = (P_t P_s f^-)(e) \\
&= (\mu_t * \mu_s * f^-)(e) = \int_K f(x) d\mu_t * \mu_s(x),
\end{aligned}$$

so that $\mu_{t+s} = \mu_t * \mu_s$.

Now let $\{k_i\} \subseteq C_c^+(K)$ be the approximate identity as in the Lemma 2.4. For any f in $C_c(K)$, $\int_K (f * k_i)(x) d\mu_t(x) = (\mu_t * (f * k_i)^-)(e) = (\mu_t * f^- * k_i^-)(e) = (P_t(f^-) * k_i^-)(e) =$

$\int_K P_t(f^-)(x)k_i(x)dm(x)$. Also for any $x \in K$ we have

$$\begin{aligned} |f * k_i(x)| &\leq \int_K |f(x * y)| \cdot k_i(y^-)\, dm(y) \\ &\leq \int_K \left(\int_K |f(t)|\, d(\delta_x * \delta_y)(t) \right) k_i(y^-)\, dm(y) \\ &\leq M \cdot \int_K k_i(y^-)\, dm(y) \\ &= M \cdot \int_K k_i(y)\, dm(y) = M \cdot \|k_i\|_1 = M, \end{aligned}$$

where $M := \sup_{t \in K} |f(t)|$. So the net $(\int_K (f * k_i)(x)\, d\mu_t(x))_{i,t}$ is bounded, because

$$|\int_K (f * k_i)(x)\, d\mu_t(x)| \leq \int_K |f * k_i(x)|\, d\mu_t(x) \leq M \cdot \mu_t(K) \leq M.$$

Therefore $(\int_K (f * k_i)(x)\, d\mu_t(x))_{i,t}$ is convergent (by passing to a subnet if necessary). Then

$$\begin{aligned} \lim_{t \to 0^+} \int_K f\, d\mu_t &= \lim_{t \to 0^+} \lim_i \int_K (f * k_i)(x)\, d\mu_t(x) \\ &= \lim_i \lim_{t \to 0^+} \int_K (f * k_i)(x)\, d\mu_t(x) \\ &= \lim_i \int_K f^-\, k_i\, dm = f^-(e) = f(e). \end{aligned}$$

This shows that $\{\mu_t\}_{t>0}$ is a convolution semigroup on K. □

For a convolution semigroup $\{\mu_t\}_{t>0}$ on K, the contraction semigroup defined by $P_t f = \mu_t * f$, $f \in L^p(K)$, is called the contraction semigroup on $L^p(K)$ induced by $\{\mu_t\}_{t>0}$.

Definition 3.8. Let $1 \leq p < \infty$ and $\{P_t\}_{t>0}$ be a strongly continuous contraction semigroup on $L^p(K)$.
(i) We define $N : D(N) \to L^p(K)$ by

$$Nf = \lim_{t \to \infty} \int_0^t P_s f\, ds, \quad (f \in D(N))$$

where $D(N) = \{f \in L^p(K) : \lim_{t \to \infty} \int_0^t P_s f ds \text{ exists in } L^p(K)\}$. The function N is called the potential operator for $\{P_t\}_{t>0}$, and is denoted by $(N, D(N))$.
(ii) For any $\lambda > 0$ the function $N_\lambda : L^p(K) \to L^p(K)$ defined by

$$N_\lambda f = \int_0^\infty e^{-\lambda t} P_t f\, dt, \quad (f \in L^p(K))$$

is a bounded operator with domain $L^p(K)$ and of norm $\|N_\lambda\| \leq \frac{1}{\lambda}$ ([1], Proposition 11.10).
We define $N_0 : D(N_0) \to L^p(K)$ by

$$N_0 f = \lim_{\lambda \to 0^+} N_\lambda f, \quad (f \in D(N_0))$$

where $D(N_0) = \{f \in L^p(K) : \lim_{\lambda \to 0^+} N_\lambda f \text{ exists in } L^p(K)\}$. The function N_0 is called the zero-resolvent for $\{P_t\}_{t>0}$ and is denoted by $(N_0, D(N_0))$.
(iii) If $\{\mu_t\}_{t>0}$ is a convolution semigroup then for any $\lambda > 0$ we define the measure $\rho_\lambda \in M(K)$ by

$$\rho_\lambda(\varphi) = \int_0^\infty e^{-\lambda t} \mu_t(\varphi)\, dt \quad (\varphi \in C_c(K)).$$

$\{\rho_\lambda\}_{\lambda>0}$ is called the resolvent of measures on K. For basic properties of $\{\rho_\lambda\}_{\lambda>0}$ refer to [2].

Proposition 3.9. Let $1 \le p < \infty$ and $\{P_t\}_{t>0}$ be a strongly continuous contraction semigroup on $L^p(K)$.
(*i*) $D(N_0)$ is dense in $L^p(K)$ if and only if $\lim_{\lambda\to 0^+} \lambda N_\lambda f = 0$ for all $f \in L^p(K)$.
(*ii*) $D(N)$ is dense in $L^p(K)$ if and only if $\lim_{t\to\infty} P_t f = 0$ for all $f \in L^p(K)$.

Proof. Refer to the Propositions 11.9 and 11.14 of [1]. □

Theorem 3.10. Let K be a commutative strong hypergroup, $\{P_t\}_{t>0}$ be the contraction semigroup on $L^2(K)$ induced by $\{\mu_t\}_{t>0}$ and ψ be the continuous negative definite function on $\hat{K}$ associated with $\{\mu_t\}_{t>0}$.
(*i*) The domain of zero-resolvent $(N_0, D(N_0))$ of $\{P_t\}_{t>0}$ is dense in $L^2(K)$ if and only if $\psi \neq 0$ locally almost everywhere on $\hat{K}$.
(*ii*) The domain of potential operator $(N, D(N))$ of $\{P_t\}_{t>0}$ is dense in $L^2(K)$ if and only if $Re\psi \neq 0$ locally almost everywhere on $\hat{K}$.

Proof. (*i*) Since ψ is a negative definite function with $Re\psi \ge 0$, then $\psi(\mathbf{1}) \ge 0$. We prove the theorem in two cases. Put $B = \{\xi \in \hat{K} : \psi(\xi) = \psi(\mathbf{1})\}$.

Case 1. $\psi(\mathbf{1}) = 0$.
For any $k \in C_c(\hat{K})$ and any $\lambda > 0$ we have $N_\lambda(\check{k}) = \int_0^\infty e^{-\lambda t} P_t(\check{k}) dt$ and $N_\lambda(\check{k}) \in L^2(K)$. Then for any $\xi \in \hat{K}$,

$$\begin{aligned}
(N_\lambda(\check{k}))\hat{}(\xi) &= \int_K \overline{\xi(x)} N_\lambda(\check{k})(x)\, dm(x) \\
&= \int_K \int_0^\infty \overline{\xi(x)} e^{-\lambda t} (\mu_t * \check{k})(x)\, dt\, dm(x) \\
&= \int_K \int_0^\infty \int_K \int_{\hat{K}} \overline{\xi(x)} e^{-\lambda t} k(\chi) \overline{\chi(y)} \chi(x)\, d\pi(\chi)\, d\mu_t(y)\, dt\, dm(x) \\
&= \int_0^\infty e^{-\lambda t} \int_K \overline{\xi(x)} \int_{\hat{K}} k(\chi)\chi(x) \int_K \overline{\chi(y)}\, d\mu_t(y)\, d\pi(\chi)\, dm(x)\, dt \\
&= \int_0^\infty e^{-\lambda t} \int_{\hat{K}} k(\chi) e^{-t\psi(\chi)} \int_K \overline{\xi(x)} \chi(x)\, dm(x)\, d\pi(\chi)\, dt \\
&= \int_0^\infty e^{(-\lambda-\psi(\xi))t} k(\xi) M_\xi^{-1} dt = \frac{k(\xi)}{M_\xi} \frac{1}{\lambda + \psi(\xi)},
\end{aligned}$$

where $M_\xi^{-1} = \int_K |\xi(x)|^2 dm(x)$. So by ([6], 7.3I),

$$\|N_\lambda(\check{k})\|_2^2 = \|(N_\lambda(\check{k}))\hat{}\|_2^2 = \int_{\hat{K}} |\frac{k(\xi)}{M_\xi} \frac{1}{\lambda + \psi(\xi)}|^2 d\pi(\xi).$$

We have

$$|\frac{k(\xi)}{M_\xi} \frac{1}{\lambda + \psi(\xi)}|^2 \le |k(\xi)|^2 \frac{\lambda^2}{(\lambda + Re\psi(\xi))^2} \le |k(\xi)|^2,$$

since $Re\psi \ge 0$. If $\psi \neq 0$ locally almost everywhere then B is a locally null set and so by using of the dominated convergence theorem,

$$\lim \|\lambda N_\lambda(\check{k})\|_2^2 = \int_{\hat{K}} \lim |\frac{k(\xi)}{M_\xi} \frac{\lambda}{\lambda + \psi(\xi)}|^2 dm(\xi) = 0$$

as $\lambda \to 0^+$. The latter relation shows that $\{\check{k} : k \in C_c(\hat{K})\} \subseteq \{f \in L^2(K) : \lim_{\lambda\to 0^+} \lambda N_\lambda f = 0\}$. But the set $\{\check{k} : k \in C_c(\hat{K})\}$ is dense in $L^2(\hat{K})$ ([2], p. 85) and the set $\{f \in L^2(K) :$

$\lim_{\lambda\to 0}\lambda N_\lambda f = 0\}$ is closed ([1], p. 82). Then for each $f \in L^2(K)$, $\lim_{\lambda\to 0^+}\lambda\, N_\lambda f = 0$ and so by Proposition 3.9(i) $D(N_0)$ is dense in $L^2(K)$.

Conversely, if B is not locally null then by Theorem 2.2.45(h) of [2] $B^\perp$ is compact. So there exist $f, g \in C_c^+(K)$ such that $g \equiv 1$ on $B^\perp$ and $g \leq f * f^-$. So $1 \leq f * f^-$ on $B^\perp$. On the other hand for any $\lambda > 0$,

$$\begin{aligned}\operatorname{supp}(\rho_\lambda * \rho_\lambda^-) &\subseteq (\operatorname{supp}(\rho_\lambda) * \operatorname{supp}(\rho_\lambda)^-)^c\\ &\subseteq S * S^-\\ &\subseteq B^\perp * B^\perp \subseteq B^\perp,\end{aligned}$$

where $S = \operatorname{supp}(\rho_\lambda) = (\bigcup_{t\geq 0}\operatorname{supp}(\mu_t))^c$. For any $\lambda > 0$ we have

$$\begin{aligned}&\textstyle\int_K |\rho_\lambda * f(u)|^2 dm(u)\\ &= \int_K\int_K\int_K f(s^- * u)\, f(t^- * u) d\rho_\lambda(s) d\rho_\lambda(t) dm(u)\\ &= \int_K\int_K\int_K f_s(u)\, f_{t^-}(u) dm(u) d\rho_\lambda(s^-) d\rho_\lambda(t)\\ &= \int_K\int_K\int_K f(u)\, f_{t^-}(s^- * u) dm(u) d\rho_\lambda(s^-) d\rho_\lambda(t)\\ &= \int_K\int_K\int_K f(u)\, f_u(s * t) dm(u) d\rho_\lambda(s) d\rho_\lambda(t^-)\\ &= \int_K\int_K\int_K\int_K f^-(u^-) f(u * x) dm(u) d\delta_s * \delta_t(x) d\rho_\lambda(s) d\rho_\lambda(t^-)\\ &= \int_K\int_K\int_K f * f^-(x) d\delta_s * \delta_t(x) d\rho_\lambda(s) d\rho_\lambda(t^-)\\ &= \int_K\int_K f * f^-(s * t) d\rho_\lambda(s) d\rho_\lambda^-(t)\\ &= \int_K f * f^-(x) d\rho_\lambda * \rho_\lambda^-(t).\end{aligned}$$

The latter integral is greater than (or equal with) $\int_K d\rho_\lambda * \rho_\lambda^-(x) = \rho_\lambda(K)^2 = \frac{1}{\lambda^2}$, since

$$\rho_\lambda(K) = \hat{\rho_\lambda}(\mathbf{1}) = \int_0^\infty e^{-\lambda t}\hat{\mu_t}(\mathbf{1}) dt = \int_0^\infty e^{-\lambda t} e^{-t\psi(\mathbf{1})} dt = \frac{1}{\lambda}.$$

But

$$\begin{aligned}\rho_\lambda * f(x) = \int_K f(x * y^-)\, d\rho_\lambda(y) &= \int_0^\infty e^{-\lambda t}\int_K f(x * y^-)\, d\mu_t(y) dt\\ &= \int_0^\infty e^{-\lambda t}(\mu_t * f)(x) dt\\ &= N_\lambda f(x).\end{aligned}$$

Then $\|\lambda\, N_\lambda f\|_2 \geq 1$ and by Proposition 3.9(i), this implies that $D(N_0)$ is not dense in $L^2(K)$.

Case 2. $\psi(1) \neq 0$.

For any $f \in L^2(K)$,

$$\begin{aligned}\|\lambda N_\lambda f\|_2 = \|\int_0^\infty \lambda e^{-\lambda t} P_t f dt\|_2 &\le \int_0^\infty \lambda e^{-\lambda t}\|P_t f\|_2 dt \\ &\le \int_0^\infty \lambda e^{-\lambda t} e^{-t\psi(\mathbf{1})}\|f\|_2 dt \\ &= \frac{\lambda}{\lambda+\psi(\mathbf{1})}\|f\|_2 \to 0,\end{aligned}$$

as $\lambda \to 0^+$, since $\|P_t f\|_2 = \|\mu_t * f\|_2 \le \|\mu_t\|\,\|f\|_2 = \widehat{\mu_t}(\mathbf{1})\|f\|_2 = e^{-t\psi(\mathbf{1})}\|f\|_2$. Then $D(N_0)$ is dense in $L^p(K)$. On the other hand by Theorem 2.2 and Proposition 1.3(a) in [7] we have $\{\xi \in \hat{K} : \psi(\xi) = 0\} = \emptyset$ and so that $\psi \neq 0$ locally almost everywhere.

(ii) Put $D = \{\xi \in \hat{K} : Re\psi(\xi) = \psi(\mathbf{1})\}$ and again consider two cases.

Case 1. $\psi(\mathbf{1}) = 0$.

By continuity of ψ, D is closed. For any $\phi \in C_c(\hat{K})$,

$$\begin{aligned}(\check{\phi})\hat{}(\xi) = \int_K \overline{\xi(x)}\check{\phi}(x)\,dm(x) &= \int_K\int_{\hat{K}} \overline{\xi(x)}\phi(\chi)\chi(x)\,d\pi(\chi)dm(x) \\ &= \int_{\hat{K}} \phi(\chi)\int_K \overline{\xi(x)}\chi(x)\,dm(x)d\pi(\chi) \\ &= \int_K \phi(\xi)|\xi(x)|^2\,dm(x) = \phi(\xi)\,M_\xi^{-1},\end{aligned}$$

and then

$$(P_t\check{\phi})\hat{}(\xi) = (\mu_t * \check{\phi})\hat{}(\xi) = \widehat{\mu_t}(\xi)(\check{\phi})\hat{}(\xi) = \widehat{\mu_t}(\xi)\phi(\xi)M_\xi^{-1} = \frac{e^{-t\psi(\xi)}\phi(\xi)}{M_\xi}.$$

So

$$\|P_t\check{\phi}\|_2^2 = \|(P_t\check{\phi})\hat{}\|_2^2 = \int_{\hat{K}} |\frac{e^{-t\psi(\xi)}\phi(\xi)}{M_\xi}|^2\,d\pi(\xi).$$

We have

$$|\frac{e^{-t\psi(\xi)}\phi(\xi)}{M_\xi}|^2 \le e^{-tRe\psi(\xi)}|\phi(\xi)|^2 \le |\phi(\xi)|^2.$$

If $Re\psi \neq 0$ locally almost everywhere then D is locally null, and so by dominated convergence theorem we have $\lim_{t\to\infty} P_t\check{\phi} = 0$ in $L^2(K)$ (note that by Theorem 2.2 in [7], $Re\psi \ge 0$). Thus $\{\check{\phi} : \phi \in C_c(\hat{K})\} \subseteq \{f \in L^2(K) : \lim_{t\to\infty} P_t f = 0\}$. But as before the set $\{\check{\phi} : \phi \in C_c(\hat{K})\}$ is dense in $L^2(K)$ and the set $E_0 := \{f \in L^2(K) : \lim_{t\to\infty} P_t f = 0\}$ is closed because if f is in its closure then for any $\epsilon > 0$ there is a $g \in E_0$ such that $\|f-g\|_2 < \frac{\epsilon}{2}$ and there is a $t_0 > 0$ such that for any $t \in [0, t_0]$, $\|P_t g\|_2 < \frac{\epsilon}{2}$. Then for any $t \in [0, t_0]$ we have

$$\|P_t f\|_2 \le \|P_t f - P_t g\|_2 + \|P_t g\|_2 \le \|P_t\|\,\|f-g\|_2 + \|P_t g\|_2 < \epsilon,$$

since $\|P_t\| \le 1$. So $f \in E_0$. Therefore for any $f \in L^2(K), \lim_{t\to\infty} P_t f = 0$. Then by Proposition 3.9(ii), $D(N)$ is dense in $L^2(K)$.

Conversely if D is not locally null, then by Proposition 2.2.45(h) in [2] $D^\perp$ is compact, and by [9] it is a subhypergroup of K. For any $t > 0$ we have $\mathrm{supp}(\mu_t) \subseteq D^\perp$. Then

$$\begin{aligned}\mathrm{supp}(\mu_t * \mu_t^-) &= (\mathrm{supp}(\mu_t) * (\mathrm{supp}(\mu_t))^-)^c \\ &\subseteq (D^\perp * D^\perp)^c \subseteq (D^\perp)^c = D^\perp.\end{aligned}$$

On the other hand for $f \in C_c(K)$ with $f * f^- \geq 1$ on $D^\perp$,

$$\|P_t f\|_2^2 = \int_K |\mu_t * f(x)|^2 dm(x) = \int_K f * f^-(x)\, d\mu_t * \mu_t^-(x).$$

The last integral is greater than (or equal with) $\int_K \mu_t * \mu_t^-(x) = (\mu_t(K))^2 = 1$ (note that $\mu_t(K) = \widehat{\mu_t}(\mathbf{1}) = e^{-t\psi(\mathbf{1})} = 1$). Thus $\lim_{t\to\infty} P_t f \neq 0$. Now Proposition 3.9($ii$) shows that $D(N)$ is not dense in $L^2(K)$.

Case 2. $\psi(\mathbf{1}) \neq 0$.

For any $f \in L^2(K)$ we have

$$\|P_t f\|_2 = \|\mu_t * f\|_2 \leq e^{-t\psi(\mathbf{1})}\|f\|_2 \to 0,$$

as $t \to \infty$, see the case 2 of (i). Then $D(N)$ is dense in $L^2(K)$. Also as in the case 2 of (i), $\{\xi \in \hat{K} : Re\psi(\xi) = 0\} = \emptyset$ and so $Re\psi \neq 0$ locally almost everywhere. □

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