# FUZZY LOCATION PROBLEMS WITH TRIANGULAR NORM: EXISTENCE AND STABILITY

### MASAMICHI KON

Received March 11, 2008; revised May 22, 2008

ABSTRACT. A fuzzy max-T location problem is considered. The fuzzy max-T location problem is a generalization of a fuzzy maximin location problem by allowing in the objective function arbitrary triangular norms instead of the triangular norm defined by the minimum operation. Then we give conditions for the existence of its optimal solutions, and derive a relationship between its optimal solutions and efficient solutions of a fuzzy multicriteria location problem. Furthermore, we give some properties of triangular norms, and for triangular norms, we investigate the stability of optimal solutions of the fuzzy max-T location problem.

1 Introduction and preliminaries In a general continuous location model, finitely many points called demand points in  $\mathbb{R}^n$ , modeling existing facilities or customers, are given. Let  $\mathbf{d}_i \in \mathbb{R}^n$ ,  $i = 1, 2, \dots, \ell \geq 2$  be demand points. We put  $I \equiv \{1, 2, \dots, \ell\}$ . Then a problem to locate a new facility in  $\mathbb{R}^n$  is called a single facility location problem. If one prefers the location of the facility near demand points, then the problem is formulated as follows:

(1) 
$$\min_{\boldsymbol{x}\in\mathbb{R}^n} f(\gamma_1(\boldsymbol{x}-\boldsymbol{d}_1),\gamma_2(\boldsymbol{x}-\boldsymbol{d}_2),\cdots,\gamma_\ell(\boldsymbol{x}-\boldsymbol{d}_\ell))$$

where  $\boldsymbol{x} \in \mathbb{R}^n$  is the variable location of the facility. It is often assumed that  $f: \mathbb{R}^{\ell} \to \mathbb{R}$ is non-decreasing and convex or that  $f: \mathbb{R}^{\ell} \to \mathbb{R}^{\ell}$  satisfies  $f(\boldsymbol{y}) = \boldsymbol{y}$  for all  $\boldsymbol{y} \in \mathbb{R}^{\ell}$ . It is also often assumed that  $\gamma_i: \mathbb{R}^n \to \mathbb{R}, i \in I$  are norms or gauges, and each  $\gamma_i(\boldsymbol{x} - \boldsymbol{d}_i), i \in I$ represents the distance from  $\boldsymbol{d}_i$  to  $\boldsymbol{x}$ . In this paper, it is assumed that  $\gamma_i, i \in I$  are gauges.

Let  $B \subset \mathbb{R}^n$  be a compact convex set containing the origin in its interior. A gauge  $\gamma : \mathbb{R}^n \to \mathbb{R}$  for B is defined by  $\gamma(\mathbf{x}) \equiv \inf\{\lambda > 0 : \mathbf{x} \in \lambda B\}$  for  $\mathbf{x} \in \mathbb{R}^n$ ; see [4] and [12].

Formulation (1) is natural if one prefers the location of the facility near demand points. However, for the location of the facility, degrees of satisfaction with respect to demand points may be different even if distances from demand points to the facility are the same. Furthermore, for example, if the facility is an airport, then one may not prefer the location of the facility near demand points because of the noise. In order to deal with such situations, we consider membership functions, which represent degrees of satisfaction for the location of the facility with respect to demand points, and a maximization problem with an objective function involving membership functions. It is assumed that membership functions  $\mu_i$ :  $\mathbb{R}$  $\rightarrow [0,1] \equiv \{x \in \mathbb{R}: 0 \leq x \leq 1\}, i \in I$  are given. For each  $x \in \mathbb{R}^n$  and  $i \in I$ , the value  $\mu_i(\gamma_i(x - d_i))$  represents the degree of satisfaction for the location x with respect to the demand point  $d_i$ . For convenience, it is assumed that  $\mu_i(x) = 0, i \in I$  for x < 0. We put  $\overline{\mu_i}(x) \equiv \mu_i(\gamma_i(x - d_i)), i \in I$  for  $x \in \mathbb{R}^n$ . For each  $i \in I$ , let  $A_i$  and  $\overline{A_i}$  be fuzzy sets on  $\mathbb{R}$  and  $\mathbb{R}^n$  with membership functions  $\mu_i$  and  $\overline{\mu_i}$ , respectively. Then for each  $i \in I, A_i$ represents the fuzzy set of desirable distances from  $d_i$  to the facility, and  $\overline{A_i}$  represents the

<sup>2000</sup> Mathematics Subject Classification. 90B85.

Key words and phrases. fuzzy location problem, triangular norm, gauge.

fuzzy set of desirable locations of the facility with respect to  $d_i$ . A fuzzy maximin location problem (FMMP) is formulated as follows:

(2) 
$$\max_{\boldsymbol{x}\in\mathbb{R}^n}\mu_{\text{FMMP}}(\boldsymbol{x})\equiv\min\left\{\mu_1(\gamma_1(\boldsymbol{x}-\boldsymbol{d}_1)),\cdots,\mu_\ell(\gamma_\ell(\boldsymbol{x}-\boldsymbol{d}_\ell))\right\}.$$

For example, FMMP with block norm and asymmetric rectilinear distance are considered in [7] and [9], respectively. For each  $\boldsymbol{x} \in \mathbb{R}^n$ , the value  $\mu_{\text{FMMP}}(\boldsymbol{x})$  represents the degree of total satisfaction for the location  $\boldsymbol{x}$ , which is the degree of satisfaction for  $\boldsymbol{x}$  with respect to all demand points. The fuzzy set on  $\mathbb{R}^n$  with a membership function  $\mu_{\text{FMMP}}$  is denoted by  $\bigcap_{i \in I} \overline{A}_i$ , which is the usual intersection of fuzzy sets  $\overline{A}_i, i \in I$ , and it represents the fuzzy set of desirable locations of the facility with respect to all demand points. Therefore FMMP is a problem to find the location of the facility which maximizes the degree of total satisfaction. A *fuzzy max-T location problem* (FMTP), which is a generalization of FMMP and our main problem, is formulated as follows:

(3) 
$$\max_{\boldsymbol{x} \in \mathbb{P}^n} \mu_{\text{FMTP}}(\boldsymbol{x}) \equiv T\left(\mu_1(\gamma_1(\boldsymbol{x} - \boldsymbol{d}_1)), \cdots, \mu_\ell(\gamma_\ell(\boldsymbol{x} - \boldsymbol{d}_\ell))\right)$$

where  $T: [0,1]^{\ell} \to [0,1]$  is an extension of a triangular norm (t-norm for short), which is a binary operation on [0,1], to an  $\ell$ -ary operation on [0,1]. It is a generalization of the minimum operation in FMMP, and its precise definition will be given in section 2. FMTP can be interpreted in the same way as FMMP. For each  $\boldsymbol{x} \in \mathbb{R}^n$ , the value  $\mu_{\text{FMTP}}(\boldsymbol{x})$ represents the degree of total satisfaction for the location  $\boldsymbol{x}$  under the t-norm T as an operation. The fuzzy set on  $\mathbb{R}^n$  with a membership function  $\mu_{\text{FMTP}}$  is denoted by  $(\cap_T)_{i\in I}\overline{A_i}$ , which is the intersection of fuzzy sets  $\overline{A_i}, i \in I$  under the t-norm T, and it represents the fuzzy set of desirable locations of the facility with respect to all demand points under the t-norm T. Therefore FMTP is a problem to find the location of the facility which maximizes the degree of total satisfaction under the t-norm T. Let  $S_{\text{FMMP}}^*$  and  $S_{\text{FMTP}}^*$  be sets of all optimal solutions of FMMP and FMTP, respectively. We also consider a *fuzzy multicriteria location problem* (FMCP) formulated as follows:

(4) 
$$\max_{\boldsymbol{x}\in\mathbb{R}^n}\boldsymbol{\mu}_{\text{FMCP}}(\boldsymbol{x}) \equiv (\mu_1(\gamma_1(\boldsymbol{x}-\boldsymbol{d}_1)),\cdots,\mu_\ell(\gamma_\ell(\boldsymbol{x}-\boldsymbol{d}_\ell)))^T$$

We define  $\gamma(\boldsymbol{x}) \equiv (\gamma_1(\boldsymbol{x}-\boldsymbol{d}_1), \gamma_2(\boldsymbol{x}-\boldsymbol{d}_2), \cdots, \gamma_\ell(\boldsymbol{x}-\boldsymbol{d}_\ell))^T$  for  $\boldsymbol{x} \in \mathbb{R}^n$  and  $\boldsymbol{\mu}(\boldsymbol{y}) \equiv (\mu_1(y_1), \mu_2(y_2), \cdots, \mu_\ell(y_\ell))^T$  for  $\boldsymbol{y} \equiv (y_1, y_2, \cdots, y_\ell)^T \in \mathbb{R}^\ell$ . Then  $\boldsymbol{\mu}_{\text{FMCP}} = \boldsymbol{\mu} \circ \boldsymbol{\gamma}$ , where  $\boldsymbol{\mu} \circ \boldsymbol{\gamma}$  is the composite function of  $\boldsymbol{\mu}$  and  $\boldsymbol{\gamma}$ . A point  $\boldsymbol{x}_0 \in \mathbb{R}^n$  is called an *efficient solution* of FMCP if there is no  $\boldsymbol{x} \in \mathbb{R}^n$  such that  $\boldsymbol{\mu}_{\text{FMCP}}(\boldsymbol{x}) \geq \boldsymbol{\mu}_{\text{FMCP}}(\boldsymbol{x}_0)$  and  $\boldsymbol{\mu}_{\text{FMCP}}(\boldsymbol{x}) \neq \boldsymbol{\mu}_{\text{FMCP}}(\boldsymbol{x}_0)$ . Let  $F_{\text{E}}$  be the set of all efficient solutions of FMCP. FMCP is considered, for example, in [6].

In [10] and [11], fuzzy maximin, max-T and multicriteria problems are considered. These problems are, respectively, ones that  $\mu_i(\gamma_i(\boldsymbol{x} - \boldsymbol{d}_i)), i \in I$  in (2), (3) and (4) are replaced by  $\mu_i(\boldsymbol{x}), i \in I$  as functions from  $\mathbb{R}^n$  into [0, 1].

In this paper, the fuzzy max-T location problem is considered mainly. Then we give conditions for the existence of its optimal solutions, and derive a relationship between its optimal solutions and efficient solutions of the fuzzy multicriteria location problem. Furthermore, we give some properties of triangular norms, and for triangular norms, we investigate the stability of optimal solutions of the fuzzy max-T location problem.

**2** Triangular Norms In this section, we give some properties of triangular norms which are used in FMTP.

A triangular norm (t-norm for short) is a binary operation T on [0, 1], that is, a function  $T : [0, 1]^2 \to [0, 1]$ , such that for all  $x, y, z \in [0, 1]$ , the following four axioms are satisfied:

(T1) T(x,y) = T(y,x) (commutativity), (T2) T(x,T(y,z)) = T(T(x,y),z) (associativity), (T3)  $T(x,y) \leq T(x,z)$  whenever  $y \leq z$  (monotonicity) and (T4) T(x,1) = x (boundary condition); see [5].

**Example 1.** The following are two of basic t-norms. For  $x, y \in [0, 1]$ ,

(i) 
$$T_{\rm M}(x,y) \equiv \min\{x,y\},$$
 (minimum)

(ii)

$$T_{\rm D}(x,y) \equiv \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1, \\ 0 & \text{otherwise.} \end{cases}$$
 (drastic product)

For any t-norm T, it can be seen easily that  $T_{\text{D}}(x,y) \leq T(x,y) \leq T_{\text{M}}(x,y)$  for all  $x, y \in [0,1]$ . Let T be a t-norm. From the commutativity (T1) and the associativity (T2), we define its extension to more than two arguments by

$$T^{k+1}(x_1, x_2, \cdots, x_{k+2}) \equiv T(T^k(x_1, x_2, \cdots, x_{k+1}), x_{k+2})$$

for  $x_i \in [0,1], i = 1, 2, \dots, k+2$ , where  $T^1(x_1, x_2) \equiv T(x_1, x_2)$ . If there is no danger of misunderstanding, the upper index  $\ell - 1$  of  $T^{\ell-1}$  is omitted and we write T instead of  $T^{\ell-1}$ .

**Example 2.** For  $x_1, x_2, \dots, x_{\ell} \in [0, 1]$ ,

(i) 
$$T_{\mathrm{M}}(x_1, x_2, \cdots, x_\ell) = \min\{x_1, x_2, \cdots, x_\ell\},\$$

(ii)

$$T_{\rm D}(x_1, x_2, \cdots, x_\ell) = \begin{cases} x_i & \text{if } x_j = 1, \forall j \neq i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that FMTP with  $T_{\rm M}$  reduces to FMMP. Let T be a t-norm, and we put  $x_i = \mu_i(\gamma_i(\boldsymbol{x} - \boldsymbol{d}_i)), i \in I$  for  $\boldsymbol{x} \in \mathbb{R}^n$ . Then the value  $T(x_1, x_2, \dots, x_\ell)$  represents the degree of total satisfaction for the location  $\boldsymbol{x}$ . When  $T = T_{\rm M}$ , the degree of total satisfaction is the minimum among all degrees of satisfaction with respect to demand points. When  $T = T_{\rm D}$ , it is the minimum among all degrees of satisfaction with respect to demand points if all degrees of satisfaction with respect to demand points if all degrees of satisfaction with respect to demand points if all observe to demand points except for some  $\boldsymbol{d}_i, i \in I$  are 1, otherwise it is 0.

Some of the following Theorem 1-3 about t-norms seem to be known. In fact, Theorem 1 is similar to Proposition 8.3 in [5]. However, within our knowledge, no literature contains the same results as Theorem 1-3. So, we give their proofs for completeness.

**Theorem 1.** Let  $T_{\lambda} : [0,1]^{\ell} \to [0,1], \lambda \in \Lambda$  and  $T : [0,1]^{\ell} \to [0,1]$  be t-norms, where  $\Lambda \subset [-\infty,\infty]$  is a nonempty interval and  $[-\infty,\infty] \equiv \mathbb{R} \cup \{-\infty,\infty\}$ . If T is continuous and  $T_{\lambda}$  converges (pointwise) to T as  $\lambda$  approaches  $\lambda_0$ , then  $T_{\lambda}$  converges uniformly to T as  $\lambda$  approaches  $\lambda_0$ , where  $\lambda, \lambda_0 \in \Lambda$ .

**Proof.** We shall show only the case  $\ell = 2$  and  $\lambda_0 \in \mathbb{R}$ . It can be shown similarly in the other case. Fix any  $\varepsilon > 0$  and  $(x_0, y_0)^T \in [0, 1]^2$ . Because of the continuity of T, there exists  $\delta_{x_0y_0} > 0$  (which depends on  $\varepsilon$ ) such that for all  $(x, y)^T \in [0, 1]^2 \cap$  $([x_0 - \delta_{x_0y_0}, x_0 + \delta_{x_0y_0}] \times [y_0 - \delta_{x_0y_0}, y_0 + \delta_{x_0y_0}]), |T(x, y) - T(x_0, y_0)| < \frac{\varepsilon}{3}$ . Writing  $\underline{x}_0$  $= \max\{x_0 - \delta_{x_0y_0}, 0\}, \ \underline{y}_0 = \max\{y_0 - \delta_{x_0y_0}, 0\}, \ \overline{x}_0 = \min\{x_0 + \delta_{x_0y_0}, 1\}$  and  $\overline{y}_0 =$  $\min\{y_0 + \delta_{x_0y_0}, 1\}$ , we get  $T(\overline{x}_0, \overline{y}_0) - T(\underline{x}_0, \underline{y}_0) = T(\overline{x}_0, \overline{y}_0) - T(x_0, y_0) + T(x_0, y_0) T(\underline{x}_0, \underline{y}_0) \leq \frac{2\varepsilon}{3}$ . From the pointwise convergence of  $T_\lambda$  to T as  $\lambda$  approaches  $\lambda_0$ , there exists  $\eta > 0$  such that for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta, \lambda_0 + \eta), |T_\lambda(\underline{x}_0, \underline{y}_0) - T(\underline{x}_0, \underline{y}_0)| < \frac{\varepsilon}{3}$  and

$$\begin{split} |T_{\lambda}(\overline{x}_{0},\overline{y}_{0}) - T(\overline{x}_{0},\overline{y}_{0})| &< \frac{\varepsilon}{3}, \text{ where } (a,b) \equiv \{x \in \mathbb{R} : a < x < b\} \text{ for } a,b \in \mathbb{R} \text{ with } a < b. \text{ Fix any } \lambda \in \Lambda \cap (\lambda_{0} - \eta,\lambda_{0} + \eta). \text{ For all } (x,y)^{T} \in [0,1]^{2} \cap ([x_{0} - \delta_{x_{0}y_{0}},x_{0} + \delta_{x_{0}y_{0}}] \times [y_{0} - \delta_{x_{0}y_{0}},y_{0} + \delta_{x_{0}y_{0}}]) \text{ with } T_{\lambda}(x,y) < T(x,y), \text{ we have } T(x,y) - T_{\lambda}(x,y) \leq T(\overline{x}_{0},\overline{y}_{0}) - T_{\lambda}(\underline{x}_{0},\underline{y}_{0}) \leq T(\overline{x}_{0},\overline{y}_{0}) - T(\underline{x}_{0},\underline{y}_{0}) + |T(\underline{x}_{0},\underline{y}_{0}) - T_{\lambda}(\underline{x}_{0},\underline{y}_{0})| < \varepsilon. \text{ For all } (x,y)^{T} \in [0,1]^{2} \cap ([x_{0} - \delta_{x_{0}y_{0}},x_{0} + \delta_{x_{0}y_{0}}] \times [y_{0} - \delta_{x_{0}y_{0}},y_{0} + \delta_{x_{0}y_{0}}]) \text{ with } T_{\lambda}(x,y) \geq T(x,y), \text{ we have } T_{\lambda}(x,y) - T(x,y) \leq T_{\lambda}(\overline{x}_{0},\overline{y}_{0}) - T(\underline{x}_{0},\underline{y}_{0}) \leq |T_{\lambda}(\overline{x}_{0},\overline{y}_{0}) - T(\overline{x}_{0},\overline{y}_{0})| + T(\overline{x}_{0},\overline{y}_{0}) - T(\underline{x}_{0},\underline{y}_{0}) < \varepsilon. \text{ Since, as a compact set, the unit square } [0,1]^{2} \text{ can be covered by a finite number of rectangles of the form } [0,1]^{2} \cap ((x - \frac{\delta_{xy}}{2}, x + \frac{\delta_{xy}}{2}) \times (y - \frac{\delta_{xy}}{2}, y + \frac{\delta_{xy}}{2})), \text{ there exists } \eta_{\varepsilon} > 0 \text{ such that for all } \lambda \in \Lambda \cap (\lambda_{0} - \eta_{\varepsilon}, \lambda_{0} + \eta_{\varepsilon}) \text{ and } (x,y)^{T} \in [0,1]^{2}, |T_{\lambda}(x,y) - T(x,y)| < \varepsilon. \end{split}$$

**Theorem 2.** Let  $T_{\lambda}^{1} : [0,1]^{2} \to [0,1], \lambda \in \Lambda$  and  $T^{1} : [0,1]^{2} \to [0,1]$  be t-norms, where  $\Lambda \subset [-\infty,\infty]$  is a nonempty interval. If  $T^{1}$  is continuous and  $T_{\lambda}^{1}$  converges to  $T^{1}$  as  $\lambda$  approaches  $\lambda_{0}$ , then  $T_{\lambda}^{\ell-1}$  converges to  $T^{\ell-1}$  as  $\lambda$  approaches  $\lambda_{0}$ , where  $\lambda, \lambda_{0} \in \Lambda$ .

**Proof.** We shall show only the case  $\lambda_0 \in \mathbb{R}$  by induction on  $\ell$ . It can be shown similarly in the other case. When  $\ell = 2$ , from the assumption,  $T_{\lambda}^{\ell-1}$  converges to  $T^{\ell-1}$  as  $\lambda$  approaches  $\lambda_0$ . Suppose that  $T_{\lambda}^{\ell-1}$  converges to  $T^{\ell-1}$  as  $\lambda$  approaches  $\lambda_0$  for  $\ell \geq 2$ , and we shall show that  $T_{\lambda}^{\ell}$  converges to  $T^{\ell}$  as  $\lambda$  approaches  $\lambda_0$ . Fix any  $(x_1, \cdots, x_\ell, x_{\ell+1})^T \in [0, 1]^{\ell+1}$  and  $\varepsilon > 0$ . From the continuity of  $T^1$ , there exists  $\delta > 0$  such that for all  $x \in [0, 1] \cap (T^{\ell-1}(x_1, \cdots, x_\ell) - \delta, T^{\ell-1}(x_1, \cdots, x_\ell) + \delta)$ ,  $|T^1(x, x_{\ell+1}) - T^1(T^{\ell-1}(x_1, \cdots, x_\ell), x_{\ell+1})| < \frac{\varepsilon}{2}$ . Since  $T_{\lambda}^{\ell-1}(x_1, \cdots, x_\ell)$  converges to  $T^{\ell-1}(x_1, \cdots, x_\ell)$  as  $\lambda$  approaches  $\lambda_0$ , there exists  $\eta_1 > 0$  such that for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta_1, \lambda_0 + \eta_1)$ ,  $|T_{\lambda}^{\ell-1}(x_1, \cdots, x_\ell) - T^{\ell-1}(x_1, \cdots, x_\ell)| < \delta$ . Since  $T^1$  is continuous,  $T_{\lambda}^1$  converges uniformly to  $T^1$  as  $\lambda$  approaches  $\lambda_0$  from Theorem 1. Thus there exists  $\eta_2 > 0$  such that for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta_2, \lambda_0 + \eta_2)$  and  $x \in [0, 1]$ ,  $|T_{\lambda}^1(x, x_{\ell+1}) - T^1(x, x_{\ell+1})| < \frac{\varepsilon}{2}$ . We put  $\eta = \min\{\eta_1, \eta_2\} > 0$ . Then for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta, \lambda_0 + \eta)$ , we have  $|T_{\lambda}^\ell(x_1, \cdots, x_\ell, x_{\ell+1}) - T^\ell(x_1, \cdots, x_\ell), x_{\ell+1})| = |T_{\lambda}^1(T_{\lambda}^{\ell-1}(x_1, \cdots, x_\ell), x_{\ell+1})|$ ,  $x_{\ell+1}| + |T^1(T_{\lambda}^{\ell-1}(x_1, \cdots, x_\ell), x_{\ell+1}) - T^1(T^{\ell-1}(x_1, \cdots, x_\ell), x_{\ell+1})| < |T_{\lambda}^1(T_{\lambda}^{\ell-1}(x_1, \cdots, x_\ell), x_{\ell+1})| < \varepsilon$ .

**Theorem 3.** Let  $T^1 : [0,1]^2 \to [0,1]$  be a t-norm.

- (i) If  $T^1$  is upper semicontinuous, then  $T^{\ell-1}$  is also upper semicontinuous.
- (ii) If  $T^1$  is lower semicontinuous, then  $T^{\ell-1}$  is also lower semicontinuous.
- (iii) If  $T^1$  is continuous, then  $T^{\ell-1}$  is also continuous.

## Proof.

(i) We proceed by induction on  $\ell$ . When  $\ell = 2$ ,  $T^{\ell-1}$  is upper semicontinuous from the assumption. Suppose that  $T^{\ell-1}$  is upper semicontinuous for  $\ell \ge 2$ , and we shall show that  $T^{\ell}$  is also upper semicontinuous. Fix any  $(x_1^0, \cdots, x_{\ell}^0, x_{\ell+1}^0)^T \in [0, 1]^{\ell+1}$  and  $\varepsilon > 0$ . Since  $T^1$  is upper semicontinuous, there exists  $\delta > 0$  such that for all  $(x, y)^T \in [0, 1]^2 \cap ((T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) - \delta, T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) + \delta) \times (x_{\ell+1}^0 - \delta, x_{\ell+1}^0 + \delta)), T^1(x, y) < T^1(T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0), x_{\ell+1}^0) + \varepsilon$ . From the monotonicity of  $T^1$ , this inequality holds for all  $(x, y)^T \in [0, 1]^2 \cap ([0, T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) + \delta) \times [0, x_{\ell+1}^0 + \delta))$ , where  $[a, b) \equiv \{x \in \mathbb{R} : a \le x < b\}$  for  $a, b \in \mathbb{R}$  with a < b. Since  $T^{\ell-1}$  is upper semicontinuous, there exists  $\eta > 0$  such that for all  $(x_1, \cdots, x_{\ell})^T \in [0, 1]^\ell \cap ((x_1^0 - \eta, x_1^0 + \eta) \times \cdots \times (x_{\ell}^0 - \eta, x_{\ell}^0 + \eta))$ ,  $T^{\ell-1}(x_1, \cdots, x_{\ell}) < T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) + \delta$ . Thus for all  $(x_1, \cdots, x_{\ell}, x_{\ell+1})^T \in [0, 1]^{\ell+1} \cap ((x_1^0 - \eta, x_1^0 + \eta) \times \cdots \times (x_{\ell}^0 - \eta, x_{\ell}^0 + \eta))$ ,  $x_{\ell+1} = T^1(T^{\ell-1}(x_1, \cdots, x_{\ell}), x_{\ell+1}) < T^1(T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0), x_{\ell+1}^0) + \varepsilon = T^\ell(x_1^0, \cdots, x_{\ell}^0)$ 

 $x_{\ell+1}^0$  +  $\varepsilon$ . Therefore  $T^{\ell}$  is upper semicontinuous.

(ii) We proceed by induction on  $\ell$ . When  $\ell = 2$ ,  $T^{\ell-1}$  is lower semicontinuous from the assumption. Suppose that  $T^{\ell-1}$  is lower semicontinuous for  $\ell \ge 2$ , and we shall show that  $T^{\ell}$  is also lower semicontinuous. Fix any  $(x_1^0, \cdots, x_{\ell}^0, x_{\ell+1}^0)^T \in [0, 1]^{\ell+1}$  and  $\varepsilon > 0$ . Since  $T^1$  is lower semicontinuous, there exists  $\delta > 0$  such that for all  $(x, y)^T \in [0, 1]^2 \cap ((T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) - \delta, T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) + \delta) \times (x_{\ell+1}^0 - \delta, x_{\ell+1}^0 + \delta)), T^1(x, y) > T^1(T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0), x_{\ell+1}^0) - \varepsilon$ . From the monotonicity of  $T^1$ , this inequality holds for all  $(x, y)^T \in [0, 1]^2 \cap ((T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) - \delta, 1] \times (x_{\ell+1}^0 - \delta, 1])$ , where  $(a, b] \equiv \{x \in \mathbb{R} : a < x \le b\}$  for  $a, b \in \mathbb{R}$  with a < b. Since  $T^{\ell-1}$  is lower semicontinuous, there exists  $\eta > 0$  such that for all  $(x_1, \cdots, x_{\ell})^T \in [0, 1]^\ell \cap ((x_1^0 - \eta, x_1^0 + \eta) \times \cdots \times (x_{\ell}^0 - \eta, x_{\ell}^0 + \eta))$ ,  $T^{\ell-1}(x_1, \cdots, x_{\ell}) > T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0) - \delta$ . Thus for all  $(x_1, \cdots, x_{\ell}, x_{\ell+1})^T \in [0, 1]^{\ell+1} \cap ((x_1^0 - \eta, x_1^0 + \eta) \times \cdots \times (x_{\ell}^0 - \eta, x_{\ell}^0 + \eta))$ ,  $x_{\ell+1} = T^1(T^{\ell-1}(x_1, \cdots, x_{\ell}), x_{\ell+1}) > T^1(T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0, x_{\ell+1}) > T^1(T^{\ell-1}(x_1^0, \cdots, x_{\ell}^0, x_{\ell+1}) - \varepsilon)$ . Therefore  $T^\ell$  is lower semicontinuous.

(iii) We have the conclusion immediately from (i) and (ii).

**3** Existence of optimal solutions of FMTP In this section, we give conditions for  $S^*_{\text{FMTP}}$  to be nonempty, and derive a relationship between  $S^*_{\text{FMTP}}$  and  $F_{\text{E}}$ .

Let  $\mu$  be a function from  $\mathbb{R}^n$  into [0,1]. For  $\alpha \in (0,1]$ , the upper level set of  $\mu$ ,  $[\mu]_{\alpha} \equiv \{ \boldsymbol{x} \in \mathbb{R}^n : \mu(\boldsymbol{x}) \geq \alpha \}$ , is called  $\alpha$ -cut of  $\mu$ . If  $\alpha$ -cut of  $\mu$  is bounded for all  $\alpha \in (0,1]$ , then the fuzzy set on  $\mathbb{R}^n$  with a membership function  $\mu$  is said to be *bounded*.

The following theorem gives sufficient conditions for the existence of optimal solutions of FMTP, and also gives a relationship between those optimal solutions and efficient solutions of FMCP.

**Theorem 4.** If all  $\mu_i$ ,  $i \in I$  are upper semicontinuous and  $A_j$  is bounded for some  $j \in I$ and T is an upper semicontinuous t-norm, then  $S^*_{\text{FMTP}} \neq \emptyset$ . Moreover, if  $\max_{\boldsymbol{x} \in \mathbb{R}^n} \mu_{\text{FMTP}}(\boldsymbol{x})$ > 0, then  $S^*_{\text{FMTP}} \cap F_{\text{E}} \neq \emptyset$ .

**Proof.** Put  $\overline{\gamma}_i(\boldsymbol{x}) \equiv \gamma_i(\boldsymbol{x} - \boldsymbol{d}_i), i \in I$  for  $\boldsymbol{x} \in \mathbb{R}^n$ . First, we shall show that  $(\cap_T)_{i \in I} \overline{A}_i$ is bounded. Fix any  $\alpha \in (0, 1]$ . Since  $A_j$  is bounded,  $[\mu_j]_{\alpha}$  is bounded. Thus  $[\mu_j \circ \overline{\gamma}_j]_{\alpha} = \{\boldsymbol{x} \in \mathbb{R}^n : \mu_j(\overline{\gamma}_j(\boldsymbol{x})) \geq \alpha\} = \{\boldsymbol{x} \in \mathbb{R}^n : \overline{\gamma}_j(\boldsymbol{x}) \in [\mu_j]_{\alpha}\}$  is bounded. Therefore  $\{\boldsymbol{x} \in \mathbb{R}^n : \min\{\mu_1(\overline{\gamma}_1(\boldsymbol{x})), \mu_2(\overline{\gamma}_2(\boldsymbol{x})), \cdots, \mu_\ell(\overline{\gamma}_\ell(\boldsymbol{x}))\} \geq \alpha\} = \cap_{i \in I}\{\boldsymbol{x} \in \mathbb{R}^n : \mu_i(\overline{\gamma}_i(\boldsymbol{x})) \geq \alpha\}$  $= \cap_{i \in I}[\mu_i \circ \overline{\gamma}_i]_{\alpha}$  is bounded. Since, for  $\boldsymbol{x} \in [\mu_{\mathrm{FMTP}}]_{\alpha}, \alpha \leq T(\mu_1(\overline{\gamma}_1(\boldsymbol{x})), \mu_2(\overline{\gamma}_2(\boldsymbol{x})), \cdots, \mu_\ell(\overline{\gamma}_\ell(\boldsymbol{x}))\}$ , we see that  $[\mu_{\mathrm{FMTP}}]_{\alpha} \subset \cap_{i \in I}[\mu_i \circ \overline{\gamma}_i]_{\alpha}$ . Consequently,  $[\mu_{\mathrm{FMTP}}]_{\alpha}$  is bounded.

Next, in order to show that  $\mu_{\text{FMTP}}$  is upper semicontinuous, we shall show that for  $\boldsymbol{x}_0 \in \mathbb{R}^n$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\mu_{\text{FMTP}}(\boldsymbol{x}) \leq \mu_{\text{FMTP}}(\boldsymbol{x}_0) + \varepsilon$  for all  $\boldsymbol{x} \in N_{\delta}(\boldsymbol{x}_0) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta\}$ , where  $\|\cdot\|$  is Euclidean norm. For each  $i \in I$ , since  $\mu_i$  is upper semicontinuous and  $\overline{\gamma}_i$  is continuous,  $\mu_i \circ \overline{\gamma}_i$  is upper semicontinuous. Put  $y_{0i} \equiv \mu_i(\overline{\gamma}_i(\boldsymbol{x}_0))$ ,  $i \in I$  and  $\boldsymbol{y}_0 \equiv (y_{01}, y_{02}, \cdots, y_{0\ell})^T$ . Since T is upper semicontinuous on  $[0, 1]^{\ell}$ , there exists  $\eta > 0$  such that

(5) 
$$T(y_1, y_2, \cdots, y_\ell) \le T(y_{01}, y_{02}, \cdots, y_{0\ell}) + \varepsilon$$

for all  $\boldsymbol{y} \in N_{\eta}(\boldsymbol{y}_0) \equiv \{\boldsymbol{y} \in [0,1]^{\ell} : \|\boldsymbol{y} - \boldsymbol{y}_0\| < \eta\}$ , where  $\boldsymbol{y} \equiv (y_1, y_2, \cdots, y_{\ell})^T$ . Since  $\mu_i \circ \overline{\gamma}_i$ ,  $i \in I$  are upper semicontinuous, there exists  $\delta > 0$  such that  $\mu_i(\overline{\gamma}_i(\boldsymbol{x})) \le \mu_i(\overline{\gamma}_i(\boldsymbol{x}_0)) + \frac{\eta}{2\sqrt{\ell}}$  for all  $\boldsymbol{x} \in N_{\delta}(\boldsymbol{x}_0)$  and  $i \in I$ . By the monotonicity of the t-norm,

(6) 
$$T(\mu_1(\overline{\gamma}_1(\boldsymbol{x})), \mu_2(\overline{\gamma}_2(\boldsymbol{x})), \cdots, \mu_\ell(\overline{\gamma}_\ell(\boldsymbol{x}))) \le T(z_1, z_2, \cdots, z_\ell)$$

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where  $z_i \equiv \min\{1, \mu_i(\overline{\gamma}_i(\boldsymbol{x}_0)) + \frac{\eta}{2\sqrt{\ell}}\}, i \in I$ . Since  $(z_1, z_2, \dots, z_\ell)^T \in N_\eta(\boldsymbol{y}_0), \mu_{\text{FMTP}}(\boldsymbol{x}) \leq \mu_{\text{FMTP}}(\boldsymbol{x}_0) + \varepsilon$  from (5) and (6). Therefore  $[\mu_{\text{FMTP}}]_\alpha$  is compact. Consequently,  $\mu_{\text{FMTP}}$  attains its maximum on  $\mathbb{R}^n$ , that is,  $S^*_{\text{FMTP}} \neq \emptyset$ .

In order to prove the last part of the theorem, suppose that  $\boldsymbol{x}^* \in S^*_{\text{FMTP}}$  and  $\alpha^* \equiv \mu_{\text{FMTP}}(\boldsymbol{x}^*) > 0$ . Then  $S^*_{\text{FMTP}} = [\mu_{\text{FMTP}}]_{\alpha^*}$ . Since  $\alpha^* > 0$ ,  $S^*_{\text{FMTP}}$  is compact from the above discussion. Put  $g \equiv \sum_{i \in I} \mu_i \circ \overline{\gamma}_i$ . Since all  $\mu_i \circ \overline{\gamma}_i$ ,  $i \in I$  are upper semicontinuous, g is also upper semicontinuous. Thus there exists  $\boldsymbol{x}_0 \in S^*_{\text{FMTP}}$  such that  $g(\boldsymbol{x}_0) = \max_{\boldsymbol{x} \in S^*_{\text{FMTP}}} g(\boldsymbol{x})$ . Assume that  $\boldsymbol{x}_0 \notin F_{\text{E}}$  in order to show that  $\boldsymbol{x}_0 \in F_{\text{E}}$ . Then there exists  $\boldsymbol{x}_1 \in \mathbb{R}^n$  such that  $\mu \circ \gamma(\boldsymbol{x}_1) \ge \mu \circ \gamma(\boldsymbol{x}_0)$  and  $\mu \circ \gamma(\boldsymbol{x}_1) \ne \mu \circ \gamma(\boldsymbol{x}_0)$ . Since  $g(\boldsymbol{x}_1) > g(\boldsymbol{x}_0)$ , we see that  $\boldsymbol{x}_1 \notin S^*_{\text{FMTP}}$  by the definition of  $\boldsymbol{x}_0$ . Since, by the monotonicity of the t-norm,  $\alpha^* = \mu_{\text{FMTP}}(\boldsymbol{x}_0) = T(\mu_1(\overline{\gamma}_1(\boldsymbol{x}_0)), \mu_2(\overline{\gamma}_2(\boldsymbol{x}_0)), \cdots, \mu_\ell(\overline{\gamma}_\ell(\boldsymbol{x}_0))) \le T(\mu_1(\overline{\gamma}_1(\boldsymbol{x}_1)), \mu_2(\overline{\gamma}_2(\boldsymbol{x}_1)), \cdots, \mu_\ell(\overline{\gamma}_\ell(\boldsymbol{x}_1))) = \mu_{\text{FMTP}}(\boldsymbol{x}_1) \le \alpha^*$ , we have  $\mu_{\text{FMTP}}(\boldsymbol{x}_1) = \alpha^*$ , which contradicts that  $\boldsymbol{x}_1 \notin S^*_{\text{FMTP}}$ . Therefore  $\boldsymbol{x}_0 \in S^*_{\text{FMTP}} \cap F_{\text{E}}$ , that is,  $S^*_{\text{FMTP}} \cap F_{\text{E}} \ne \emptyset$ .

The following corollary gives sufficient conditions for the existence of optimal solutions of FMMP, and also gives a relationship between those optimal solutions and efficient solutions of FMCP.

**Corollary 1.** If all  $\mu_i$ ,  $i \in I$  are upper semicontinuous and  $A_j$  is bounded for some  $j \in I$ , then  $S^*_{\text{FMMP}} \neq \emptyset$ . Moreover, if  $\max_{\boldsymbol{x} \in \mathbb{R}^n} \mu_{\text{FMMP}}(\boldsymbol{x}) > 0$ , then  $S^*_{\text{FMMP}} \cap F_E \neq \emptyset$ .

**Proof.** Since FMMP is FMTP with  $T_{\rm M}$  which is continuous, we have the conclusion immediately from Theorem 4.

The following example shows that neither the condition all  $\mu_i$ ,  $i \in I$  are upper semicontinuous nor the condition  $A_j$  is bounded for some  $j \in I$  in Theorem 4 and Corollary 1 can be eliminated.

**Example 3.** Set n = 2,  $d_1 = (0, 0)^T$ ,  $d_2 = (1, 0)^T$ , and assume that  $\gamma_1$  and  $\gamma_2$  are the same Euclidean norm.

(i) If

$$\mu_1(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise,} \end{cases} \quad \mu_2(x) = \begin{cases} 2^{-x} & \text{if } x \ge 0, \\ 0 & \text{otherwise} \end{cases}$$

then  $\mu_1$  is not upper semicontinuous, and we see that  $\mu_{\text{FMMP}}(\boldsymbol{x}) < 1$  for all  $\boldsymbol{x} \in \mathbb{R}^2$  and that  $\sup_{\boldsymbol{x} \in \mathbb{R}^2} \mu_{\text{FMMP}}(\boldsymbol{x}) = 1$ . In this case,  $\mu_{\text{FMMP}}$  does not attain its maximum on  $\mathbb{R}^2$ . (ii) If

$$\mu_1(x) = \mu_2(x) = \begin{cases} 1 - 2^{-x} & \text{if } x \ge 0, \\ 0 & \text{otherwise} \end{cases}$$

then neither  $A_1$  nor  $A_2$  is bounded, and we see that  $\mu_{\text{FMMP}}(\boldsymbol{x}) < 1$  for all  $\boldsymbol{x} \in \mathbb{R}^2$  and that  $\sup_{\boldsymbol{x} \in \mathbb{R}^2} \mu_{\text{FMMP}}(\boldsymbol{x}) = 1$ . In this case,  $\mu_{\text{FMMP}}$  does not attain its maximum on  $\mathbb{R}^2$ .

The following example shows that the condition a t-norm T is upper semicontinuous in Theorem 4 can not be eliminated.

**Example 4.** Set n = 2,  $d_1 = (0, 0)^T$ ,  $d_2 = (1, 0)^T$ , and assume that  $\gamma_1$  and  $\gamma_2$  are the same rectilinear norm. Put

$$\mu_1(x) = \mu_2(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Then both  $\mu_1$  and  $\mu_2$  are upper semicontinuous, and both  $A_1$  and  $A_2$  are bounded. Moreover, put

$$T(x,y) = \begin{cases} 0 & \text{if } x + y < 1 \text{ or } x = y = 0.5, \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

Then T is a t-norm constructed by using Proposition 3.63 in [5], and it is not upper semicontinuous at  $(0.5, 0.5)^T$ . We see that  $\mu_{\text{FMTP}}(\boldsymbol{x}) < 0.5$  for all  $\boldsymbol{x} \in \mathbb{R}^2$  and that  $\sup_{\boldsymbol{x} \in \mathbb{R}^2} \mu_{\text{FMTP}}(\boldsymbol{x}) = 0.5$ . In this case,  $\mu_{\text{FMTP}}$  does not attain its maximum on  $\mathbb{R}^2$ .

The following example shows that neither the condition  $\max_{\boldsymbol{x} \in \mathbb{R}^n} \mu_{\text{FMTP}}(\boldsymbol{x}) > 0$  in Theorem 4 nor the condition  $\max_{\boldsymbol{x} \in \mathbb{R}^n} \mu_{\text{FMMP}}(\boldsymbol{x}) > 0$  in Corollary 1 can be eliminated.

**Example 5.** Set n = 2,  $d_1 = (0, 0)^T$ ,  $d_2 = (1, 0)^T$ , and assume that  $\gamma_1$  and  $\gamma_2$  are the same Euclidean norm. Put  $\mu_1(x) = 0$  for all  $x \in \mathbb{R}$  and

$$\mu_2(x) = \begin{cases} 1 - 2^{-x} & \text{if } x \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then both  $\mu_1$  and  $\mu_2$  are continuous, and  $A_1$  is bounded. In this case, we see that  $S^*_{\text{FMMP}} = \mathbb{R}^2$ ,  $\max_{\boldsymbol{x} \in \mathbb{R}^2} \mu_{\text{FMMP}}(\boldsymbol{x}) = 0$  and  $F_{\text{E}} = \emptyset$ .

**4** Stability of optimal solutions of FMTP In this section, we investigate the stability of optimal solutions of FMTP for t-norms.

Let  $\Lambda \subset [-\infty, \infty]$  be a nonempty interval and  $\{T_{\lambda}\}_{\lambda \in \Lambda}$  be a class of t-norms. We define  $h : \mathbb{R}^n \times \Lambda \to [0, 1]$  by

$$h(\boldsymbol{x},\lambda) \equiv T_{\lambda} \left( \mu_1(\gamma_1(\boldsymbol{x}-\boldsymbol{d}_1)), \mu_2(\gamma_2(\boldsymbol{x}-\boldsymbol{d}_2)), \cdots, \mu_\ell(\gamma_\ell(\boldsymbol{x}-\boldsymbol{d}_\ell)) \right)$$

for  $\boldsymbol{x} \in \mathbb{R}^n$  and  $\lambda \in \Lambda$ . For each  $\lambda \in \Lambda$ ,  $h(\cdot, \lambda) : \mathbb{R}^n \to [0, 1]$  is the objective function of FMTP with the t-norm  $T_{\lambda}$ . Then we define the optimal value function  $\phi : \Lambda \to [0, 1]$  and the optimal set mapping  $\Phi : \Lambda \to \mathbb{R}^n$  by

$$\phi(\lambda) \equiv \sup\{h(\boldsymbol{x},\lambda) : \boldsymbol{x} \in \mathbb{R}^n\}$$

and

$$\Phi(\lambda) \equiv \{ \boldsymbol{x} \in \mathbb{R}^n : \phi(\lambda) = h(\boldsymbol{x}, \lambda) \}$$

for  $\lambda \in \Lambda$ , respectively, where the symbol  $\rightsquigarrow$  stands for a set-valued mapping. For each  $\lambda \in \Lambda$ , if there exists  $\boldsymbol{x} \in \mathbb{R}^n$  such that  $\phi(\lambda) = h(\boldsymbol{x}, \lambda)$ , then  $\phi(\lambda)$  is the optimal value of FMTP with the t-norm  $T_{\lambda}$  and  $\Phi(\lambda)$  is the set of all optimal solutions of FMTP with the t-norm  $T_{\lambda}$ .

**Theorem 5.** If  $T_{\lambda}$  converges to  $T_{\lambda_0}$  as  $\lambda$  approaches  $\lambda_0$  and  $T_{\lambda_0}$  is continuous, then  $\phi$  is continuous at  $\lambda_0$ , where  $\lambda, \lambda_0 \in \Lambda$ .

**Proof.** We shall show only the case  $\lambda_0 \in \mathbb{R}$ . It can be shown similarly in the other case. Fix any  $\varepsilon > 0$ . Form Theorem 1,  $T_{\lambda}$  converges uniformly to  $T_{\lambda_0}$  as  $\lambda$  approaches  $\lambda_0$ . Thus there exists  $\eta > 0$  such that for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta, \lambda_0 + \eta)$  and  $\mathbf{x} \in \mathbb{R}^n$ ,  $|h(\mathbf{x}, \lambda) - h(\mathbf{x}, \lambda_0)| < \frac{\varepsilon}{2}$ . Fix any  $\lambda \in \Lambda \cap (\lambda_0 - \eta, \lambda_0 + \eta)$ . Since  $\sup\{h(\mathbf{x}, \lambda_0) - \frac{\varepsilon}{2} : \mathbf{x} \in \mathbb{R}^n\} \le \sup\{h(\mathbf{x}, \lambda) : \mathbf{x} \in \mathbb{R}^n\} \le \sup\{h(\mathbf{x}, \lambda_0) + \frac{\varepsilon}{2} : \mathbf{x} \in \mathbb{R}^n\}$ , we have  $\phi(\lambda_0) - \frac{\varepsilon}{2} \le \phi(\lambda) \le \phi(\lambda_0) + \frac{\varepsilon}{2}$ . Therefore for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta, \lambda_0 + \eta)$ ,  $|\phi(\lambda) - \phi(\lambda_0)| \le \frac{\varepsilon}{2} < \varepsilon$ .

Let  $\mu$  be a function from  $\mathbb{R}$  into [0,1]. The set  $\operatorname{supp}(\mu) \equiv \{x \in \mathbb{R} : \mu(x) > 0\}$  is called the *support* of  $\mu$ . The fuzzy set on  $\mathbb{R}$  with a membership function  $\mu$  is said to be *support bounded* if the support of  $\mu$  is bounded.  $\Phi$  is said to be *uniformly bounded* around  $\lambda_0 \in \Lambda$  if there exists a neighborhood of  $\lambda_0$ ,  $V \subset \Lambda$ , such that the union  $\bigcup_{\lambda \in V} \Phi(\lambda) \subset \mathbb{R}^n$  is bounded.  $\Phi$  is said to be *upper semicontinuous* at  $\lambda_0 \in \Lambda$  if  $\Phi$  is uniformly bounded around  $\lambda_0$  and  $\mathbf{x}_0 \in \Phi(\lambda_0)$  for any sequence  $\{\lambda_k\} \subset \Lambda$ , which converges to  $\lambda_0$ , and  $\{\mathbf{x}_k\} \subset \mathbb{R}^n$  with  $\mathbf{x}_k \in \Phi(\lambda_k) (k = 1, 2, \cdots)$ , which converges to  $\mathbf{x}_0 \in \mathbb{R}^n$ .  $\Phi$  is said to be *lower semicontinuous* at  $\lambda_0 \in \Lambda$  if for any sequence  $\{\lambda_k\} \subset \Lambda$ , which converges to  $\lambda_0$ , and any  $\mathbf{x}_0 \in \Phi(\lambda_0)$ , there exist an integer  $k_0 > 0$  and a sequence  $\{x_k\} \subset \mathbb{R}^n$  such that  $\{x_k\}$  converges to  $x_0$  and  $x_k \in \Phi(\lambda_k) (k \geq k_0)$ .  $\Phi$  is said to be *continuous* at  $\lambda_0 \in \Lambda$  if  $\Phi$  is upper and lower semicontinuous at  $\lambda_0$ .

**Theorem 6.** Assume that all  $\mu_i, i \in I$  are continuous on  $[0, \infty) \equiv \{x \in \mathbb{R} : x \geq 0\}$  and that  $A_j$  is support bounded for some  $j \in I$  and that  $T_{\lambda}$  converges to  $T_{\lambda_0}$  as  $\lambda$  approaches  $\lambda_0$  and that  $T_{\lambda_0}$  is continuous, where  $\lambda, \lambda_0 \in \Lambda$ . If  $\Phi(\lambda_0) \neq \emptyset$  and  $\phi(\lambda_0) > 0$ , then  $\Phi$  is upper semicontinuous at  $\lambda_0$ . Moreover, if there exists a neighborhood of  $\lambda_0, V \subset \Lambda$ , such that  $\Phi(\lambda) \neq \emptyset$  for all  $\lambda \in V$  and if  $\Phi(\lambda_0)$  is a singleton, then  $\Phi$  is continuous at  $\lambda_0$ .

**Proof.** We shall show only the case  $\lambda_0 \in \mathbb{R}$ . It can be shown similarly in the other case. First, we shall show that  $\Phi$  is uniformly bounded around  $\lambda_0$ . From Theorem 5,  $\phi$  is continuous at  $\lambda_0$ . Since  $\phi(\lambda_0) > 0$ , there exists  $\eta_0 > 0$  such that  $\phi(\lambda) > 0$  for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta_0, \lambda_0 + \eta_0)$ . On the other hand,  $h(\boldsymbol{x}, \lambda) = 0$  for  $\boldsymbol{x} \notin \{\boldsymbol{z} \in \mathbb{R}^n : \gamma_j(\boldsymbol{z} - \boldsymbol{d}_j) \in \operatorname{supp}(\mu_j)\}$ . Thus  $\Phi(\lambda) \subset \{\boldsymbol{z} \in \mathbb{R}^n : \gamma_j(\boldsymbol{z} - \boldsymbol{d}_j) \in \operatorname{supp}(\mu_j)\}$  for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta_0, \lambda_0 + \eta_0)$ . Since  $\operatorname{supp}(\mu_j)$  is bounded,  $\{\boldsymbol{z} \in \mathbb{R}^n : \gamma_j(\boldsymbol{z} - \boldsymbol{d}_j) \in \operatorname{supp}(\mu_j)\}$  is bounded. Therefore  $\Phi$  is uniformly bounded around  $\lambda_0$ .

Next, fix any sequence  $\{\lambda_k\} \subset \Lambda$ , which converges to  $\lambda_0$ , and  $\{\boldsymbol{x}_k\} \subset \mathbb{R}^n$  with  $\boldsymbol{x}_k \in \Phi(\lambda_k)(k = 1, 2, \cdots)$ , which converges to  $\boldsymbol{x}_0 \in \mathbb{R}^n$ . Then we shall show that h is continuous at  $(\boldsymbol{x}_0, \lambda_0)$ . Fix any  $\varepsilon > 0$ . Since  $h(\cdot, \lambda_0) : \mathbb{R}^n \to [0, 1]$  is continuous, there exists a neighborhood of  $\boldsymbol{x}_0, U \subset \mathbb{R}^n$ , such that for all  $\boldsymbol{x} \in U$ ,  $|h(\boldsymbol{x}, \lambda_0) - h(\boldsymbol{x}_0, \lambda_0)| < \frac{\varepsilon}{2}$ . Since, from Theorem 1,  $T_\lambda$  converges uniformly to  $T_{\lambda_0}$  as  $\lambda$  approaches  $\lambda_0$ , there exists  $\eta > 0$  such that for all  $\lambda \in \Lambda \cap (\lambda_0 - \eta, \lambda_0 + \eta)$  and  $\boldsymbol{x} \in \mathbb{R}^n, |h(\boldsymbol{x}, \lambda) - h(\boldsymbol{x}, \lambda_0)| < \frac{\varepsilon}{2}$ . Thus for all  $(\boldsymbol{x}, \lambda) \in U \times (\Lambda \cap (\lambda_0 - \eta, \lambda_0 + \eta)), |h(\boldsymbol{x}, \lambda) - h(\boldsymbol{x}_0, \lambda_0)| \leq |h(\boldsymbol{x}, \lambda) - h(\boldsymbol{x}, \lambda_0)| + |h(\boldsymbol{x}, \lambda_0) - h(\boldsymbol{x}_0, \lambda_0)| < \varepsilon$ . Therefore h is continuous at  $(\boldsymbol{x}_0, \lambda_0)$ . Since  $\phi(\lambda_0) = \lim_{k \to \infty} \phi(\lambda_k) = \lim_{k \to \infty} h(\boldsymbol{x}_k, \lambda_k) = h(\boldsymbol{x}_0, \lambda_0)$ , we have  $\boldsymbol{x}_0 \in \Phi(\lambda_0)$ . Therefore  $\Phi$  is upper semicontinuous at  $\lambda_0$ .

Finally, suppose that  $\Phi(\lambda_0) = \{x_0\}$  and that  $\Phi$  is not lower semicontinuous at  $\lambda_0$ . Then there exists a sequence  $\{\lambda_k\} \subset \Lambda$ , which converges to  $\lambda_0$ , such that for any sequence  $\{x_k\} \subset \mathbb{R}^n$  with  $x_k \in \Phi(\lambda_k) (k \ge k_0)$  for some positive integer  $k_0$ ,  $\{x_k\}$  does not converge to  $x_0$ . Fix any sequence  $\{x_k\} \subset \mathbb{R}^n$  with  $x_k \in \Phi(\lambda_k) (k \ge k_0)$  for some positive integer  $k_0$ . Since  $\{x_k\}$  does not converge to  $x_0$ , there exists a subsequence  $\{x_{k(p)}\}$  of  $\{x_k\}$  such that

$$\begin{split} & \lim_{p \to \infty} \| \boldsymbol{x}_{k(p)} - \boldsymbol{x}_0 \| > 0, \\ & \boldsymbol{x}_{k(p)} \in \Phi(\lambda_{k(p)}) \quad (p \in \mathbb{N}), \\ & \lambda_{k(p)} \to \lambda_0 \quad (p \to \infty), \end{split}$$

where  $\mathbb{N}$  is the set of all natural numbers. Since  $\Phi$  is uniformly bounded around  $\lambda_0$ , there exists a subsequence  $\{\boldsymbol{x}_{k(p,q)}\}$  of  $\{\boldsymbol{x}_{k(p)}\}$  such that  $\{\boldsymbol{x}_{k(p,q)}\}$  converges to some  $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ . In this case, we have

$$\lim_{q \to \infty} \inf \| \boldsymbol{x}_{k(p,q)} - \boldsymbol{x}_0 \| > 0, \quad (*) \\ \boldsymbol{x}_{k(p,q)} \in \Phi(\lambda_{k(p,q)}) \quad (q \in \mathbb{N}), \\ \lambda_{k(p,q)} \to \lambda_0 \quad (q \to \infty).$$

Since  $\Phi$  is upper semicontinuous at  $\lambda_0$ , we have  $\overline{x} \in \Phi(\lambda_0)$ , that is,  $\overline{x} = x_0$ , which contradicts (\*).

**5** Some discussions It is known that t-norms are the special case of aggregation operators. A fuzzy location problem with an aggregation operator instead of a t-norm can be formulated. It is a problem that a t-norm T in (3) is replaced by an aggregation operator A. An aggregation operator is a function  $A : \bigcup_{k \in \mathbb{N}} [0,1]^k \to [0,1]$  such that: (A1)  $A(x_1, x_2, \cdots, x_k) \leq A(y_1, y_2, \cdots, y_k)$  whenever  $x_i \leq y_i$  for all  $i \in \{1, 2, \cdots, k\}$ , (A2) A(x) = x for all  $x \in [0, 1]$  and (A3)  $A(0, 0, \dots, 0) = 0$  and  $A(1, 1, \dots, 1) = 1$ . The aggregation operator A is said to be commutative if  $A(x_1, x_2, \dots, x_k) = A(x_{\alpha(1)}, x_{\alpha(2)}, \dots, x_{\alpha(k)})$  for all  $k \in \mathbb{N}$  with  $k \geq 2$  and all  $x_1, x_2, \dots, x_k \in [0, 1]$  and all permutations  $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(k))$  of  $(1, 2, \dots, k)$ . The aggregation operator A is said to be associative if  $A(x_1, \dots, x_k, y_1, \dots, y_m) = A(A(x_1, \dots, x_k), A(y_1, \dots, y_m))$  for all  $k, m \in \mathbb{N}$  and all  $x_1, \dots, x_k, y_1, \dots, y_m \in [0, 1]$ . An element  $e \in [0, 1]$  is called a neutral element of the aggregation operator A if  $A(x_1, \dots, x_{i-1}, e, x_{i+1}, \dots, x_k) = A(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$  for all  $k \in \mathbb{N}$  with  $k \geq 2$  and all  $i \in \{1, 2, \dots, k\}$  and all  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k \in [0, 1]$ . Each t-norm is a commutative, associative aggregation operator with neutral element 1. Moreover, a commutative, associative aggregation operator with neutral element  $e \in [0, 1]$  is a t-norm if and only if e = 1; see [5].

FMMP (2) is a problem to find a location which maximizes the worst degree of satisfaction among degrees of satisfaction for the location of the facility with respect to demand points, where the worst degree of satisfaction is measured by the minimum operation. FMTP (3) has the same interpretation as FMMP (2), where the worst degree of satisfaction is measured by a t-norm. Therefore FMTP (3) is a natural generalization of FMMP (2) in the sense of the interpretation.

Let A be an aggregation operator. When the degree of total satisfaction for the location  $x \in \mathbb{R}^n$ , which represents the worst degree of satisfaction, is evaluated by  $A(\mu_1(\gamma_1(x - x)))$  $(d_1)$ ,  $\cdots$ ,  $\mu_\ell(\gamma_\ell(\boldsymbol{x} - \boldsymbol{d}_\ell))$ , (i) the commutativity of  $\boldsymbol{A}$  means that the degree of total satisfaction with respect to demand points  $d_1, d_2, \cdots, d_\ell$  is the same value as the degree of total satisfaction with respect to demand points  $d_{\alpha(1)}, d_{\alpha(2)}, \cdots, d_{\alpha(\ell)}$  for all permutations  $\alpha = (\alpha(1), \alpha(2), \cdots, \alpha(\ell))$  of  $(1, 2, \cdots, \ell)$ , (ii) the associativity of **A** means that the worst degree of satisfaction with respect to demand points  $d_1, d_2, \cdots, d_\ell$  is the same value as the worst degree of satisfaction of the worst degree of satisfaction with respect to  $d_1, \dots, d_k$ and the worst degree of satisfaction with respect to  $d_{k+1}, \dots, d_{\ell}$  for all  $k \in \mathbb{N}$  with  $1 \leq \ell$  $k \leq \ell - 1$  and (iii) "A has a neutral element 1" means that the worst degree of satisfaction is independent of any demand point whose degree of satisfaction is one. We would like to emphasize that these properties (i), (ii) and (iii) are desirable and needed for a fuzzy location problem with an aggregation operator as a natural generalization of FMMP (2) in the sense of the interpretation. As mentioned before, commutative, associative aggregation operators with neutral element 1 are t-norms. Therefore in the sense of the interpretation, a fuzzy location problem with an aggregation operator which is a t-norm is a natural generalization of FMMP (2), but a fuzzy location problem with an aggregation operator which is not a t-norm is not a natural generalization of FMMP (2). However, it should be noted that a fuzzy location problem with an aggregation operator, which is not a t-norm, seems to be another interesting problem which has an interpretation different from that of FMMP (2) and FMTP (3).

**6** Conclusions In this paper, we dealt with fuzzy maximin, max-T and multicriteria location problems. Our main problem was the fuzzy max-T location problem. First, we gave some properties of triangular norms. Next, we gave sufficient conditions for the existence of optimal solutions of the fuzzy maximin and max-T location problems, and derived a relationship between those optimal solutions and efficient solutions of the fuzzy multicriteria location problem. Finally, for triangular norms, we investigated the stability of optimal solutions of the fuzzy max-T location problem.

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GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY, HIROSAKI UNIVERSITY, 3 BUNKYO, HIROSAKI, AOMORI, 036-8561, JAPAN *E-mail:* masakon@cc.hirosaki-u.ac.jp