THE STRUCTURE, APPROXIMATE IDENTITIES, AND DUALS OF ℓ^p-MUNN ALGEBRAS

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Abstract. In the present paper we first introduce the notion of $\mathcal{LM}_I^p(\mathcal{A})$ of a Banach space $\mathcal{A}$, where I is an index set, and $1 \le p < \infty$. In the case where $\mathcal{A}$ is a Banach algebra and $1 \le p \le 2$ we find necessary and sufficient conditions for which $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity. We also find the second dual of $\mathcal{LM}_I^p(\mathcal{A})$ for an arbitrary index set I. In the case where $1 \le p \le 2$ and $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity we prove that $\mathcal{LM}_I^p(\mathcal{A})^{**}$ is unital if and only if $\mathcal{A}^{**}$ is unital.

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Introduction The properties of ℓ^1-Munn algebras is investigated by G. H. Esslamzadeh (for example, see [1]). The aim of the present paper is to introduce and investigate the properties of ℓ^p-Munn algebras. The organization of this paper is as follows. The preliminaries and notations are given in section 1. In section 2 we introduce and investigate the structure of $\mathcal{LM}_I^p(\mathcal{A})$, where $\mathcal{A}$ is a Banach space, I is an index set, and $1 \le p < \infty$. The Banach space $\mathcal{LM}_I^p(\mathcal{A})$ is the vector space of all $I \times I$-matrices A over $\mathcal{A}$ such that $\|A\|_p = \left(\sum_{i,j \in I} \|A_{ij}\|^p\right)^{\frac{1}{p}} < \infty$. We find necessary and sufficient conditions for which $\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra. We prove that if $\mathcal{A}$ is a Banach algebra such that $\mathcal{A}^2 \ne 0$, then $\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra if and only if $1 \le p \le 2$. We also prove that for $1 \le p \le 2$, the Banach algebra $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity if and only if I is finite and $\mathcal{A}$ has a bounded approximate identity. Finally, in section 3 for an arbitrary index set I, we study the second dual of $\mathcal{LM}_I^p(\mathcal{A})$ over a unital Banach algebra $\mathcal{A}$. In the case where $1 \le p \le 2$ and $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity we prove that $\mathcal{LM}_I^p(\mathcal{A})^{**}$ is unital if and only if $\mathcal{A}^{**}$ is unital.

1 Preliminaries Let $\mathcal{A}$ be an algebra, a norm $\| \, . \, \|$ on $\mathcal{A}$ is said to be *submultiplicative* if

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in \mathcal{A}).$$

In this case the pair $(\mathcal{A}, \| \, . \, \|)$ is called a *normed algebra*. A complete normed algebra is called a *Banach algebra*. The *radical* of $\mathcal{A}$ is the subset $\mathrm{rad}(\mathcal{A})$ of $\mathcal{A}$ given by

$$\mathrm{rad}(\mathcal{A}) = \cap\{\ker(\phi) | \; \phi \in \Sigma_{\mathcal{A}}\},$$

where $\Sigma_{\mathcal{A}}$ is the class of all irreducible representations of $\mathcal{A}$. $\mathcal{A}$ is called *semi-simple* if $\mathrm{rad}(\mathcal{A})=\{0\}$.

If $\mathcal{A}$ admits a unit $e_{\mathcal{A}}$ (i.e. $ae_{\mathcal{A}} = e_{\mathcal{A}}a = a$, for all $a \in \mathcal{A}$) and $\|e_{\mathcal{A}}\| = 1$ we say that $\mathcal{A}$ is a *unital Banach algebra*.

Let $(\mathcal{A}, \| \, . \, \|)$ be a normed algebra. A *left* (respectively, *right*) *approximate identity* for $\mathcal{A}$ is a net (e_α) in $\mathcal{A}$ such that $\lim_\alpha e_\alpha a = a$ ($\lim_\alpha ae_\alpha = a$, respectively) for each $a \in \mathcal{A}$. An *approximate identity* for $\mathcal{A}$ is a net (e_α) which is both a left and a right approximate identity. An approximate identity is bounded by M, whenever $M > 0$ and $\sup_\alpha \|e_\alpha\| \leq M$.

Let $\mathcal{A}$ be an arbitrary Banach algebra. The *first* and *second Arens multiplications* on $\mathcal{A}^{**}$ that we denote by Δ and . respectively, are defined by the following relations:

$$\langle f\Delta a, b\rangle = \langle f, ab\rangle, \qquad \langle a.f, b\rangle = \langle f, ba\rangle,$$

$$\langle m\Delta f, a\rangle = \langle m, f\Delta a\rangle, \quad \langle f.m, b\rangle = \langle m, b.f\rangle,$$
$$\langle m\Delta n, f\rangle = \langle m, n\Delta f\rangle, \quad \langle m.n, f\rangle = \langle n, f.m\rangle,$$

where $a, b \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$.

2 The structure of the Banach space $\mathcal{LM}_I^p(\mathcal{A})$ $(1 \leq p < \infty)$ over a Banach algebra $\mathcal{A}$

Definition 2.1. Let $\mathcal{A}$ be a Banach space, $1 \leq p < \infty$, and I be an arbitrary index set, let $\mathcal{LM}_I^p(\mathcal{A})$ be the vector space of all $I \times I$-matrices A over $\mathcal{A}$ such that

$$\|A\|_p = \Big(\sum_{i,j\in I} \|A_{ij}\|^p\Big)^{\frac{1}{p}} < \infty.$$

Then it is easy to check that $\mathcal{LM}_I^p(\mathcal{A})$ with scaler multiplication, matrix addition, and the norm $\| \, . \, \|_p$ is a Banach space.

The space $\mathcal{LM}_I^1(\mathcal{A})$ over a unital Banach algebra $\mathcal{A}$ is called ℓ^1-*Munn algebra* (see [1]).

In the rest of the paper we suppose that $\mathcal{A}$ is a Banach algebra.

Theorem 2.2. *Let $1 \leq p \leq 2$. The Banach space $\mathcal{LM}_I^p(\mathcal{A})$ with matrix multiplication and the norm $\| \,.\, \|_p$ is a Banach algebra.*

Proof. Let $A, B \in \mathcal{LM}_I^p(\mathcal{A})$, and $i, j \in I$. Since $1 \leq p \leq 2$, so for q with $\frac{1}{p} + \frac{1}{q} = 1$, $q \geq 2 \geq p$. Hence $\ell^p(I) \subseteq \ell^q(I)$ and $\|f\|_q \leq \|f\|_p$ ($f \in \ell^p(I)$). Now, we have

$$\begin{aligned}
\left(\sum_{k\in I} \|A_{ik}\|\|B_{kj}\|\right)^p &= \|(\|A_{ik}\|)_k(\|B_{kj}\|)_k\|_1^p \\
&\leq \|(\|A_{ik}\|)_k\|_p^p\|(\|B_{kj}\|)_k\|_q^p \\
&\leq \|(\|A_{ik}\|)_k\|_p^p\|(\|B_{kj}\|)_k\|_p^p \\
&= \left(\sum_{k\in I} \|A_{ik}\|^p\right)\left(\sum_{l\in I} \|B_{lj}\|^p\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|AB\|_p^p &= \sum_{i,j\in I} \left\|\sum_{k\in I} A_{ik}B_{kj}\right\|^p \\
&\leq \sum_{i,j\in I} \left(\sum_{k\in I} \|A_{ik}\|\|B_{kj}\|\right)^p \\
&\leq \sum_{i,j\in I} \left(\sum_{k\in I} \|A_{ik}\|^p\right)\left(\sum_{l\in I} \|B_{lj}\|^p\right) \\
&= \left(\sum_{i,k\in I} \|A_{ik}\|^p\right)\left(\sum_{j,l\in I} \|B_{lj}\|^p\right) \\
&= \|A\|_p^p\|B\|_p^p.
\end{aligned}$$

Hence $\|AB\|_p \leq \|A\|_p\|B\|_p$. It shows that $\| \,.\, \|_p$ is an algebra norm. Hence $\mathcal{LM}_I^p(\mathcal{A})$ is a Banach algebra. $\square$

Proposition 2.3. *Let I be an infinite set and $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^2 \neq 0$. Then for each $2 < p < \infty$, $\mathcal{LM}_I^p(\mathcal{A})$ is not an algebra.*

Proof. Since $\mathcal{A}^2 \neq 0$, so there exist $a, b \in \mathcal{A}$ such that $ab \neq 0$. Let $\{i_n\}_{n\in\mathbb{N}}$ be an infinite subset of distinct elements of I. Define the $I \times I$-matrix A over $\mathcal{A}$ by $A_{i_1 i_n} = \frac{1}{\sqrt{n}}a$ ($n \in \mathbb{N}$) and $A_{ij} = 0$ for other $i, j \in I$. Also define the $I \times I$-matrix B over $\mathcal{A}$ by $B_{i_n i_1} = \frac{1}{\sqrt{n}}b$ ($n \in \mathbb{N}$) and $B_{ij} = 0$ for other $i, j \in I$. It is easy to see that $A, B \in \mathcal{LM}_I^p(\mathcal{A})$. But AB is not even well defined, since

$$(AB)_{i_1 i_1} = \sum_{n\in\mathbb{N}} A_{i_1 i_n}B_{i_n i_1} = \left(\sum_{n\in\mathbb{N}} \frac{1}{n}\right) ab.$$

□

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra and $1 \le p \le 2$.*

(i) *If $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity, then $\mathcal{A}$ has a bounded approximate identity and the index set I is finite.*

(ii) *If $\mathcal{A}$ has a bounded approximate identity and the index set I finite, then $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity.*

Proof. (i) Let $\{E_\alpha |\ \alpha \in \Lambda\}$ be a bounded approximate identity for $\mathcal{LM}_I^p(\mathcal{A})$ bounded by $M > 0$. For $i, j \in I$, and $a \in \mathcal{A}$, let $\varepsilon_{ij}a$ be an $I \times I$ matrix over $\mathcal{A}$ that has a as its (i, j)-th entry and 0 elsewhere. For $i, j \in I$, and $a \in \mathcal{A}$

$$\begin{aligned}
0 &= \lim_\alpha \|E_\alpha . a\varepsilon_{ij} - a\varepsilon_{ij}\|_p^p = \lim_\alpha \Big\| \sum_{k,l\in I} (E_\alpha)_{kl}\varepsilon_{kl} a\varepsilon_{ij} - a\epsilon_{ij} \Big\|_p^p \\
&= \lim_\alpha \Big\| \sum_{k\in I} (E_\alpha)_{ki} a\varepsilon_{kj} - a\epsilon_{ij} \Big\|_p^p \\
&= \lim_\alpha \Big\| \sum_{k\in I, k\neq i} (E_\alpha)_{ki} a\varepsilon_{kj} + (E_\alpha)_{ii} a\varepsilon_{ij} - a\varepsilon_{ij} \Big\|_p^p \\
&= \lim_\alpha \left(\sum_{k\in I, k\neq i} \|(E_\alpha)_{ki} a\|^p + \|(E_\alpha)_{ii} a - a\|^p \right) \\
&\ge \limsup_\alpha \|(E_\alpha)_{ii} a - a\|^p.
\end{aligned}$$

Thus $\lim_\alpha \|(E_\alpha)_{ii}a - a\| = 0$. If we let $e_i^\alpha = (E_\alpha)_{ii}$, then for every $a \in \mathcal{A}$, $\lim_\alpha \|e_i^\alpha a - a\| = \lim_\alpha \|(E_\alpha)_{ii} a - a\| = 0$. Similarly one can prove that $\lim_\alpha \|a e_i^\alpha - a\| = 0$. Moreover,

$$\|e_i^\alpha\| = \|(E_\alpha)_{ii}\| \le \|(E_\alpha)\|_p \le M.$$

Therefore $\{e_i^\alpha |\ \alpha \in \Lambda\}$ is a bounded approximate identity for $\mathcal{A}$.

Let $a \in \mathcal{A}$ such that $\|a\| = 1$. Suppose on the contrary that I is infinite. Let $(E_\alpha)_\alpha$ be an approximate identity for $\mathcal{LM}_I^p(\mathcal{A})$. For every finite subset F of I, define A_F by $(A_F)_{ii} = a$ if $i \in F$, $(A_F)_{ii} = 0$ if $i \in I - F$ and $(A_F)_{ij} = 0$ if $i \neq j$ then

$$\begin{aligned}
(\text{Card } F)^{\frac{1}{p}} &= \Big(\sum_{i\in F} \|a\|^p \Big)^{\frac{1}{p}} \\
&= \|A_F\|_p \\
&= \lim_\alpha \|A_F E_\alpha\|_p \\
&= \lim_\alpha \Big(\sum_{i\in F, j\in I} \|a(E_\alpha)_{ij}\|^p \Big)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{\alpha} \Big(\sum_{i\in F, j\in I} \|a\|^p \|(E_\alpha)_{ij}\|^p \Big)^{\frac{1}{p}} \\
&= \liminf_{\alpha} \Big(\sum_{i\in F, j\in I} \|(E_\alpha)_{ij}\|^p \Big)^{\frac{1}{p}} \\
&\leq \liminf_{\alpha} \|E_\alpha\|_p .
\end{aligned}
$$

Therefore $\lim_\alpha \|E_\alpha\|_p = \infty$. Thus the Banach algebra $\mathcal{LM}_I^p(\mathcal{A})$ does not have a bounded approximate identity. This is a contradiction. Therefore the index set I is finite.

(ii) Let $|I| = n$ and $\{e^\alpha|\ \alpha \in \Lambda\}$ be a bounded approximate identity for $\mathcal{A}$, bounded by $K > 0$. Define $\{E^\alpha|\ \alpha \in \Lambda\}$ in $\mathcal{LM}_I^p(\mathcal{A})$ by $(E_\alpha)_{ii} = e_\alpha$, and $(E_\alpha)_{ij} = 0$, if $i \neq j$ for $i, j \in I$. Now, for every $A \in \mathcal{LM}_I^p(\mathcal{A})$

$$
\begin{aligned}
\lim_{\alpha} \|AE_\alpha - A\|_p^p &= \lim_{\alpha} \sum_{i,j\in I} \|(AE_\alpha)_{ij} - A_{ij}\|^p \\
&= \lim_{\alpha} \sum_{i,j\in I} \Big\| \sum_{k\in I} A_{ik}(E_\alpha)_{kj} - A_{ij} \Big\|^p \\
&= \lim_{\alpha} \sum_{i,j\in I} \|A_{ij} e_\alpha - A_{ij}\|^p \\
&= 0.
\end{aligned}
$$

By a similar method one can prove that $\lim_\alpha \|E_\alpha A - A\|_p = 0$, for every $A \in \mathcal{LM}_I^p(\mathcal{A})$. Moreover,

$$
\|E_\alpha\|_p^p = \sum_{i,j\in I} \|(E_\alpha)_{ij}\|^p = \sum_{i\in I} \|(E_\alpha)_{ii}\|^p \leq n.M^p,
$$

and so $\|E_\alpha\|_p \leq n^{\frac{1}{p}} M$. Therefore $\{E^\alpha|\ \alpha \in \Lambda\}$ is a bounded approximate identity for $\mathcal{LM}_I^p(\mathcal{A})$. □

Corollary 2.5. *Let $\mathcal{A}$ be a Banach algebra with unit $e_{\mathcal{A}}$, and $1 \leq p \leq 2$. The following conditions are equivalent:*

(i) *If $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity.*

(ii) *I is finite.*

Remark 2.6. If I is finite, then $\mathcal{LM}_I^p(\mathcal{A})$ is semisimple if and only if $\mathcal{A}$ is semisimple (see for example [3] section IX.2 exercise 13).

3 Duals of $\mathcal{LM}_I^p(\mathcal{A})$ for $1 \leq p \leq 2$.

Let X be an arbitrary Banach space and $\mathcal{A}$ be a Banach algebra and I be an arbitrary index set. It is well known that $\ell^p(I, X)^* = \ell^q(I, X^*)$ whenver $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.1. *Let I be an arbitrary index set and $1 \leq p \leq 2$. Then $\mathcal{LM}_I^p(\mathcal{A})^{**}$ is topologically algebra isomorphic to $\mathcal{LM}_I^p(\mathcal{A}^{**})$ when both of $\mathcal{A}^{**}$ and $\mathcal{LM}_I^p(\mathcal{A})^{**}$ are equipped with the first [second] Arens product.*

Proof. If we define $\psi : \mathcal{LM}_I^p(\mathcal{A})^* \rightarrow \ell^p(I \times I, \mathcal{A}^*)$ by $\langle \psi(F)(i,j), a\rangle = \langle F, a\varepsilon_{ij}\rangle$, where $a\varepsilon_{ij}$ is an $I \times I$ matrix on $\mathcal{A}$ which has a as (i,j)- th entry and 0 elsewhere. We will denote $\psi(F)$ by $\widetilde{F}$. It is now obvious what we mean by $\ell^p(I \times I, \mathcal{A}^{**})$ and its elements. We can identify the two Banach spaces $\mathcal{LM}_I^p(\mathcal{A})^{**}$ and $\mathcal{LM}_I^p(\mathcal{A}^{**})$. Therefore we need only to show that the linear isomorphism $\psi : \mathcal{LM}_I^p(\mathcal{A})^{**} \rightarrow \mathcal{LM}_I^p(\mathcal{A}^{**})$ is multiplicative. Throughout we will use the fact that the restriction of the Arens product of $\mathcal{A}^{**}$ to $\mathcal{A}$ agrees with the multiplication of $\mathcal{A}$. Let $A, X \in \mathcal{LM}_I^p(\mathcal{A})$, $F \in \mathcal{LM}_I^p(\mathcal{A})^*$ and $M, N \in \mathcal{LM}_I^p(\mathcal{A})^{**}$, then we have

$$\begin{aligned} \langle (\widetilde{F\Delta A})_{ij}, X_{ij}\rangle &= \langle F\Delta A, X_{ij}\varepsilon_{ij}\rangle \\ &= \langle F, A(X_{ij}\varepsilon_{ij})\rangle \\ &= \sum_k \langle F, A_{ki}X_{ij}\varepsilon_{kj}\rangle \\ &= \sum_k \langle \widetilde{F}_{kj}, A_{ki}X_{ij}\rangle \\ &= \sum_k \langle \widetilde{F}_{kj}\Delta A_{ki}, X_{ij}\rangle. \end{aligned}$$

Thus

$$(\widetilde{F\Delta A})_{ij} = \sum_k \widetilde{F}_{kj}\Delta A_{ik}.$$

Now applying this relation to $\widetilde{M\Delta F}$ we have

$$\begin{aligned} \langle (\widetilde{M\Delta F})_{ij}, A_{ij}\rangle &= \langle M\Delta F, A_{ij}\varepsilon_{ij}\rangle \\ &= \langle M, F\Delta(A_{ij}\varepsilon_{ij})\rangle \\ &= \sum_k \langle \widetilde{M}_{jk}, \widetilde{F}_{ik}\Delta A_{ij}\rangle \\ &= \sum_k \langle \widetilde{M}_{jk}\Delta \widetilde{F}_{ik}, A_{ij}\rangle. \end{aligned}$$

So

$$(\widetilde{M\Delta F})_{ij} = \sum_k \widetilde{M}_{jk}\Delta \widetilde{F}_{ik}.$$

Using this relation to $\widetilde{N\Delta M}$ we obtain

$$\begin{aligned}\langle(\widetilde{N\Delta M})_{ij}, \widetilde{F}_{ij}\rangle &= \langle N\Delta M, \widetilde{F}_{ij}\varepsilon_{ij}\rangle \\ &= \langle N, M\Delta(\widetilde{F}_{ij}\varepsilon_{ij})\rangle \\ &= \sum_k \langle \widetilde{N}_{ik}, \widetilde{M}_{kj}\Delta\widetilde{F}_{ij}\rangle \\ &= \langle \sum_k \widetilde{N}_{ik}\Delta\widetilde{M}_{kj}, \widetilde{F}_{ij}\rangle.\end{aligned}$$

Thus

$$(\widetilde{N\Delta M})_{ij} = \sum_k \widetilde{N}_{ik}\Delta\widetilde{M}_{kj}.$$

It is now easy to see that $\psi(N\Delta M) = \psi(N)\Delta\psi(M)$. Similarly we can prove that $\psi(N.M) = \psi(N).\psi(M)$. $\square$

Theorem 3.2. *Let $1 \leq p \leq 2$ and $\mathcal{A}$ be a Banach algebra such that $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity. Then $\mathcal{LM}_I^p(\mathcal{A})^{**}$ is unital if and only if $\mathcal{A}^{**}$ is unital.*

Proof. We first note that since $\mathcal{LM}_I^p(\mathcal{A})$ has a bounded approximate identity, from Theorem 2.4 it follows that I is finite. So without loss of generality we may assume that $p = 2$. Then from the following identities,

$$\mathcal{LM}_I^2(\mathcal{A})^* = \ell^2(I \times I, \mathcal{A}^*)$$

$$\mathcal{LM}_I^2(\mathcal{A})^*\mathcal{LM}_I^2(\mathcal{A}) = \ell^2(I \times I, \mathcal{A}^*\mathcal{A})$$
$$\mathcal{LM}_I^2(\mathcal{A})\mathcal{LM}_I^2(\mathcal{A})^* == \ell^2(I \times I, \mathcal{A}\mathcal{A}^*),$$

we conclude that $\mathcal{A}^*$ factors on one side if and only if $\mathcal{LM}_I^p(\mathcal{A})^*$ factors on the same side. Now the result follows from Proposition 2.2 of [4]. $\square$

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References

[1] Esslamzadeh, G. H; *Banach algebra structure and amenability of a class of matrix algebras with applications*, J. Funct. Anal. 161 (1999), 364-383.

[2] Esslamzadeh, G. H; *Duals and topological center of a class of matrix algebras with applications*, Proc. Amer. Math. Soc. 128 (2000), 3493-3503.

[3] Hungerford, T. W; *Algebra*, Reprint of the 1974 original. Graduate Texts in Mathematics, 73. Springer-Verlag, New York, Berlin, 1980.

[4] Lau, A. T. M and Ülger, A; *Topological centers of certain dual algebras*, Trans. Amer. Math. Soc. 348 (1996), 1191-1212.

[5] Palmer, T; *Banach algebras and the general theory of $*$-algebras* Vol. I, Cambridge Univ. Press, 1994.

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