FIXED POINT RESULTS FOR ĆIRIĆ TYPE CONTRACTIONS ON A SET WITH TWO SEPARATING GAUGE STRUCTURES

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Abstract. The purpose of this article is to present some local and global fixed point results (existence of the fixed point, well-posedness for the fixed point problem, homotopy theorem) for Ćirić type contractions on a set with two separating gauge structures.

1 Preliminaries Throughout this paper X will denote a gauge space endowed with a separating gauge structure $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$, where A is a directed set (see [4] for definitions). Let $\mathbb{N} := \{0, 1, 2, \cdots\}$ and let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

A sequence (x_n) of elements in X is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_n, x_{n+p}) \le \varepsilon$ for all $n \ge N$ and $p \in \mathbb{N}^*$. The sequence (x_n) is called convergent if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_0, x_n) \le \varepsilon$ for all $n \ge N$.

A gauge space is called sequentially complete if any Cauchy sequence is convergent. A subset of X is said to be sequentially closed if it contains the limit of any convergent sequence of its elements.

If $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$ are two separating gauge structures (A, B are directed sets), then for $r = \{r_\beta\}_{\beta\in B} \in (0,\infty)^B$ and $x_0 \in X$ we will denote by $\overline{B}^p_q(x_0, r)$ the closure of $B_q(x_0, r)$ in $(X, \mathcal{P})$, where

$$B_q(x_0, r) = \{x \in X \ : q_\beta(x_0, x) < r_\beta \text{ for all } \beta \in B\}.$$

Let $P((X,\mathcal{P}))$ be the set of all nonempty subsets of X endowed with the convergence given by the family $\mathcal{P}$. We will use the following symbols when there is no confusion:

$$P(X) := \{Y \in \mathcal{P}(X) : \ Y \ne \emptyset\}; P_b(X) := \{Y \in P(X) : \ Y \text{ is bounded }\};$$

$$P_{cl}(X) := \{Y \in P(X) : \ Y \text{ is closed }\}.$$

Let us define the gap functional between Y and Z in the $(X,\mathcal{Q})$ gauge space

$$D_\beta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ D_\beta(Y,Z) = \inf\{q_\beta(y, z) \mid y \in Y,\ z \in Z\}$$

(in particular, if $x_0 \in X$ then $D_\beta(x_0, Z) := D_\beta(\{x_0\}, Z))$ and the (generalized) Pompeiu-Hausdorff functional

$$H_\beta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_\beta(Y,Z) = \max\{\sup_{y\in Y} D_\beta(y, Z), \sup_{z\in Z} D_\beta(Y, z)\}.$$

If $F : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for F if and only if $x \in F(x)$. The set $Fix(F) := \{x \in X | x \in F(x)\}$ is called the fixed point set of F, while $SFix(F) := \{x \in X | \{x\} = F(x)\}$ denotes the strict fixed point set of F.

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Recall that, in 1972, L.B. Ćirić ([3]) proved that if (X,d) is a complete metric space, $F : X \to P_{cl}(X)$ is a multivalued operator and there exists $\alpha \in (0,1)$ such that, for every $x, y \in X$:

$$H(F(x),F(y)) \leq \alpha \max\{d(x,y), D(x,F(x)), D(y,F(y)), \frac{1}{2}[D(x,F(y)) + D(y,F(x))]\},$$

then $Fix(F) \neq \emptyset$ and for every $x \in X$ and $y \in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(1) $x_0 = x,\ x_1 = y$;

(2) $x_{n+1} \in F(x_n),\ n \in \mathbb{N}$;

(3) $x_n \xrightarrow{d} x^* \in F(x^*)$, for every $n \to \infty$.

The aim of this paper is to present some (local and global) fixed point results (existence of the fixed point, well-posedness for the fixed point problem, homotopy theorem) for Ćirić type contractions on a set with two separating gauge structures. The results of the paper extend and generalize some previous theorems given in R.P. Agarwal, J. Dshalalow, D. O'Regan [1], L.B. Ćirić [3], M. Frigon [6] and [7], T. Lazăr, D. O'Regan, A. Petruşel [8] and they are related to the works A. Chiş, R. Precup [2] and D. O'Regan, R.P. Agarwal, D. Jiang [9].

2 The main results Our first result is a local version of Ćirić's theorem ([3]) on a set with two separating gauge structures. The results relies on the concept of multivalued admissible contraction (in the sense of Frigon, see Frigon [6], [7] and R.P. Agarwal, J. Dshalalow, D. O'Regan [1]) of Ćirić type.

Theorem 2.1 *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$, $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$ (A, B are directed sets), $r = \{r_\beta\}_{\beta\in B} \in (0,\infty)^B$, $x_0 \in X$ and $F : (\overline{B}^p_q(x_0,r),\mathcal{P}) \to P((X,\mathcal{P})$ be a multivalued operator with closed graph. We suppose that:*

(i) $(X,\mathcal{P})$ is a sequentially complete gauge space;
(ii) there exist a function $\psi : A \to B$ and $c = \{c_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ such that

$$p_\alpha(x,y) \leq c_\alpha \cdot q_{\psi(\alpha)}(x,y),\ \text{for every } \alpha \in A \text{ and } x,y \in \overline{B}^p_q(x_0,r).$$

(iii) there exists $\{a_\beta\}_{\beta\in B} \in (0,1)^B$ such that, for every $\beta \in B$, the following implication holds: for each $x, y \in \overline{B}^p_q(x_0,r)$ we have:

$$H_\beta(F(x),F(y)) \leq a_\beta \cdot M^F_\beta(x,y),$$

where

$$M^F_\beta(x,y) := \max\{q_\beta(x,y), D_\beta(x,F(x)), D_\beta(y,F(y)), \frac{1}{2}[D_\beta(x,F(y)) + D_\beta(y,F(x))]\}.$$

In addition, assume that the following conditions are satisfied:

$$D_\beta(x_0,F(x_0)) < (1-a_\beta)r_\beta,\ \text{for each } \beta \in B; \tag{2.1}$$

(2.2) *for every* $x \in \overline{B}^p_q(x_0, r)$ *and every* $k = \{k_\beta\}_{\beta \in B} \in (0, \infty)^B$ *there exists* $y \in F(x)$ *such that* $q_\beta(x, y) \leq D_\beta(x, F(x)) + k_\beta$, *for every* $\beta \in B$.

Then:

(i) $Fix(F) \neq \emptyset$;

(ii) if, additionally, $SFix(F) \neq \emptyset$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B}^p_q(x_0, r)$ *is such that* $D_\beta(x_n, F(x_n)) \to 0$ *as* $n \to \infty$ *for every* $\beta \in B$, *then* $q_\beta(x_n, x) \to 0$ *as* $n \to \infty$, *for every* $\beta \in B$, *where* $x \in SFix(F)$ *(i.e., the fixed point problem is well-posed in the generalized sense for* F *with respect to* D_β, *for every* $\beta \in B$*). Moreover, we also have that* $p_\alpha(x_n, x) \to 0$ *as* $n \to \infty$, *for every* $\alpha \in A$.

Proof. From (2.1) we may choose $x_1 \in F(x_0)$ with
$$q_\beta(x_0, x_1) < (1 - a_\beta)r_\beta \text{ for every } \beta \in B.$$
Thus $x_1 \in \overline{B}^p_q(x_0, r)$.

Hence, there exists $\{a_\beta\}_{\beta\in B} \in (0, 1)^B$ such that, for each $\beta \in B$, we have:

$$\begin{aligned} H_\beta(F(x_0), F(x_1)) &\leq a_\beta \cdot M^F_\beta(x_0, x_1) \\ &= a_\beta \cdot \max\{q_\beta(x_0, x_1), D_\beta(x_0, F(x_0)), D_\beta(x_1, F(x_1)), \\ &\qquad \frac{1}{2}[D_\beta(x_0, F(x_1)) + D_\beta(x_1, F(x_0))]\} \\ &\leq a_\beta \cdot \max\{q_\beta(x_0, x_1), D_\beta(x_1, F(x_1)), \\ &\qquad \frac{1}{2}[q_\beta(x_0, x_1) + D_\beta(x_1, F(x_1))]\} \\ &= a_\beta \cdot \max\{q_\beta(x_0, x_1), D_\beta(x_1, F(x_1))\}. \end{aligned}$$

Let $N^F_\beta(x_0, x_1) := \max\{q_\beta(x_0, x_1), D_\beta(x_1, F(x_1))\}$.

If $N^F_\beta(x_0, x_1) = D_\beta(x_1, F(x_1))$, then

$$H_\beta(F(x_0), F(x_1)) \leq a_\beta \cdot D_\beta(x_1, F(x_1)) \leq a_\beta \cdot H_\beta(F(x_0), F(x_1))$$

which is a contradiction, since $\{a_\beta\}_{\beta\in B} \in (0, 1)^B$. Thus $N^F_\beta(x_0, x_1) = q_\beta(x_0, x_1)$. Then, we have:

$$H_\beta(F(x_0), F(x_1)) \leq a_\beta \cdot q_\beta(x_0, x_1), \text{ for every } \beta \in B.$$

Denote $k_\beta := a_\beta[(1 - a_\beta)r_\beta - q_\beta(x_0, x_1)]$ for every $\beta \in B$.

On the other hand, by (2.2) there exists $x_2 \in F(x_1)$ such that

$$q_\beta(x_1, x_2) \leq D_\beta(x_1, F(x_1)) + k_\beta, \text{ for every } \beta \in B.$$

Hence

$$q_\beta(x_1, x_2) \leq H_\beta(F(x_0), F(x_1)) + k_\beta \leq a_\beta q_\beta(x_0, x_1) + k_\beta = a_\beta(1 - a_\beta)r_\beta \text{ for every } \beta \in B.$$

Moreover we have

$$\begin{aligned} q_\beta(x_0, x_2) &\leq q_\beta(x_0, x_1) + q_\beta(x_1, x_2) \\ &< (1 - a_\beta)r_\beta + a_\beta(1 - a_\beta)r_\beta \\ &= (1 - a^2_\beta)r_\beta < r_\beta. \end{aligned}$$

Therefore $x_2 \in \overline{B}^p_q(x_0, r)$.

By the same procedure, we obtain the elements $x_{n+1} \in F(x_n)$ for $n \in \{1, 2, 3, ...\}$ having the following properties:

(a) $q_\beta(x_{n-1}, x_n) \le a_\beta^{n-1} q_\beta(x_0, x_1) \le a_\beta^{n-1}(1-a_\beta)r_\beta,$ for every $n \in \mathbb{N}^*$ and $\beta \in B$;

(b) $q_\beta(x_0, x_n) < (1-a_\beta^n)r_\beta,$ for every $n \in \mathbb{N}^*$ and $\beta \in B$.

From (a) it is immediate that $\{x_n\}$ is $\mathcal{Q}$-Cauchy. Now (ii) implies that the sequence $\{x_n\}$ is $\mathcal{P}$-Cauchy too, hence it is $\mathcal{P}$-convergent to some $x \in \overline{B}_q^p(x_0, r)$, from (i). It now remains to show that $x \in F(x)$. This follows since $F : (\overline{B}_q^p(x_0, r), \mathcal{P}) \to P((X, \mathcal{P})$ has closed graph.

Let $x \in SFix(F)$. We now show that the fixed point problem is well-posed. Let $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B}_q^p(x_0, r)$ be such that $D_\beta(x_n, F(x_n)) \to 0$ as $n \to \infty$ for every $\beta \in B$. Since x is a strict fixed point for F, we have $q_\beta(x_n, x) \le D_\beta(x_n, F(x_n)) + H_\beta(F(x_n), F(x))$. Then:

$$\begin{aligned}
q_\beta(x_n, x) &\le D_\beta(x_n, F(x_n)) + a_\beta M_\beta^F(x_n, x) \\
&= D_\beta(x_n, F(x_n)) + a_\beta \max\{q_\beta(x_n, x), D_\beta(x_n, F(x_n)), \\
&\qquad\qquad \frac{1}{2}[D_\beta(x_n, F(x)) + D_\beta(x, F(x_n))]\} \\
&\le D_\beta(x_n, F(x_n)) + a_\beta \max\{q_\beta(x_n, x), D_\beta(x_n, F(x_n)), \\
&\qquad\qquad \frac{1}{2}[q_\beta(x_n, x) + q_\beta(x, x_n) + D_\beta(x_n, F(x_n))]\} \\
&\le D_\beta(x_n, F(x_n)) + a_\beta \max\{D_\beta(x_n, F(x_n)), q_\beta(x_n, x) + \frac{1}{2}D_\beta(x_n, F(x_n))\}.
\end{aligned}$$

If $\max\{D_\beta(x_n, F(x_n)), q_\beta(x_n, x) + \frac{1}{2}D_\beta(x_n, F(x_n))\} = D_\beta(x_n, F(x_n))$, then

$$q_\beta(x_n, x) \le (1 + a_\beta)D_\beta(x_n, F(x_n)).$$

If $\max\{D_\beta(x_n, F(x_n)), q_\beta(x_n, x) + \frac{1}{2}D_\beta(x_n, F(x_n))\} = q_\beta(x_n, x) + \frac{1}{2}D_\beta(x_n, F(x_n))$ then

$$q_\beta(x_n, x) \le D_\beta(x_n, F(x_n)) + a_\beta q_\beta(x_n, x) + \frac{a_\beta}{2}D_\beta(x_n, F(x_n))$$

and we have

$$q_\beta(x_n, x) \le \frac{2 + a_\beta}{2(1 - a_\beta)}D_\beta(x_n, F(x_n)).$$

From (ii), we also obtain that $p_\alpha(x_n, x) \to 0$ as $n \to +\infty$, for every $\alpha \in A$. Thus the proof is complete. □

We will obtain now a global version of the theorem above.

Theorem 2.2 *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$, $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$ and $F : (X, \mathcal{P}) \to P((X, \mathcal{P})$ be a multivalued operator with closed graph. We suppose that:*

(i) $(X, \mathcal{P})$ is a sequentially complete gauge space;
(ii) there exists a function $\psi : A \to B$ and $c = \{c_\alpha\}_{\alpha\in A} \in (0, \infty)^A$ such that

$$p_\alpha(x, y) \le c_\alpha \cdot q_{\psi(\alpha)}(x, y), \text{ for every } \alpha \in A \text{ and } x, y \in X.$$

(iii) there exists $\{a_\beta\}_{\beta\in B} \in (0, 1)^B$ such that, for each $x, y \in X$, we have:

$$H_\beta(F(x), F(y)) \le a_\beta \cdot M_\beta(x, y), \text{ for every } \beta \in B.$$

In addition, assume that

$$\begin{aligned}
&\text{for every } x \in X \text{ and every } k = \{k_\beta\}_{\beta\in B} \in (0, \infty)^B \text{ there exists} \\
&y \in F(x) \text{ such that } q_\beta(x, y) \le D_\beta(x, F(x)) + k_\beta, \text{ for every } \beta \in B.
\end{aligned} \tag{2.3}$$

Then F has a fixed point. Furthermore, if $SFix(F) \neq \emptyset$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$ is such that $D_\beta(x_n, F(x_n)) \to 0$ as $n \to \infty$ for every $\beta \in B$, then $q_\beta(x_n, x) \to 0$ as $n \to \infty$, where $x \in SFix(F)$ (i.e. the fixed point problem is well-posed in the generalized sense for F with respect to D_β, for every $\beta \in B$).

Proof. Let $x_0 \in X$ be arbitrary and choose $r = \{r_\beta\}_{\beta\in B} \in (0,\infty)^B$ such that $D_\beta(x_0, F(x_0)) < (1 - a_\beta)r_\beta$, for each $\beta \in B$. Next, the proof follows from Theorem 2.1. □.

Another global result is based on the concept of multivalued admissible contraction (in the sense of Espínola and Petruşel, see [5]) of Ćirić type.

Theorem 2.3 *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$, $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$ (A, B are directed sets) and let $F : (X, \mathcal{P}) \to P((X, \mathcal{P})$ be a multivalued operator with closed graph. We suppose that:*

(i) $(X, \mathcal{P})$ is a sequentially complete gauge space;

(ii) there exists a function $\psi : A \to B$ and $c = \{c_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ such that

$$p_\alpha(x, y) \leq c_\alpha \cdot q_{\psi(\alpha)}(x, y), \text{ for every } \alpha \in A \text{ and } x, y \in X;$$

(iii) there exists $\{a_\beta\}_{\beta\in B} \in (0, 1)^B$ such that for every $\beta \in B$ we have:

$$H_\beta(F(x), F(y)) \leq a_\beta \cdot M_\beta(x, y), \text{ for each } x, y \in X.$$

(iv) for every $x, y \in X$, every $u \in F(x)$ and every $k = \{k_\beta\}_{\beta\in B} \in (1,\infty)^B$ there exists $v \in F(y)$ such that $q_\beta(u, v) \leq k_\beta \cdot H_\beta(F(x), F(y))$, for every $\beta \in B$;

Then $Fix(F) \neq \emptyset$. Furthermore, if $SFix(F) \neq \emptyset$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$ is such that $D_\beta(x_n, F(x_n)) \to 0$ as $n \to \infty$ for every $\beta \in B$, then $q_\beta(x_n, x) \to 0$ as $n \to \infty$, where $x \in SFix(F)$ (i.e. the fixed point problem is well-posed in the generalized sense for F with respect to D_β, for every $\beta \in B$).

Proof. Let $x_0 \in X$ and $x_1 \in F(x_0)$ be arbitrary. For every $k = \{k_\beta\}_{\beta\in B} \in (1,\infty)^B$, by (iv), there exists $x_2 \in F(x_1)$ such that

$$q_\beta(x_1, x_2) \leq k_\beta H_\beta(F(x_0), F(x_1)), \text{ for each } \beta \in B.$$

Then:

$$\begin{aligned} q_\beta(x_1, x_2) &\leq k_\beta H_\beta(F(x_0), F(x_1)) \\ &\leq k_\beta a_\beta M^F_\beta(x_0, x_1) \\ &= k_\beta a_\beta \max\{q_\beta(x_0, x_1), D_\beta(x_0, F(x_0)), D_\beta(x_1, F(x_1)), \frac{1}{2}D_\beta(x_0, F(x_1))\}. \end{aligned}$$

We introduce the following notation:

$$\Gamma := \max\{q_\beta(x_0, x_1), D_\beta(x_0, F(x_0)), D_\beta(x_1, F(x_1)), \frac{1}{2}D_\beta(x_0, F(x_1))\}$$

and we choose $k = \{k_\beta\}_{\beta\in B} \in (1,\infty)^B$ such that $1 < k_\beta < \frac{1}{a_\beta}$, for each $\beta \in B$.

If $\Gamma = q_\beta(x_0, x_1)$ then $q_\beta(x_1, x_2) \leq k_\beta a_\beta q_\beta(x_0, x_1)$.

If $\Gamma = D_\beta(x_0, F(x_0))$ then since $D_\beta(x_0, F(x_0)) \leq q_\beta(x_0, x_1)$ we have $q_\beta(x_1, x_2) \leq k_\beta a_\beta q_\beta(x_0, x_1)$.

If $\Gamma = D_\beta(x_1, F(x_1))$ then $q_\beta(x_1, x_2) \leq k_\beta a_\beta D_\beta(x_1, F(x_1)) \leq k_\beta a_\beta q_\beta(x_1, x_2)$, which is a contradiction since $1 < k_\beta < \frac{1}{a_\beta}$, for each $\beta \in B$.

If $\Gamma = \frac{1}{2}D_\beta(x_0, F(x_1))$ then

$$\begin{aligned} q_\beta(x_1, x_2) &\leq k_\beta a_\beta \frac{1}{2} D_\beta(x_0, F(x_1)) \leq \frac{k_\beta a_\beta}{2} q_\beta(x_0, x_2) \\ &\leq \frac{k_\beta a_\beta}{2}[q_\beta(x_0, x_1) + q_\beta(x_1, x_2)]. \end{aligned}$$

Hence, we obtain that

$$q_\beta(x_1,x_2) \le \frac{k_\beta a_\beta}{2-k_\beta a_\beta} q_\beta(x_0,x_1).$$

Then

$$\begin{aligned}\Gamma &= \frac{1}{2}D_\beta(x_0,F(x_1)) \le \frac{1}{2}q_\beta(x_0,x_2) \le \frac{1}{2}[q_\beta(x_0,x_1)+q_\beta(x_1,x_2)]\\ &\le \frac{1}{2}[1+\frac{k_\beta a_\beta}{2-k_\beta a_\beta}]q_\beta(x_0,x_1) = \frac{1}{2-k_\beta a_\beta}q_\beta(x_0,x_1) < q_\beta(x_0,x_1),\end{aligned}$$

which is a contradiction with the definition of Γ.

Thus in all cases we have that

$$q_\beta(x_1,x_2) \le k_\beta a_\beta q_\beta(x_0,x_1).$$

By induction, we will obtain a sequence $(x_n)_{n\in\mathbb{N}}$ of succesive approximations for F starting from x_0, satisfying the following assertion:

$$q_\beta(x_n,x_{n+1}) \le (k_\beta a_\beta)^n q_\beta(x_0,x_1), \text{ for every } n\in\mathbb{N}^* \text{ and } \beta\in B.$$

For each $n,p\in\mathbb{N}^*$ and for every $\beta\in B$ we have

$$\begin{aligned}q_\beta(x_n,x_{n+p}) &\le q_\beta(x_n,x_{n+1})+...+q_\beta(x_{n+p-1},x_{n+p})\\ &\le [1+...+(k_\beta a_\beta)^{p-1}]\cdot(k_\beta a_\beta)^n q_\beta(x_0,x_1)\\ &= \frac{1-(k_\beta a_\beta)^p}{1-k_\beta a_\beta}\cdot(k_\beta a_\beta)^n q_\beta(x_0,x_1) \le \frac{(k_\beta a_\beta)^n}{1-k_\beta a_\beta}q_\beta(x_0,x_1).\end{aligned}$$

Letting $n\to+\infty$ and taking into account (i), we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{Q}$-Cauchy. Thus, by (ii), the sequence is convergent in $(X,\mathcal{P})$. Denote by u its limit. Notice that $u\in Fix(F)$ since the operator $F:(X,\mathcal{P})\to P((X,\mathcal{P})$ has closed graph. The second part follows in a similar way as in Theorem 2.1. The proof is complete. □

For the particular case of a unique gauge structure, we get the following data dependence result.

Theorem 2.4 *Let X be a nonempty set endowed with a separating gauge structure $\mathcal{P}=\{p_\alpha\}_{\alpha\in A}$, and let $F_1,F_2:X\to P(X)$ be two multivalued operators with closed graph. We suppose that:*

(i) *$(X,\mathcal{P})$ is a sequentially complete gauge space;*
(ii) *there exists $\{a_\alpha^{(i)}\}_{\alpha\in A}\in(0,1)^A$ such that, for each $x,y\in X$, we have:*

$$H_\alpha(F_i(x),F_i(y)) \le a_\alpha^{(i)}\cdot M_\alpha^{F_i}(x,y), \text{ for every } \alpha\in A, \text{ for } i\in\{1,2\}.$$

In addition, assume that:

(a) for every $x,y\in X$ every $u\in F_i(x)$ and every $k:=\{k_\alpha\}_{\alpha\in B}\in(1,\infty)^A$ there exists $v\in F_i(x)$ such that $q_\alpha(u,v)\le k_\alpha H_\alpha(F_i(x),F_i(y))$, for every $\alpha\in A$, for $i\in\{1,2\}$;

(b) for every $x\in X$ every $u\in F_1(x)$ and every $k:=\{k_\alpha\}_{\alpha\in B}\in(1,\infty)^A$ there exists $v\in F_2(x)$ such that $q_\alpha(u,v)\le k_\alpha H_\alpha(F_1(x),F_2(x))$, for every $\alpha\in A$;

Then:

1)

$$Fix(F_i) \in P_{cl}(X), \text{ for } i \in \{1,2\};$$

2) If there exists $\eta := \{\eta_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ such that $H_\alpha(F_1(x), F_2(x)) \le \eta_\alpha$ for each $\alpha \in A$, then

$$H_\alpha(Fix(F_1), Fix(F_2)) \le \frac{\eta_\alpha}{1-\max\{a_\alpha^{(1)}, a_\alpha^{(2)}\}}, \text{ for each } \alpha \in A.$$

Proof. 1) The existence of the fixed point follows from Theorem 2.3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $Fix(F)$ (where F is F_1 or F_2), such that x_n converges to x^*. Denote $a_\alpha := a_\alpha^{(1)}$ if $F := F_1$ and $a_\alpha := a_\alpha^{(2)}$ if $F := F_2$. Then, we have:
$D_\alpha(x^*, F(x^*)) \le d_\alpha(x^*, x_n) + H_\alpha(F(x_n), F(x^*)) \le d_\alpha(x^*, x_n) + a_\alpha \max\{d_\alpha(x^*, x_n), D_\alpha(x^*, F(x^*)), \frac{1}{2}[D_\alpha(x^*, F(x_n)) + D_\alpha(x_n, F(x^*))]\} \le d_\alpha(x^*, x_n) + a_\alpha \max\{d_\alpha(x^*, x_n) + \frac{1}{2}D_\alpha(x^*, F(x^*)), D_\alpha(x^*, F(x^*))\}$, for each $\alpha \in A$. Then, we immediately get that $D_\alpha(x^*, F(x^*)) = 0$, for each $\alpha \in A$ and hence $x^* \in Fix(F)$.

2) For the second conclusion, let $x_0 \in Fix(F_1)$ and $k_\alpha \in (1, \min\{\frac{1}{a_\alpha^{(1)}}, \frac{1}{a_\alpha^{(2)}}\})$ be arbitrary chosen. Then, as in the proof of Theorem 2.3, there exists a convergent sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F_2 (i.e., $x_{n+1} \in F_2(x_n), n \in \mathbb{N}$) starting from x_0 and $x_1 \in F_2(x_0)$ with the following property:

$$q_\alpha(x_n, x_{n+p}) \le \frac{(k_\alpha a_\alpha^{(2)})^n}{1-k_\alpha a_\alpha^{(2)}} q_\alpha(x_0, x_1), \text{ for } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

Letting $p \to +\infty$ we get that

$$q_\alpha(x_n, x_2^*) \le \frac{(k_\alpha a_\alpha^{(2)})^n}{1-k_\alpha a_\alpha^{(2)}} q_\alpha(x_0, x_1), \text{ for } n \in \mathbb{N}.$$

As before, $x_2^* \in Fix(F_2)$. For $n := 0$ in the above relation, we get that

$$q_\alpha(x_0, x_2^*) \le \frac{1}{1-k_\alpha a_\alpha^{(2)}} q_\alpha(x_0, x_1).$$

Since $x_0 \in F_1(x_0)$, by (b), there exists $x_1 \in F_2(x_0)$ such that $q_\alpha(x_0, x_1) \le k_\alpha \eta_\alpha$, for each $\alpha \in A$. Thus,

$$q_\alpha(x_0, x_2^*) \le \frac{k_\alpha \eta_\alpha}{1-k_\alpha a_\alpha^{(2)}}, \text{ for each } \alpha \in A.$$

Thus

$$D_\alpha(x_0, Fix(F_2)) \le \frac{k_\alpha \eta_\alpha}{1-k_\alpha a_\alpha^{(2)}}, \text{ for each } \alpha \in A,$$

and since $x_0 \in FixF_1$ is arbitrary we have

$$\sup_{y\in Fix(F_1)} D_\alpha(y, Fix(F_2)) \le \frac{k_\alpha \eta_\alpha}{1-k_\alpha a_\alpha^{(2)}}, \text{ for each } \alpha \in A.$$

By the above relation and by a similar approach, with the roles of F_1 and F_2 reversed, we obtain that

$$H_\alpha(Fix(F_1), Fix(F_2)) \le \frac{k_\alpha \eta_\alpha}{1-k_\alpha \max\{a_\alpha^{(1)}, a_\alpha^{(2)}\}}, \text{ for each } \alpha \in A.$$

Letting, for each $\alpha \in A$, $k_\alpha \searrow 1$ we get the conclusion. $\square$

In what follows we will present a homotopy result for Ćirić type contractions on a set with two separating gauge structures.

Theorem 2.5 *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$, such that $(X, \mathcal{P})$ is a sequentially complete gauge space. Suppose there exists a function $\psi : A \to B$ and $c = \{c_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ such that $p_\alpha(x, y) \leq c_\alpha \cdot q_{\psi(\alpha)}(x, y)$ for every $\alpha \in A$ and $x, y \in X$. Let U be an open subset of $(X, \mathcal{Q})$. Let $G : \overline{U}^p \times [0, 1] \to P(X, \mathcal{P})$ be a multivalued operator such that the following assumptions are satisfied:*

(i) $x \notin G(x, t)$, for each $x \in \partial U$ and each $t \in [0, 1]$;

(ii) there exists $\{a_\beta\}_{\beta\in B} \in (0, 1)^B$ for every $\beta \in B$ such that for each $x, y \in \overline{U}$ we have

$$H_\beta(G(x, t), G(y, t)) \leq a_\beta \cdot M_\beta^{G(\cdot,t)}(x, y),$$

where

$$\begin{aligned} M_\beta^{G(\cdot,t)}(x, y) &= \max\{q_\beta(x, y), D_\beta(x, G(x, t)), D_\beta(y, G(y, t)), \\ &\quad \frac{1}{2}[D_\beta(x, G(y, t)) + D_\beta(y, G(x, t))]\}. \end{aligned}$$

(iii) there exists a continuous function $\phi : [0, 1] \to \mathbb{R}$ such that

$$H_\beta(G(x, t), G(x, s)) \leq |\phi(t) - \phi(s)|, \text{ for all } t, s \in [0, 1] \text{ and each } x \in \overline{U}^p;$$

(iv) $G : (\overline{U}^p, \mathcal{P})\times [0, 1] \to P(X, \mathcal{P})$ is closed.

(v) for each $t \in [0, 1]$, for every $x \in \overline{U}^p$ and every $k = \{k_\beta\}_{\beta\in B} \in (0,\infty)^B$ there exists $y \in G(x, t)$ such that

$$q_\beta(x, y) \leq D_\beta(x, G(x, t)) + k_\beta, \text{ for every } \beta \in B.$$

Then $G(\cdot, 0)$ has a fixed point if and only if $G(\cdot, 1)$ has a fixed point.

Proof. Suppose that $z \in Fix(G(\cdot, 0))$. From (i) we have that $z \in U$. Consider the set:

$$E := \{(t, x) \in [0, 1] \times U : x \in G(x, t)\}.$$

Since $(0, z) \in E$, we have that $E \neq \emptyset$. We introduce a partial order defined on E

$$(t, x) \leq (s, y) \text{ if and only if } t \leq s \text{ and } q_\beta(x, y) \leq \frac{2}{1 - a_\beta}[\phi(s) - \phi(t)].$$

Let M be a totally ordered subset of E, $t^* := \sup\{t\ : (t, x) \in M\}$ and $(t_n, x_n)_{n\in\mathbb{N}^*} \subset M$ be a sequence such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Then

$$q_\beta(x_m, x_n) \leq \frac{2}{1 - a_\beta}[\phi(t_m) - \phi(t_n)], \text{ for each } m, n \in \mathbb{N}^*,\ m > n.$$

Letting $m, n \to +\infty$ we obtain that $q_\beta(x_m, x_n) \to 0$, thus $(x_n)_{n\in\mathbb{N}^*}$ is $\mathcal{Q}$-Cauchy, so it is $\mathcal{P}$-Cauchy too. Denote by $x^* \in (X, \mathcal{P})$ its limit. We know that $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}^*$ and G is $\mathcal{P}$-closed. Therefore we have that $x^* \in G(x^*, t^*)$. From (i) we note that $x^* \in U$. Thus $(t^*, x^*) \in E$.

From the fact that M is totally ordered we have that $(t,x) \leq (t^*,x^*)$, for each $(t,x) \in M$. Thus (t^*,x^*) is an upper bound of M. We can apply Zorn's Lemma, so E admits a maximal element $(t_0,x_0) \in E$. We now prove that $t_0 = 1$.

Suppose that $t_0 < 1$. Let $r = \{r_\beta\}_{\beta \in B} \in (0,\infty)^B$ and $t \in]t_0,1]$ be such that $B_q(x_0,r_\beta) \subset U$ and $r_\beta := \frac{2}{1-a_\beta}[\phi(t)-\phi(t_0)]$ for every $\beta \in B$. Then for each $\beta \in B$ we have

$$\begin{aligned} D_\beta(x_0,G(x_0,t)) &\leq D_\beta(x_0,G(x_0,t_0)) + H_\beta(G(x_0,t_0),G(x_0,t)) \\ &\leq \phi(t)-\phi(t_0) = \frac{r_\beta(1-a_\beta)}{2} < (1-a_\beta)r_\beta. \end{aligned}$$

Since $\overline{B}^p_q(x_0,r_\beta) \subset \overline{U}^p$, the closed multivalued operator $G(\cdot,t) : \overline{B}^p_q(x_0,r) \to P(X,\mathcal{P})$ satisfies the assumptions of Theorem 2.1, for all $t \in [0,1]$. Hence there exists $x \in \overline{B}^p_q(x_0,r_\beta)$ such that $x \in G(x,t)$. Thus $(t,x) \in E$. However we know that

$$q_\beta(x_0,x) \leq r_\beta = \frac{2}{1-a_\beta}[\phi(t)-\phi(t_0)],$$

so we have that

$$(t_0,x_0) < (t,x),$$

which contradicts the maximality of (t_0,x_0). Thus $t_0 = 1$ and the proof is complete. □

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