STRONG CONVERGENCE THEOREMS BY A HYBRID STEEPEST DESCENT METHOD FOR COUNTABLE NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce a new iterative procedure for finding a solution of the variational inequality problem over the intersection of fixed point sets of infinite nonexpansive mappings in a Hilbert space and then discuss the strong convergence of the iterative procedure.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping A of C into H is called *monotone* if $\langle x-y, Ax-Ay\rangle \geq 0$ for all $x, y \in C$. The *variational inequality problem* for A is to find $z \in C$ such that

$$\langle y-z, Az\rangle \geq 0$$

for all $y \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. A mapping A of C into H is called *strongly monotone* if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha \|x-y\|^2$$

for all $x, y \in C$. Such A is called α-strongly monotone. A mapping T of C into itself is called *nonexpansive* if

$$\|Tx-Ty\| \leq \|x-y\|$$

for all $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T. The well-known iterative procedure for finding a solution of the variational inequality problem may be the *projected gradient method* [3, 14]: $x_1 \in C$ and

$$x_{n+1} = P_C(I-\rho A)x_n \tag{1}$$

for $n = 1, 2, \ldots$, where P_C is the metric projection of H onto C and ρ is a positive real number. Indeed, when A is strongly monotone and Lipschitzian, the sequence $\{x_n\}$ generated by (1) converges strongly to a unique solution of $VI(C, A)$. However, the projected gradient method requires the use of the metric projection P_C of which the closed form expression is not known. In order to reduce the complexity which is caused by P_C, Yamada [13] introduced the following iterative procedure called the *hybrid steepest descent method*: $x_1 \in H$ and

$$x_{n+1} = (I-\lambda_n \rho A)Tx_n \tag{2}$$

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for all $n = 1, 2, \dots,$ where $\{\lambda_n\}$ is a sequence in $(0, 1]$ and ρ is a positive real number. He showed that the sequence $\{x_n\}$ generated by (2) converges strongly to a unique solution of $VI(F(T), A)$.

On the other hand, Kimura and Takahashi [4] established a weak convergence theorem for an infinite family of nonexpansive mappings which is connected with the feasibility problem and generalizes the result of Takahashi and Shimoji [11]. Shimoji and Takahashi [5] also proved a strong convergence theorem for an infinite family of nonexpansive mappings by using the methods of proofs of Shioji and Takahashi [6] and Atsushiba and Takahashi [2].

The purpose of the present paper is to prove a strong convergence theorem for finding a solution of the variational inequality problem over the intersection of fixed point sets of infinite nonexpansive mappings $\{T_n\}$ in a real Hilbert space. We deal with the following iterative scheme:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (I - \lambda_n \rho A) W_n x_n \text{ for all } n = 1, 2, \dots, \end{cases}$$

where $\{W_n\}$ is a sequence of W-mappings generated by nonexpansive mappings $T_n, T_{n-1}, \dots, T_1$ of H into itself, A is a strongly monotone and Lipschitzian mapping of H into itself, $\{\lambda_n\} \subset (0, 1]$ and $\rho > 0$.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. We denote the strong convergence and the weak convergence of x_n to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. A mapping T of C into itself is *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{z \in H : Tz = z\}$ and by $R(T)$ the range of T. We know that if C is a bounded closed convex subset of H and T is a nonexpansive mapping of C into itself, $F(T)$ is nonempty. It is also well-known that $F(T)$ is a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C and

$$\langle x - P_C x, P_C x - y \rangle \geq 0 \text{ for all } y \in C. \tag{3}$$

In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \iff u = P_C(u - \rho A u)$$

for all $\rho > 0$, where A is a monotone mapping of C into H.

A mapping A of C into H is called *strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|x - y\|^2$$

for all $x, y \in C$. Such A is called α-strongly monotone. If $A : C \to H$ is α-strongly monotone and β-Lipschitzian, then without loss of generality we can assume $\alpha < \beta$. The following lemma is in [14].

Lemma 2.1 ([14]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $\alpha, \beta, \rho > 0$. Suppose that A is an α-strongly monotone and β-Lipschitzian mapping of C into H and $\rho \in (0, 2\alpha/\beta^2)$. Then*

$$\|P_C(I - \rho A)x - P_C(I - \rho A)y\| \leq \sqrt{1 - \rho(2\alpha - \rho\beta^2)}\, \|x - y\|$$

for all $x, y \in C$. In particular, $P_C(I - \rho A)$ is a contraction of C into itself.

Remark 1. In this lemma, $\rho(2\alpha - \rho\beta^2)$ is actually in the interval $(0, 1)$. In fact, it is easy that $\rho(2\alpha - \rho\beta^2) > 0$. We have that

$$\begin{aligned} \alpha < \beta &\Longrightarrow \alpha^2 - \beta^2 < 0 \\ &\Longrightarrow \beta^2\rho^2 - 2\alpha\rho + 1 > 0 \\ &\Longrightarrow \rho(2\alpha - \rho\beta^2) < 1. \end{aligned}$$

The following theorem is due to [14]; see also [3].

Theorem 2.1 (Projected gradient method). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $\alpha, \beta > 0$. Suppose that A is an α-strongly monotone and β-Lipschitzian mapping of C into H. Then the following hold:*

(i) *$VI(C, A)$ has its unique solution $u^* \in C$;*

(ii) *for any $x_1 \in C$ and $\rho \in (0, 2\alpha/\beta^2)$, the sequence $\{x_n\}$ generated by*

$$x_{n+1} = P_C(I - \rho A)x_n \text{ for } n \in \mathbb{N}$$

converges strongly to a unique solution u^* of $VI(C, A)$.

Motivated by Theorem 2.1, Yamada [13] proved the following theorem.

Theorem 2.2 (Hybrid steepest descent method). *Let H be a real Hilbert space, let T be a nonexpansive mapping on H such that $F(T)$ is nonempty and let $\alpha, \beta > 0$. Suppose that A is an α-strongly monotone and β-Lipschitzian mapping of $R(T)$ into H. Then, $VI(F(T), A)$ has its unique solution $u^* \in C$. Further, for any $x_1 \in H$ and $\rho \in (0, 2\alpha/\beta^2)$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = (I - \lambda_n \rho A)Tx_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to a unique solution u^* of $VI(F(T), A)$, where $\{\lambda_n\}$ is a sequence of $(0, 1]$ satisfying

(C1) $\lim_{n\to\infty} \lambda_n = 0$;

(C2) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(C3) $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$.

The above condition (C3) can be generalized to the following condition by Xu [12]:

(C4) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Let $T_1, T_2, \dots$ be mappings on H and let $\gamma_1, \gamma_2, \dots$ be real numbers such that $0 \leq \gamma_i \leq 1$ for every $i \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we define a mapping W_n on H as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \gamma_n T_n U_{n,n+1} + (1 - \gamma_n)I, \\ U_{n,n-1} &= \gamma_{n-1} T_{n-1} U_{n,n} + (1 - \gamma_{n-1})I, \\ &\ \ \vdots \\ U_{n,k} &= \gamma_k T_k U_{n,k+1} + (1 - \gamma_k)I, \\ U_{n,k-1} &= \gamma_{k-1} T_{k-1} U_{n,k} + (1 - \gamma_{k-1})I, \end{aligned}$$

$$\vdots$$
$$U_{n,2} = \gamma_2 T_2 U_{n,3} + (1-\gamma_2)I,$$
$$W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I.$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\gamma_n, \gamma_{n-1}, \dots, \gamma_1$; see [8], [10] and [11]. The following lemma was proved in [11].

Lemma 2.2 ([11]). *Let H be a real Hilbert space. Let $T_1, T_2, \dots, T_n$ be nonexpansive mappings on H such that $\bigcap_{i=1}^{n} F(T_i)$ is nonempty and let $\gamma_1, \gamma_2, \dots, \gamma_n$ be real numbers such that $0 < \gamma_i < 1$ for $i = 1, 2, \dots, n$. For any $n \in \mathbb{N}$, let W_n be the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\gamma_n, \gamma_{n-1}, \dots, \gamma_1$. Then W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$.*

For $k \in \mathbb{N}$, from Lemma 3.2 in [5], we define mappings $U_{\infty,k}$ and W on H as follows:

$$U_{\infty,k}x = \lim_{n\to\infty} U_{n,k}x$$

and

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for all $x \in H$. Such a mapping W is called the W-mapping generated by $T_1, T_2, \dots$ and $\gamma_1, \gamma_2, \dots$. We know the following two lemmas:

Lemma 2.3 ([5]). *Let H be a real Hilbert space. Let $T_1, T_2, \dots$ be nonexpansive mappings on H such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and let $\gamma_1, \gamma_2, \dots$ be real numbers such that $0 < \gamma_i < 1$ for all $i \in \mathbb{N}$. Let W be the W-mapping generated by $T_1, T_2, \dots$ and $\gamma_1, \gamma_2, \dots$. Then W is nonexpansive and $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.4 ([7]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Let μ be a mean on $\mathbb{N}$, i.e., a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf_{n\in\mathbb{N}} a_n \le \mu(f) \le \sup_{n\in\mathbb{N}} a_n$$

for each $f = (a_1, a_2, \dots) \in l^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. So, a Banach limit μ is a mean on $\mathbb{N}$ satisfying $\mu_n(a_n) = \mu_n(a_{n+1})$. Let $f = (a_1, a_2, \dots) \in l^\infty$ with $a_n \to a$ as $n \to \infty$ and let μ be a Banach limit on $\mathbb{N}$. Then $\mu(f) = \mu_n(a_n) = a$. We also know the following lemma [6].

Lemma 2.5 ([6]). *Let a be a real number and let $(a_1, a_2, \dots) \in l^\infty$ such that $\mu_n(a_n) \le a$ for all Banach limit μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \le 0$. Then, $\limsup_{n\to\infty} a_n \le a$.*

The following lemma is proved in [1].

Lemma 2.6 ([1]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of real numbers with $\limsup_n \beta_n \le 0$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\sum_{n=1}^{\infty} \gamma_n < \infty$. Suppose that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n = 0$.

The following lemma is in [9, 10].

Lemma 2.7. *In a real Hilbert space H, the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle x+y, y\rangle$$

for all $x, y \in H$.

The following theorem plays a crucial role for our main theorem.

Theorem 2.3 ([13]). *Let H be a real Hilbert space and $\alpha, \beta > 0$. Let W be a nonexpansive mapping on H such that $F(W)$ is nonempty and let A be an α-strongly monotone and β-Lipschitzian mapping of $R(W)$ into H. For $\rho \in (0, 2\alpha/\beta^2)$, define $S_n : H \to H$ and C_f by*

$$S_n = (I - \lambda_n \rho A)W \text{ for all } n \in \mathbb{N},$$
$$C_f = \left\{ x \in H : \|x - f\| \le \frac{\|\rho A(Wf)\|}{r} \right\} \text{ for all } f \in F(W),$$

where $\{\lambda_n\} \subset (0,1]$ and $r = 1 - \sqrt{1 - \rho(2\alpha - \rho\beta^2)} \in (0,1)$. Then the following holds:

(i) *For each $n \in \mathbb{N}$, S_n is a contraction which has a unique fixed point $u_n \in \bigcap_{f \in F(W)} C_f$.*
(ii) *Suppose that the sequence $\{\lambda_n\} \subset (0,1]$ satisfies $\lim_{n\to\infty} \lambda_n = 0$. Let u_n be a unique fixed point of S_n, that is, $u_n = S_n u_n = W u_n - \lambda_n \rho A(W u_n)$. Then the sequence $\{u_n\}$ converges strongly to a unique solution u^* of $VI(F(W), A)$.*

3. Main theorems

In this section, we show a strong convergence theorem for finding a solution of the variational inequality problem over the intersection of fixed point sets of infinite nonexpansive mappings. Before proving the theorem, we need the following lemma which is essentially used in the proof.

Lemma 3.1. *Let H be a real Hilbert space and let $\alpha, \beta > 0$. Let $T_1, T_2, \dots$ be nonexpansive mappings on H such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and let $\gamma_1, \gamma_2, \dots$ be real numbers such that $0 < a \le \gamma_i \le b < 1$ for all $i = 1, 2, \dots$ and some $a, b \in (0,1)$ with $a \le b$. For any $n \in \mathbb{N}$, let W_n be the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\gamma_n, \gamma_{n-1}, \dots, \gamma_1$ and let A be an α-inverse strongly monotone and β-Lipschitzian mapping on H. Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = (I - \lambda_n \rho A) W_n x_n \tag{4}$$

for all $n \in \mathbb{N}$, where $\rho \in (0, 2\alpha/\beta^2)$, and $\{\lambda_n\} \subset (0,1]$ satisfies $\lim_{n\to\infty} \lambda_n = 0$. Then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof. Putting $T_n = (I - \lambda_n \rho A) W_n$, we can rewrite (4) to $x_{n+1} = T_n x_n$. Let $u \in \bigcap_{n=1}^{\infty} F(T_n)$. It follows from Lemma 2.2 that

$$\begin{aligned}
\|x_{n+1} - u\| &= \|T_n x_n - u\| \\
&\le \|T_n x_n - T_n u\| + \|T_n u - u\| \\
&\le (1 - \lambda_n r)\|x_n - u\| + \|(I - \lambda_n \rho A) W_n u - W_n u\| \\
&= (1 - \lambda_n r)\|x_n - u\| + \lambda_n \rho \|Au\| \\
&= (1 - \lambda_n r)\|x_n - u\| + \lambda_n r \frac{\rho}{r} \|Au\| \\
&\le \max\left\{ \|x_n - u\|, \frac{\rho}{r}\|Au\| \right\},
\end{aligned}$$

where $r = 1 - \sqrt{1-\rho(2\alpha - \rho\beta^2)} \in (0,1)$. By induction, we get

$$\|x_n - u\| \leq \max\left\{\|x_1 - u\|, \frac{\rho}{r}\|Au\|\right\} =: K$$

and hence $\{x_n\}$ is bounded. So, $\{T_n x_n\}$ is also bounded. From (4), we note that

$$\begin{aligned}
x_{n+1} &= (I - \lambda_n \rho A) W_n x_n \\
&= \lambda_n (I - \rho A) W_n x_n + (1-\lambda_n) W_n x_n \\
&= \lambda_n (I - \rho A) W_n x_n + (1-\lambda_n)(\gamma_1 T_1 U_{n,2} x_n + (1-\gamma_1) x_n).
\end{aligned}$$

Put

$$y_n = \frac{\lambda_n (I-\rho A) W_n x_n + (1-\lambda_n)\gamma_1 T_1 U_{n,2} x_n}{\lambda_n + (1-\lambda_n)\gamma_1}.$$

Then, we have

$$\begin{aligned}
\|y_n - u\| &= \left\| \frac{\lambda_n \{(I-\rho A) W_n x_n - u\} + (1-\lambda_n)\gamma_1 (T_1 U_{n,2} x_n - u)}{\lambda_n + (1-\lambda_n)\gamma_1} \right\| \\
&\leq \frac{\lambda_n \|(I-\rho A) W_n x_n - u\| + (1-\lambda_n)\gamma_1 \|T_1 U_{n,2} x_n - u\|}{\lambda_n + (1-\lambda_n)\gamma_1} \\
&\leq \frac{1}{\lambda_n + (1-\lambda_n)\gamma_1} \{\lambda_n \|(I-\rho A) W_n x_n - (I-\rho A) W_n u\| \\
&\quad + \lambda_n \|(I-\rho A) W_n u - u\| + (1-\lambda_n)\gamma_1 \|T_1 U_{n,2} x_n - u\|\} \\
&\leq \frac{1}{\lambda_n + (1-\lambda_n)\gamma_1} \{\lambda_n (1-r) \|x_n - u\| + \lambda_n \rho \|Au\| \\
&\quad + (1-\lambda_n)\gamma_1 \|U_{n,2} x_n - u\|\} \\
&\leq \frac{\lambda_n (1-r) K + \lambda_n r K + (1-\lambda_n)\gamma_1 K}{\lambda_n + (1-\lambda_n)\gamma_1} \\
&= K.
\end{aligned}$$

So, the sequence $\{y_n\}$ is also bounded. Furthermore, we have that

$$\begin{aligned}
&\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \\
&= \limsup_{n\to\infty} \left(\left\| \frac{\lambda_{n+1} (I-\rho A) W_{n+1} x_{n+1} + (1-\lambda_{n+1})\gamma_1 T_1 U_{n+1,2} x_{n+1}}{\lambda_{n+1} + (1-\lambda_{n+1})\gamma_1} \right.\right. \\
&\qquad \left.\left. - \frac{\lambda_n (I-\rho A) W_n x_n + (1-\lambda_n)\gamma_1 T_1 U_{n,2} x_n}{\lambda_n + (1-\lambda_n)\gamma_1} \right\| - \|x_{n+1} - x_n\| \right) \\
&\leq \limsup_{n\to\infty} \left(\left\| \frac{\lambda_{n+1} (I-\rho A) W_{n+1} x_{n+1} - \lambda_{n+1} (I-\rho A) W_{n+1} x_n}{\lambda_{n+1} + (1-\lambda_{n+1})\gamma_1} \right\| \right. \\
&\qquad + \frac{\lambda_{n+1}}{\lambda_{n+1} + (1-\lambda_{n+1})\gamma_1} \|(I-\rho A) W_{n+1} x_n - (I-\rho A) W_n x_n\| \\
&\qquad + \left| \frac{\lambda_{n+1}}{\lambda_{n+1} + (1-\lambda_{n+1})\gamma_1} - \frac{\lambda_n}{\lambda_n + (1-\lambda_n)\gamma_1} \right| \|(1-\rho A) W_n x_n\| \\
&\qquad + \frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1} + (1-\lambda_{n+1})\gamma_1} \|T_1 U_{n+1,2} x_{n+1} - T_1 U_{n,2} x_{n+1}\| \\
&\qquad + \frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1} + (1-\lambda_{n+1})\gamma_1} \|T_1 U_{n,2} x_{n+1} - T_1 U_{n,2} x_n\| \\
&\qquad \left. + \left| \frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1} + (1-\lambda_{n+1})\gamma_1} - \frac{(1-\lambda_n)\gamma_1}{\lambda_n + (1-\lambda_n)\gamma_1} \right| \|T_1 U_{n,2} x_n\| - \|x_{n+1} - x_n\| \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}\|x_{n+1}-x_n\|\right.\\
&\qquad+\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}\|W_{n+1}x_n-W_nx_n\|\\
&\qquad+\left|\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{\lambda_n}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\|(1-\rho A)W_nx_n\|\\
&\qquad+\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}\|T_1U_{n+1,2}x_{n+1}-T_1U_{n,2}x_{n+1}\|\\
&\qquad+\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}\|x_{n+1}-x_n\|\\
&\qquad\left.+\left|\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{(1-\lambda_n)\gamma_1}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\|T_1U_{n,2}x_n\|-\|x_{n+1}-x_n\|\right)\\
&=\limsup_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}\|W_{n+1}x_n-W_nx_n\|\right.\\
&\qquad+\left|\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{\lambda_n}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\|(1-\rho A)W_nx_n\|\\
&\qquad+\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}\|T_1U_{n+1,2}x_{n+1}-T_1U_{n,2}x_{n+1}\|\\
&\qquad\left.+\left|\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{(1-\lambda_n)\gamma_1}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\|T_1U_{n,2}x_n\|\right)
\end{aligned}
$$

and that

$$
\begin{aligned}
\|W_{n+1}x_n-W_nx_n\| &= \|U_{n+1,1}x_n-U_{n,1}x_n\|\\
&= \|\gamma_1T_1U_{n+1,2}x_n+(1-\gamma_1)x_n-\{\gamma_1T_1U_{n,2}x_n+(1-\gamma_1)x_n\}\|\\
&= \gamma_1\|T_1u_{n+1,2}x_n-T_1U_{n,2}x_n\|\\
&\leq \gamma_1\|U_{n+1,2}x_n-U_{n,2}x_n\|\\
&= \gamma_1\|\gamma_2T_2U_{n+1,3}x_n+(1-\gamma_2)x_n-\{\gamma_2T_2U_{n,3}x_n+(1-\gamma_2)x_n)\}\|\\
&= \gamma_1\gamma_2\|T_2U_{n+1,3}x_n-T_2U_{n,3}x_n\|\\
&\leq \gamma_1\gamma_2\|U_{n+1,3}x_n-U_{n,3}x_n\|\\
&\ \ \vdots\\
&\leq \left(\prod_{i=1}^{n}\gamma_i\right)\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\|\\
&= \left(\prod_{i=1}^{n}\gamma_i\right)\|\gamma_{n+1}T_{n+1}U_{n+1,n+2}x_n+(1-\gamma_{n+1})x_n-x_n\|\\
&= \left(\prod_{i=1}^{n+1}\gamma_i\right)\|T_{n+1}x_n-x_n\|\\
&\leq b^{n+1}\|T_{n+1}x_n-x_n\|.
\end{aligned}
$$

Similarly, we have that

$$
\|T_1U_{n+1,2}x_{n+1}-T_1U_{n,2}x_{n+1}\|\leq b^n\|T_{n+1}x_{n+1}-x_{n+1}\|.
$$

So, we have

$$
\begin{aligned}
&\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\\
&\leq \limsup_{n\to\infty}\Bigg(\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}b^{n+1}\|T_{n+1}x_n-x_n\|\\
&\qquad+\left|\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{\lambda_n}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\|(1-\rho A)W_nx_n\|\\
&\qquad+\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}b^n\|T_{n+1}x_{n+1}-x_{n+1}\|\\
&\qquad+\left|\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{(1-\lambda_n)\gamma_1}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\|T_1U_{n,2}x_n\|\Bigg)\\
&\leq \limsup_{n\to\infty}\Bigg(\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}b^{n+1}\\
&\qquad+\left|\frac{\lambda_{n+1}}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{\lambda_n}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\\
&\qquad+\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}b^n\\
&\qquad+\left|\frac{(1-\lambda_{n+1})\gamma_1}{\lambda_{n+1}+(1-\lambda_{n+1})\gamma_1}-\frac{(1-\lambda_n)\gamma_1}{\lambda_n+(1-\lambda_n)\gamma_1}\right|\Bigg)L,
\end{aligned}
$$

where $L=\max\{2K, K+\|(I-\rho A)u\|\}$ with $u\in\bigcap_{n=1}^\infty F(T_n)$. Since $\lim_{n\to\infty}\lambda_n=0$ and $\gamma_n\leq b<1$ for all $n\in\mathbb{N}$, we obtain

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Now, we note that

$$x_{n+1}=(1-\lambda_n)(1-\gamma_1)x_n+\{1-(1-\lambda_n)(1-\gamma_1)\}y_n$$

for all $n\in\mathbb{N}$ and that

$$0<\liminf_{n\to\infty}(1-\lambda_n)(1-\gamma_1)\leq\limsup_{n\to\infty}(1-\lambda_n)(1-\gamma_1)<1.$$

From Lemma 2.4, we get $\lim_{n\to\infty}\|y_n-x_n\|=0$. Therefore we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\{1-(1-\lambda_n)(1-\gamma_1)\}\|y_n-x_n\|=0.$$

This completes the proof. □

We are now in a position to prove our main theorem.

Theorem 3.1. *Let H be a real Hilbert space and let $\alpha,\beta>0$. Let $T_1,T_2,\dots$ be nonexpansive mappings on H such that $\bigcap_{n=1}^\infty F(T_n)$ is nonempty and let $\gamma_1,\gamma_2,\dots$ be real numbers such that $0<a\leq\gamma_i\leq b<1$ for all $i=1,2,\dots$ and some $a,b\in(0,1)$ with $a\leq b$. For any $n\in\mathbb{N}$, let W_n be the W-mapping generated by $T_n,T_{n-1},\dots,T_1$ and $\gamma_n,\gamma_{n-1},\dots,\gamma_1$ and let A be an α-inverse strongly monotone and β-Lipschitzian mapping on H. Suppose that $\{x_n\}$ is a sequence generated by $x_1\in H$ and*

$$x_{n+1}=(I-\lambda_n\rho A)W_nx_n$$

for all $n\in\mathbb{N}$, where $\rho\in(0,2\alpha/\beta^2)$, and $\{\lambda_n\}\subset(0,1]$ satisfies (C1) and (C2), that is,

$$\lim_{n\to\infty}\lambda_n=0 \text{ and } \sum_{n=1}^\infty\lambda_n=\infty.$$

Then the sequence $\{x_n\}$ *converges strongly to a unique solution* u^* *of* $VI(\bigcap_{n=1}^{\infty} F(T_n), A)$.

Proof. As in the proof of Lemma 3.1, $\{x_n\}$ and $\{AW_nx_n\}$ are bounded. Let W be the W-mapping generated by $T_1, T_2, \dots$ and $\gamma_1, \gamma_2, \dots$ such that $Wx = \lim_{n\to\infty} W_nx$ for each $x \in H$. Using the mapping W and $\lambda_k = 1/k$ in Theorem 2.3, we have from Lemma 2.3 that there exists the sequence $\{u_k\}$ such that $\{u_k\}$ converges strongly to a unique solution u^* of $VI(\bigcap_{n=1}^{\infty} F(T_n), A)$. Then we have that for all $n, k \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - Wu_k\| &= \|W_nx_n - \lambda_n\rho AW_nx_n - Wu_k\| \\ &\le \|W_nx_n - W_nu_k\| + \|W_nu_k - Wu_k\| + \lambda_n\rho\|AW_nx_n\| \\ &\le \|x_n - u_k\| + \|W_nu_k - Wu_k\| + \lambda_n\rho\|AW_nx_n\|.\end{aligned}$$

Since $\lim_{n\to\infty}\lambda_n = 0$ and $Wu_k = \lim_{n\to\infty} W_nu_k$ for each $k \in \mathbb{N}$, for any Banach limit μ, we obtain

$$\mu_n\|x_n - Wu_k\|^2 = \mu_n\|x_{n+1} - Wu_k\|^2 \le \mu_n\|x_n - u_k\|^2. \tag{5}$$

From the definition of $\{u_k\}$, we have

$$\begin{aligned}x_n - u_k &= x_n - (Wu_k - \frac{1}{k}\rho AWu_k) \\ &= \left(1 - \frac{1}{k}\right)(x_n - Wu_k) + \frac{1}{k}(x_n - Wu_k + \rho AWu_k)\end{aligned}$$

and hence

$$\left(1 - \frac{1}{k}\right)(x_n - Wu_k) = (x_n - u_k) - \frac{1}{k}(x_n - Wu_k + \rho AWu_k).$$

So we have

$$\begin{aligned}&\left(1 - \frac{1}{k}\right)^2\|x_n - Wu_k\|^2 \\ &\ge \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - u_k, x_n - Wu_k + \rho AWu_k\rangle \\ &= \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - u_k, x_n - u_k + u_k - Wu_k + \rho AWu_k\rangle \\ &= \left(1 - \frac{2}{k}\right)\|x_n - u_k\|^2 + \frac{2}{k}\langle x_n - u_k, -u_k + Wu_k - \rho AWu_k\rangle.\end{aligned}$$

From (5), we have

$$\begin{aligned}\left(1 - \frac{1}{k}\right)^2\mu_n\|x_n - u_k\|^2 &\ge \left(1 - \frac{1}{k}\right)^2\mu_n\|x_n - Wu_k\|^2 \\ &= \left(1 - \frac{2}{k}\right)\mu_n\|x_n - u_k\|^2 \\ &\quad + \frac{2}{k}\mu_n\langle x_n - u_k, -u_k + Wu_k - \rho AWu_k\rangle\end{aligned}$$

and hence

$$\frac{1}{2k}\mu_n\|x_n - u_k\|^2 \ge \mu_n\langle x_n - u_k, -u_k + Wu_k - \rho AWu_k\rangle$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$, from $u^* \in VI(F(W), A)$ we obtain

$$\begin{aligned}0 &\ge \mu_n\langle x_n - u^*, -u^* + Wu^* - \rho AWu^*\rangle \\ &= \mu_n\langle x_n - u^*, -\rho AWu^*\rangle\end{aligned}$$

and hence

$$0 \geq \mu_n \langle x_n - u^*, -AWu^* \rangle .$$

In addition, from Lemma 3.1 we have

$$\begin{aligned} &\lim_{n\to\infty} |\langle x_{n+1} - u^*, -AWu^* \rangle - \langle x_n - u^*, -AWu^* \rangle| \\ = &\lim_{n\to\infty} |\langle x_{n+1} - x_n, -AWu^* \rangle| = 0. \end{aligned}$$

Therefore, it follows from Lemma 2.5 that

$$0 \geq \limsup_{n\to\infty} \langle x_n - u^*, -AWu^* \rangle . \tag{6}$$

Finally, we show that $\lim_{n\to\infty} \|x_n - u^*\| = 0$. In fact, from Lemma 2.7 we have

$$\begin{aligned} \|x_{n+1} - u^*\|^2 &= \|T_n x_n - u^*\|^2 \\ &= \|(T_n x_n - T_n u^*) + (T_n u^* - u^*)\|^2 \\ &\leq \|T_n x_n - T_n u^*\|^2 + 2 \langle x_{n+1} - u^*, T_n u^* - u^* \rangle \\ &= \|T_n x_n - T_n u^*\|^2 + 2 \langle x_{n+1} - u^*, W_n u^* - \lambda_n \rho A W_n u^* - u^* \rangle \\ &\leq (1 - \lambda_n r) \|x_n - u^*\|^2 + 2\lambda_n \rho \langle x_{n+1} - u^*, -AW_n u^* \rangle \\ &= (1 - \lambda_n r) \|x_n - u^*\|^2 + \lambda_n r \left\{ \frac{2\rho}{r} (\langle x_{n+1} - u^*, -AWu^* \rangle \right. \\ &\quad \left. + \langle x_{n+1} - u^*, AWu^* - AW_n u^* \rangle) \right\} \\ &\leq (1 - \lambda_n r) \|x_n - u^*\|^2 + \lambda_n r \left\{ \frac{2\rho}{r} (\langle x_{n+1} - u^*, -AWu^* \rangle \right. \\ &\quad \left. + M \|Wu^* - W_n u^*\|) \right\}, \end{aligned}$$

where $M = \beta \sup_{n\in\mathbb{N}} \|x_n - u^*\|$. From Lemma 2.6 and (6), we obtain that the sequence $\{x_n\}$ converges strongly to a unique solution u^* of $VI(\bigcap_{n=1}^{\infty} F(T_n), A)$. □

Using Theorem 3.1, we obtain the following theorem for finding a solution of the variational inequality problem over the intersection of fixed point sets of finite nonexpansive mappings.

Theorem 3.2. *Let H be a real Hilbert space and let $\alpha, \beta > 0$. Let $T_1, T_2, \ldots, T_r$ be nonexpansive mappings on H such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty and let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be real numbers such that $0 < a \leq \gamma_i \leq b < 1$ for all $i = 1, 2, \ldots, r$ and some $a, b \in (0, 1)$ with $a \leq b$. Let W be the W-mapping generated by $T_1, T_2, \ldots, T_r$ and $\gamma_1, \gamma_2, \ldots, \gamma_r$ and let A be an α-inverse strongly monotone and β-Lipschitzian mapping on H. Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = (I - \lambda_n \rho A) W x_n$$

for all $n \in \mathbb{N}$, where $\rho \in (0, 2\alpha/\beta^2)$, and $\{\lambda_n\} \subset (0, 1]$ satisfies (C1) and (C2), that is,

$$\lim_{n\to\infty} \lambda_n = 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = \infty.$$

Then, $\{x_n\}$ converges strongly to a unique solution u^* of $VI(\bigcap_{i=1}^{r} F(T_i), A)$.

The following theorem is connected with the projected gradient method; see Theorem 2.1.

Theorem 3.3. *Let H be a real Hilbert space and let $\alpha, \beta > 0$. Let $C_1, C_2, \dots$ be nonempty closed convex subsets of H such that $C := \bigcap_{n=1}^{\infty} C_n$ is nonempty, let P_{C_n} be the metric projections of H onto C_n for each $n \in \mathbb{N}$ and let $\gamma_1, \gamma_2, \dots$ be real numbers such that $0 < a \leq \gamma_i \leq b < 1$ for all $i = 1, 2, \dots$ and some $a, b \in (0,1)$ with $a \leq b$. For any $n \in \mathbb{N}$, let W_n be the W-mapping generated by $P_{C_1}, P_{C_2}, \dots$ and $\gamma_1, \gamma_2, \dots$ and let A be an α-inverse strongly monotone and β-Lipschitzian mapping on H. Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = (I - \lambda_n \rho A) W_n x_n$$

for all $n \in \mathbb{N}$, where $\rho \in (0, 2\alpha/\beta^2)$, and $\{\lambda_n\} \subset (0,1]$ satisfies (C1) and (C2), that is,

$$\lim_{n\to\infty} \lambda_n = 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = \infty.$$

Then, the sequence $\{x_n\}$ converges strongly to a unique solution u^* of $VI(C, A)$.

Proof. Since $F(W) = \bigcap_{n=1}^{\infty} F(P_{C_n}) = \bigcap_{n=1}^{\infty} C_n = C$, from Theorem 3.1 we obtain the conclusion. □

4. Application

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a bounded linear operator and let $A(C)$ be the range of A. Given an element $b \in H$, consider the following convexly constrained linear inverse problem:

$$\text{Find } z \in C \text{ such that } z \in \operatorname*{argmin}_{x\in C} \|Ax - b\|^2 . \tag{7}$$

We denote the set of solutions (7) by S_b. Then it is known that S_b is nonempty if and only if

$$P_{\overline{A(C)}}(b) \in A(C),$$

where P_C is the metric projection of H onto C. Indeed, the necessary part is trivial. Suppose that there exists $z \in C$ such that $z \in \operatorname{argmin}_{x\in C} \|Ax - b\|^2$. Then we have

$$\|Az - b\|^2 \leq \|y - b\|^2 \tag{8}$$

for all $y \in A(C)$. Let y_0 be in $\overline{A(C)}$. Then there exists a sequence $\{y_n\} \subset A(C)$ such that $y_n \to y_0$ as $n \to \infty$. So, from (8) we have

$$\|Az - b\|^2 \leq \|y_n - b\|^2 .$$

Letting $n \to \infty$, we get

$$\|Az - b\|^2 \leq \|y_0 - b\|^2 .$$

Since $y_0 \in \overline{A(C)}$ is arbitrary, we obtain $P_{\overline{A(C)}}(b) = Az \in A(C)$. If S_b is nonempty, then we know that S_b is closed and convex because of the continuity of A. In this case, S_b has a unique element $\bar{z} \in S_b$ with minimum norm, that is, $\bar{z} \in S_b$ satisfies

$$\|\bar{z}\|^2 = \min\{\|x\|^2 : x \in S_b\}. \tag{9}$$

The C-constrained pseudoinverse of A (denoted by $A_C^{\dagger}$) is defined as

$$D(A_C^{\dagger}) = \{b \in H : P_{\overline{A(C)}}(b) \in A(C)\},$$
$$A_C^{\dagger}(b) = \bar{z},\ b \in D(A_C^{\dagger}),$$

where $\bar{z} \in S_b$ is a unique solution to (9).

We now introduce the C-constrained generalized pseudoinverse of A; see [13]. Let f be a Fréchet differentiable convex function from H to $\mathbb{R}$ such that ∇f is a k-Lipschitzian and α-strongly monotone operator for some $k > 0$ and $\alpha > 0$. Under these assumptions, there exists a unique element $\bar{z}^\dagger \in S_b$ for $b \in D(A_C^\dagger)$ such that

$$f(\bar{z}^\dagger) = \min\{f(x) : x \in S_b\}. \tag{10}$$

The C-constrained generalized pseudoinverse of A associated with f (denoted by $A_{C,f}^\dagger$) is defined as

$$\begin{aligned} &D(A_{C,f}^\dagger) = D(A_C^\dagger), \\ &A_{C,f}^\dagger(b) = \bar{z}^\dagger, \ b \in D(A_{C,f}^\dagger), \end{aligned}$$

where $\bar{z}^\dagger \in S_b$ is a unique solution to (10).

We now apply our main theorem to construct the C-constrained generalized pseudoinverse $A_{C,f}^\dagger$ of A. We know from (3) that

$$\begin{aligned} z \in S_b &\iff Az = P_{\overline{A(C)}}(b) \in A(C) \\ &\iff \left\langle b - P_{\overline{A(C)}}(b), P_{\overline{A(C)}}(b) - Ax \right\rangle \geq 0 \text{ for all } x \in C \\ &\iff \langle b - Az, Az - Ax \rangle \geq 0 \text{ for all } x \in C \\ &\iff \langle A^*(Az - b), x - z \rangle \geq 0 \text{ for all } x \in C, \end{aligned}$$

where A^* is the adjoint of A. This means that for each $r > 0$,

$$\langle \{rA^*b + (I - rA^*A)z - z\}, x - z \rangle \geq 0 \text{ for all } x \in C,$$

and hence

$$P_C(rA^*b + (I - rA^*A)z) = z. \tag{11}$$

Now, assume that $\{S_b^1, S_b^2 \dots\}$ is a family of the solution sets of (7) for $\{C^1, C^2, \dots\}$ such that

$$S_b := \bigcap_{i=1}^\infty S_b^i \neq \emptyset.$$

For each $i \in \mathbb{N}$, define a mapping $T_i : H \to H$ by

$$T_i x = P_{C^i}(rA^*b + (I - rA^*A)x) \text{ for all } x \in H, \tag{12}$$

where P_{C^i} is the metric projection of H onto C^i. The following lemma was shown by Xu and Kim [12].

Lemma 4.1. *Let H be a real Hilbert space and $b \in H$. Let S_b^i be a family of the solution sets of (7) such that $\bigcap_{i=1}^\infty S_b^i \neq \emptyset$ and let T_i be a mapping of H onto C^i which is defined by (12) for each $i \in \mathbb{N}$. If $r \in (0, 2/\|A\|^2)$ and $b \in D(A_C^\dagger)$, then T_i is nonexpansive and $F(T_i) = S_b^i$ for all $i \in \mathbb{N}$.*

Using these settings, we obtain the following theorem.

Theorem 4.1. *Let H be a real Hilbert space. Let $T_1, T_2, \dots$ be mappings on H defined by (12) such that $S_b := \bigcap_{n=1}^\infty F(T_n)$ is nonempty and let $\gamma_1, \gamma_2, \dots$ be real numbers such that $0 < a \leq \gamma_i \leq b < 1$ for all $i \in \mathbb{N}$ and some a, b with $a \leq b$. For any $n \in \mathbb{N}$, let W_n be the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\gamma_n, \gamma_{n-1}, \dots, \gamma_1$ and let f be a Fréchet differentiable convex function from H to $\mathbb{R}$ such that ∇f is a k-Lipschitzian and α-strongly* *monotone mapping on H for some $k, \alpha > 0$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = (I - \lambda_n \rho \nabla f) W_n x_n \tag{13}$$

for all $n \in \mathbb{N}$, where $\rho \in (0, 2\alpha/k^2)$, and $\{\lambda_n\} \subset (0,1]$ satisfies (C1) and (C2), that is,

$$\lim_{n\to\infty} \lambda_n = 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = \infty.$$

Then, the sequence $\{x_n\}$ converges strongly to $A^{\dagger}_{S_b,f}(b)$ which is a unique solution of (10).

Proof. Put $A^{\dagger}_C(b) = \bar{z}^{\dagger}$ for $b \in D(A^{\dagger}_C)$. Then we have that

$$\begin{aligned} \bar{z}^{\dagger} \in \operatorname*{argmin}_{x\in S_b} f(x) &\iff 0 \in \nabla f(\bar{z}^{\dagger}) + N_{S_b}(\bar{z}^{\dagger}) \\ &\iff \left\langle \nabla f(\bar{z}^{\dagger}), x - \bar{z}^{\dagger} \right\rangle \geq 0 \text{ for all } x \in S_b \\ &\iff \bar{z}^{\dagger} \in VI(S_b, \nabla f), \end{aligned}$$

where N_{S_b} is the normal cone to S_b; see [10] for more details. So, it follows from Theorem 3.1 and Lemma 4.1 that the sequence $\{x_n\}$ generated by (13) converges strongly to $A^{\dagger}_{S_b,f}(b)$. □

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