# GENERALIZED ORNSTEIN-UHLENBECK PROCESSES 

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#### Abstract

We consider a family of stationary diffusion processes known as Pearson diffusions and their distributions. We show that they may be interpreted as generalized Ornstein-Uhlenbeck ( $O-U$ ) processes. We apply this result to their transition densities and hitting times.


1 Introduction In this paper we consider a family of stationary diffusion processes and their distributions known as Pearson diffusions. These distributions and diffusions have been introduced by Wong (cf. [7]) (see also [8]). Among them one can find the distribution of the well known classical Ornstein-Uhlenbeck processes but the family also contains much more involved examples. After Wong's preliminary work some subfamilies have been later investigated in different frameworks. In [9], [10], Yor relates a subclass of these diffusions to so called subordinated perpetuities in risk theory and Mathematical Finance. In [3], Madan and Yor use these diffusions to construct martingales with fixed marginals. In [1] Baldi et al. connect this same class to hitting distributions of hyperbolic Brownian motion. Studied in the statistical point of view have also been achieved, for example estimation of parameters is conducted by Nagahara in [4]. In the present paper we show that the family of Pearson distributions may be interpreted as generalized Ornstein-Uhlenbeck ( $O-U$ ) processes. Our framework is inspired by stochastic mechanics since we prove that they are the ground states of a generalized harmonic oscillator. Actually, it is a well known property of the standard O-U process that its density is the ground state of the harmonic oscillator. We apply our result to Laplace transform and distribution function of their hitting times.

The paper is organized as follows. In section 2 we first review Wong's results on Pearson diffusions. In section 3 we give two examples. In section 4 we prove that Pearson diffusions are generalized Ornstein-Uhlenbeck processes. We apply this result in section 5 to express their transition density. Section 6 contains the application to their hitting times.

2 A class of stationary Markov processes and their probability densities In his paper Wong studies the class of probability density functions $W(x)$ which satisfy the so called Pearson equation

$$
\begin{equation*}
\frac{d W(x)}{d x}=\frac{a x+b}{c x^{2}+d x+e} W(x) \tag{2.1}
\end{equation*}
$$

on an interval $\mathbf{J}=] x_{1}, x_{2}[$ such that

$$
\begin{equation*}
c x^{2}+d x+e>0 \quad \text { forall } \quad x \in \mathbf{J} . \tag{2.2}
\end{equation*}
$$

Condition (2.2) involving $B(x)$ and $\mathbf{J}$ implies that the roots of $B(x)$ play a role. In practice one takes $\mathbf{J}=] r_{1}, r_{2}$ [where the $r_{i}$ are the roots of the polynomial $B(x)$. Consequently there are several cases to consider depending on the number of real roots of $B$.

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Before proceeding further let us notice that when $a=0, b=-1, c=d=0$, and $d=1$ one obtains the exponential distribution: the function $W(x)=\mathrm{e}^{-x}$ solves (2.1). Another choice of the parameters leads to a Gamma distribution: $W(x)=\frac{x^{\alpha}}{\Gamma(\alpha+1)} \mathrm{e}^{-x}$ with $\alpha>-1$ is obtained when $a=-1, b=\alpha, c=e=0, d=1$. Normal Gaussian distributions are also obtained (cf. section 2 below).

Theorem 2.1 (cf. [7]). Let $\beta>0$ be a constant. Let us set $B(x):=\beta\left(c x^{2}+d x+e\right)$ and $A(x)=\frac{d B}{d x}+\beta(a x+b)$. Let us denote by $W$ a solution of equation (2.1) on the interval $\mathbf{J}=] x_{1}, x_{2}[$. Then $W$ is the stationary probability density of a Markov diffusion (called Pearson diffusion) whose transition density $p\left(x_{0}, t, x\right)$ is the fundamental solution of the following Fokker-Planck equation on J

$$
\begin{equation*}
\partial_{t} p=\partial_{x x}(B(x) p)-\partial_{x}(A(x) p) \tag{2.3}
\end{equation*}
$$

with boundary condition

$$
\begin{equation*}
\partial_{x}(B(x) p)-A(x) p=0 \tag{2.4}
\end{equation*}
$$

at $x=x_{1}$ and $x=x_{2}$.

Proof of Theorem 2.1 Let us sketch the proof given by Wong. By the method of separation of variables one can show that solving (2.3)-(2.4) is equivalent to look for a function $\varphi$ and a real $\lambda$ satisfying the following system $(S)$ :

$$
\begin{align*}
& \frac{d}{d x}\left(B(x) W(x) \frac{d \varphi}{d x}\right)+\lambda W(x) \varphi(x)=0  \tag{2.5}\\
& B(x) W(x) \frac{d \varphi}{d x}=0 \quad x=r_{1}, r_{2} \tag{2.6}
\end{align*}
$$

This is a well known type of problem called Sturm-Liouville problem; it is a singular SturmLiouville problem when $B$ has real roots. Then the solution $p$ of (2.3)-(2.4) can be expanded along a complete orthonormal system of eigenfunctions $\varphi$ which solve $(S)$ for different values of $\lambda$ (see remark below). Q.E.D.

Moreover using this expansion one shows the following property which enables Wong to interpret $W(x)$ as a stationary probability density.

Theorem 2.2 (cf. [7]). Let us keep the notations of Theorem 2.1. Then for all $x_{0}$

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} p\left(x_{0}, t, x\right)=W(x) \tag{2.7}
\end{equation*}
$$

Remark 2.1 The complete orthonormal system appearing in above mentioned the expansion depends on the values of the parameters; it is for instance the system of Hermite polynomials $(B(x) \equiv 1, a=-1, b=0)$, of Laguerre polynomials $(B(x)=x, a=-1, b=\alpha)$, or of Jacobi polynomials $\left(B(x)=1-x^{2}, a=-(\alpha+\gamma), b=\alpha-\gamma\right)$ with conditions on the constants $\alpha$ and $\gamma$.

## 3 Two examples of Pearson diffusions

3.1 Standard Ornstein-Uhlenbeck processes Let us choose the values of the different parameters as follows: $\beta=1, c=d=0, e=\frac{1}{2}, a=-1, b=0$. Thus we have $B(x)=\frac{1}{2}$ and $A(x)=-x$. Let us also choose $\mathbf{J}=\mathbf{R}$ which is coherent with the fact that $B(x)$ has no real root. Then $W(x)=\frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^{2}}$. The corresponding Markov diffusion is the standard gaussian Ornstein-Uhlenbeck ( O-U) process which can be defined either as the solution of the stochastic differential equation

$$
\begin{equation*}
d X_{t}=d w_{t}-X_{t} d t \tag{3.1}
\end{equation*}
$$

with initial law $W(x) d x$ (where $\left(w_{t}\right)$ is a Brownian motion) or as the process

$$
\begin{equation*}
\tilde{X}_{t}=\mathrm{e}^{-t} \int_{0}^{t} \mathrm{e}^{s} d \tilde{w}_{s}+Z \tag{3.2}
\end{equation*}
$$

where the random variable $Z$ is distributed according to the law $W(x) d x$ and is independent of the Brownian motion $\left(\tilde{w}_{t}\right)$.
3.2 Risk theory and Hitting time of Hyperbolic Brownian motion Let us now choose $\beta=1, c=e=1, d=0$ so that $B(x)=\frac{1}{2}\left(1+x^{2}\right)$. Moreover let $\mu$ and $\nu$ denote two real numbers with $\mu<0$ and set $a=\mu-\frac{1}{2}$ so that $A(x)=\left(\mu+\frac{1}{2}\right) x+\nu$. Let us also choose $\mathbf{J}=\mathbf{R}$ which is coherent with the fact that $B(x)$ has no real roots. Then the following density $W(x)$ belongs to the so called type IV Pearson diffusions.

$$
\begin{equation*}
W(x)=C_{\mu, \nu}\left(1+x^{2}\right)^{\mu-\frac{1}{2}} \mathrm{e}^{2 \nu \operatorname{arctg} x} \tag{3.3}
\end{equation*}
$$

where $C_{\mu, \nu}$ is the normalizing constant such that $\int_{\mathbf{R}} W(x) d x=1$. This density has been much studied in particular because it appears in risk theory (cf. [4], [5]). It has been proved by Yor ([9],[10]) that $W$ is the density of the invariant measure of the Markov diffusion $\left(Y_{t}^{\mu, \nu}\right)$ given by

$$
\begin{equation*}
Y_{t}^{\mu, \nu}=\mathrm{e}^{w_{t}+\mu t} \int_{0}^{t} \mathrm{e}^{-\left(w_{s}+\mu s\right)} d\left(\tilde{w}_{s}+\nu s\right) \tag{3.4}
\end{equation*}
$$

where $\left(w_{t}\right)$ and $\left(\tilde{w}_{t}\right)$ are two independent one dimensional Brownian motions.
Theorem 3.1 (cf. [9], [10]) Let $\mu<0$. Let $\left(\bar{Y}_{t}\right)$ be the process defined by

$$
\bar{Y}_{t}:=\int_{0}^{t} \mathrm{e}^{\left(w_{s}+\mu s\right)} d\left(\tilde{w}_{s}+\nu s\right)
$$

where $\left(w_{t}\right)$ and $\left(\tilde{w}_{t}\right)$ are two independent one dimensional Brownian motions. Then
(i) For all $t>0, Y_{t}$ and $\bar{Y}_{t}$ have the same law.
(ii) $W$ given by (3.3) is the density of $\bar{Y}_{\infty}=\int_{0}^{+\infty} \mathrm{e}^{\left(w_{s}+\mu s\right)} d\left(\tilde{w}_{s}+\nu s\right)$ which exists since $\mu>0$.
This density is also related to hitting times as we are now going to see. Let us recall that planar hyperbolic Brownian motion is the two dimensional process $\left(X_{t}^{1}, X_{t}^{2}\right)$ solution of

$$
\begin{align*}
d X_{t}^{1} & =X_{t}^{2} d w_{t}+\nu X_{t}^{2} d t  \tag{3.5}\\
d X_{t}^{2} & =X_{t}^{2} d \tilde{w}_{t}+\left(\mu+\frac{1}{2}\right) X_{t}^{2} d t \tag{3.6}
\end{align*}
$$

where $\left(w_{t}\right)$ and $\left(\tilde{w}_{t}\right)$ are independent real valued Brownian motions.
Theorem 3.2 (cf. [1]). Let $\mu<0$. Let $\left(X_{0}^{1}, X_{0}^{2}\right)=(0, y)$ with $y>0$. Then
(i) $\left(X_{t}^{1}, X_{t}^{2}\right) \equiv\left(y \int_{0}^{t} \mathrm{e}^{\tilde{w}_{s}+\mu s} d\left(w_{s}+\nu s\right), y \mathrm{e}^{\tilde{w}_{t}+\mu t}\right)$
(ii) $\operatorname{Let} T_{0}=\inf \left\{t>0 ; X_{t}^{2}=0\right\}$. The random variablef $T_{0}$ has the same law as $y \int_{0}^{+\infty} \mathrm{e}^{\tilde{w}_{s}+\mu s} d\left(w_{s}+\right.$ $\nu s)$.

## 4 Pearson diffusions are generalized Ornstein-Uhlenbeck processes

4.1 The harmonic oscillator The similarity between identities (3.2) and (3.4) may rise the following question: is it possible to show that every Pearson diffusion is of the same type as standard O-U process? in which sense? In order to provide an answer to this question let us recall still another description of Standard O-U. process. In the sequel $W_{0}(x)=\frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^{2}}$ and $H$ denotes the following second order linear differential operator $u \mapsto H u$ such that

$$
\begin{equation*}
H u(x)=\frac{1}{2}\left(\frac{d^{2}}{d x^{2}} u(x)-x^{2}\right) u(x) \tag{4.1}
\end{equation*}
$$

$H$ is also called the Harmonic oscillator. It is also a Schrödinger operator since $H u=$ $\frac{1}{2}\left(\frac{d^{2}}{d x^{2}} u-V u\right)$ with quadratic potential $V_{0}(x)=x^{2}$. We keep the notations of Example 1.

Theorem 4.1 Let us define $L=\frac{1}{2} \frac{d^{2}}{d x^{2}}-x \frac{d}{d x}$. The operator $H$ is self adjoint w.r.t. to Lebesgue measure whereas the operator $L$ is self adjoint w.r.t. the measure $W_{0}^{\frac{1}{2}}(x) d x$.

Theorem 4.2 Let $\mathcal{C}$ denote the mapping $\mathcal{C}: L^{2}(\mathbf{R}, d x) \rightarrow L^{2}\left(\mathbf{R}, W_{0}(x) d x\right)$ defined by $\mathcal{C} f: x \mapsto f(x) W_{0}^{-\frac{1}{2}}(x)$. Then
(i) $\mathcal{C}$ is an isometry
(ii) $\mathcal{C}^{-1} L \mathcal{C} f=H f+\frac{1}{2} f$
(iii) the largest eigenvalue of the operator $H$ is $-\frac{1}{2}$ with corresponding eigenvalue $W_{0}^{\frac{1}{2}}$.

Definition 4.1 From (ii) one says that $L$ is unitary equivalent to $H+\frac{1}{2}$.
Remark 4.1 1. Note that $L$ is the infinitesimal generator of the diffusion solving the stochastic differential equation (3.1) that is of standard $O-U$ process.
2. Because of assertion (iii), the standard $O-U$ may be considered as the ground state of the harmonic oscillator. Actually, in Physics, the operator would be -H which has the form (-Laplacian+ potential). Then $\frac{1}{2}$ is the smallest eigenvalue of $-H$ and the corresponding eigenfunction $W_{0}^{\frac{1}{2}}(x) d x$ is the ground state.
3. For subsequent generalisations, let us note the simple fact that $H+\frac{1}{2}=H-\left(-\frac{1}{2}\right)$.
4.2 Pearson diffusions as ground states of generalized harmonic oscillators We are now going to prove that a statement analogous to Theorem 4.2 holds for a general Pearson diffusion.

Theorem 4.3 Let $a<0, b$ and $B(x)=\left(c x^{2}+d x+e\right)$ be given. Let $W$ denote the solution of Pearson equation (2.1). Let us set $\alpha(x):=2 B(x)$; we denote by $\|a x+b\|_{\alpha^{-1}}^{2}$ the quantity $\frac{(a x+b)^{2}}{2 \alpha(x)}$. Then
(i) the infinitesimal generator of the Pearson diffusion associated to $a, b$ and $B$ is unitarily equivalent to the operator $H_{B}-\frac{a}{2} I$ where

$$
\begin{align*}
H_{B} u(x) & \left.=\frac{1}{2} \Delta_{\alpha} u(x)-\|a x+b\|_{\alpha^{-1}}^{2} u(x)\right)  \tag{4.2}\\
& =\frac{1}{2} \frac{d}{d x}\left(\alpha(x) \frac{d u}{d x}\right)(x)-\frac{1}{2 \alpha(x)}(a x+b)^{2} u(x) \tag{4.3}
\end{align*}
$$

(ii) $H_{B}$ is self adjoint w.r.t. Lebesgue measure
(iii) $\frac{a}{2}$ is the largest eigenvalue of the operator $H_{B}$ with corresponding eigenvalue $W^{\frac{1}{2}}$.

Remark 4.2 1. The operator $H_{B}$ has the form $\Delta_{\alpha}-V$; it is a Schrödinger operator. 2. The potential $V(x):=\|a x+b\|_{\alpha^{-1}}^{2}$ is quadratic w.r.t. the metric $\alpha$. $H_{B}$ is therefore an extension of the classical harmonic oscillator $H$.

Proof of Theorem 4.3 The proof is not difficult and is left to the reader.
5 Transition density of Pearson diffusions In this section we express the transition density of any Pearson diffusion using the operator $H_{B}$ and the function $W$.

Theorem 5.1 Let $a<0$ and $B(u)=\left(c u^{2}+d u+e\right)$ be given. Let $x, y$ be two real numbers. For $t>0$ we denote by $p(x, t, y)$ the transition density of the Pearson diffusion associated to $a, b, B$ with stationary density $W(x)$. Then

$$
\begin{equation*}
p(t, x, y)=W^{-\frac{1}{2}}(x) \mathrm{e}^{t\left(H_{B}-\frac{a}{2}\right)}(x, y) W^{\frac{1}{2}}(y) \tag{5.1}
\end{equation*}
$$

The result of this theorem relies on the following lemma..
Lemma 5.1 Let $V$ a continuous function taking positive values. Let $\mathcal{H}$ be the operator $\mathcal{H} f(x)=\frac{d}{d x}\left(B(x) \frac{d f}{d x}\right)(x)-V(x) f(x)$ and $\mathrm{e}^{t \mathcal{H}}(x, y)$ denote its kernel. Consider the process $\bar{X}_{t}:=x+\int_{0}^{t} \sqrt{2 B\left(\bar{X}_{r}\right)} d w_{r}$ where $w$ is a Brownian motion. We denote by $d_{t}^{x}(\cdot)$ the density of $\bar{X}_{t}$ w.r.t. Lebesgue measure and we define

$$
\begin{equation*}
Z_{s}:=\mathrm{e}^{-\int_{0}^{t} V\left(\bar{X}_{s}\right) d s} \mathrm{e}^{\int_{0}^{t} \frac{B^{\prime}}{\sqrt{2 B}}\left(\bar{X}_{s}\right) d w_{s}-\frac{1}{2} \int_{0}^{t}\left|\frac{B^{\prime}}{\sqrt{2 B}}\right|^{2}\left(\bar{X}_{s}\right) d s} \tag{5.2}
\end{equation*}
$$

Then

$$
\begin{equation*}
\mathrm{e}^{t \mathcal{H}}(x, y)=\mathbf{E}\left(Z_{t} \mid \bar{X}_{t}=y\right) d_{t}^{x}(y) \tag{5.3}
\end{equation*}
$$

Proof of Lemma 5.1. Let $u(t, x)$ satisfying the pde

$$
\begin{equation*}
\left(\partial_{t}-\mathcal{H}\right) u=0 \tag{5.4}
\end{equation*}
$$

with initial boundary condition $u(0, x)=f(x)$. We are going to prove that $M_{s}:=u(t-$ $\left.s, \bar{X}_{s}\right) Z_{s}$ defines a martingale on $[0, t]$. Indeed using Ito formula (cf. [2]), we obtain first

$$
\begin{align*}
d M_{s} & =\partial_{x} u\left(t-s, \bar{X}_{s}\right) Z_{s} d \bar{X}_{s}+\frac{B^{\prime}}{\sqrt{2 B}}\left(\bar{X}_{s}\right) M_{s} d w_{s}  \tag{5.5}\\
& +Z_{s}\left[-\partial_{t} u\left(t-s, \bar{X}_{s}\right)+B\left(\bar{X}_{s}\right) \partial_{x x} u\left(t-s, \bar{X}_{s}\right)\right.  \tag{5.6}\\
& \left.+B^{\prime}\left(\bar{X}_{s}\right) \partial_{x} u\left(t-s, \bar{X}_{s}\right)-V\left(\bar{X}_{s}\right) u\left(t-s, \bar{X}_{s}\right)\right] d s \tag{5.7}
\end{align*}
$$

Then using (5.4) we conclude that $\left(M_{s}\right)$ is a martingale on $[0, t]$. Therefore

$$
\begin{equation*}
u(t, x)=\mathbf{E}\left(f\left(\bar{X}_{t}\right) Z_{t}\right) \tag{5.8}
\end{equation*}
$$

There is another expression of function $u$ involving the kernel $\mathrm{e}^{t \mathcal{H}}$ of $\mathcal{H}$ namely

$$
\begin{equation*}
u(t, x)=\mathrm{e}^{t \mathcal{H}} f(x)=\int \mathrm{e}^{t \mathcal{H}}(x, y) f(y) d y \tag{5.9}
\end{equation*}
$$

Indeed, let $v(s, x):=u(t-s, x)$. This function satisfies the following pde adjoint of (5.4)

$$
\begin{equation*}
\left(\partial_{t}+\mathcal{H}\right) v=0 \tag{5.10}
\end{equation*}
$$

with terminal boundary condition $v(t, x)=f(x)$. Hence

$$
\begin{equation*}
v(s, y) \sim \int \mathrm{e}^{(t-s) \mathcal{H}}(y, z) f(z) d z \tag{5.11}
\end{equation*}
$$

Since $\mathcal{H}$ is self adjoint and $v(0, y) \equiv u(t, y)$ one obtains (5.9). By comparing the two expressions (5.8) and (5.9) of $u(t, x)$ we obtain the identity

$$
\begin{equation*}
\mathbf{E}\left(f\left(\bar{X}_{t}\right) Z_{t}\right)=\int \mathrm{e}^{t \mathcal{H}}(x, y) f(y) d y \tag{5.12}
\end{equation*}
$$

It remains to take the conditional expectation of $Z_{t}$ given $\bar{X}_{t}$ on the left-hand side to get

$$
\begin{equation*}
\int \mathbf{E}\left(Z_{t} \mid \bar{X}_{t}=y\right) d_{t}^{x}(y) f(y)=\int \mathrm{e}^{t \mathcal{H}}(x, y) f(y) d y \tag{5.13}
\end{equation*}
$$

for all $f$ continuous with compact support which implies Lemma 5.1. Q.E.D.

Proof of Theorem 5.1. In this proof we follow an argument of [6]. However in [6], the coefficient $B(x)$ is constant equal to 1 . Let $f$ be a nonnegative measurable function and $\left(X_{t}\right)$ denote the Pearson diffusion associated to $a, b, B$. By Girsanov theorem (cf. [2]),

$$
\begin{equation*}
\left.\mathbf{E}\left(f\left(X_{t}\right)\right)=\mathbf{E}\left(f\left(\bar{X}_{t}\right)\right) \mathrm{e}^{\int_{0}^{t} \theta_{s} d w_{s}-\frac{1}{2} \int_{0}^{t}\left|\theta_{s}\right|^{2} d s}\right) \tag{5.14}
\end{equation*}
$$

where $\theta_{s}=\left(\frac{B^{\prime}}{\sqrt{2 B}}+\frac{B}{\sqrt{2 B}} \frac{W^{\prime}}{W}\right)\left(\bar{X}_{s}\right.$. Ito formula applied to the process $\log W^{\frac{1}{2}}\left(\bar{X}_{t}\right)$ yields

$$
\begin{equation*}
\log W^{\frac{1}{2}}\left(\bar{X}_{t}\right)-\log W^{\frac{1}{2}}\left(\bar{X}_{0}\right)=\int_{0}^{t} \frac{1}{2} \frac{W^{\prime}}{W}\left(\bar{X}_{s}\right) d \bar{X}_{s}+\int_{0}^{t} A_{s} d s \tag{5.15}
\end{equation*}
$$

where we set $A_{s}:=B\left(\bar{X}_{s}\right)\left(\left(\frac{d^{2}}{d x^{2}} W^{\frac{1}{2}}\right) W^{-\frac{1}{2}}-\frac{1}{4}\left|\frac{W^{\prime}}{W}\right|^{2}\right)\left(\bar{X}_{s}\right)$. Moreover since $\left(H_{B}-\frac{a}{2}\right) W^{\frac{1}{2}} \equiv 0$ we obtain

$$
\begin{gathered}
\int_{0}^{t} \theta_{s} d w_{s}-\frac{1}{2} \int_{0}^{t}\left|\theta_{s}\right|^{2} d s= \\
\log W^{\frac{1}{2}}\left(\bar{X}_{t}\right)-\log W^{\frac{1}{2}}\left(\bar{X}_{0}\right)-\int_{0}^{t} V\left(\bar{X}_{s}\right) d s-\frac{a}{2} t+U_{t}
\end{gathered}
$$

where $U_{t}:=\int_{0}^{t} \frac{B^{\prime}}{\sqrt{2 B}}\left(\bar{X}_{s}\right) d w_{s}-\frac{1}{2} \int_{0}^{t}\left|\frac{B^{\prime}}{\sqrt{2 B}}\right|^{2}\left(\bar{X}_{s}\right) d s$. Therefore

$$
\begin{equation*}
\left.\mathbf{E}\left(f\left(X_{t}\right)\right)=\mathbf{E}\left(f\left(\bar{X}_{t}\right)\right) \frac{W^{\frac{1}{2}}\left(\bar{X}_{t}\right)}{W^{\frac{1}{2}}(x)} Z_{t}\right) \mathrm{e}^{-\frac{a}{2} t} \tag{5.16}
\end{equation*}
$$

where $Z_{t}$ has been introduced in Lemma 5.1. The left-hand side of this equality is equal to $\int p(t, x, y) f(y) d y$ where $p(t, x, y)$ is the transition density of the Pearson diffusion $\left(X_{t}\right)$. By conditioning by $\bar{X}_{t}$ on the right-hand side and applying Lemma 5.1 we obtain Theorem 5.1. Q.E.D.

6 Application to hitting times We are now going to prove that the unitary equivalence of Theorem 4.3 enables us to express the Laplace transform and distribution function of hitting times of Pearson diffusions. Let us begin by the Laplace transform.

Theorem 6.1 We keep the notations of Theorem 5.1. In particular $X$ is a Pearson diffusion starting from $x$. Let $r_{\lambda}$ be the kernel of the resolvent $R_{\lambda}$ of the operator $\left(H_{B}-\frac{a}{2}\right)$ which is defined as $R_{\lambda}=\int_{0}^{+\infty} \mathrm{e}^{-\left(\lambda-H_{B}+\frac{a}{2}\right) t} d t$. Let $\tau_{x}(m):=\inf \left\{s>0 ; X_{s}=m\right\}$ be the first hitting time of level $m$ by $X$. Then

$$
\begin{equation*}
\mathbf{E}\left(\mathrm{e}^{-\lambda \tau_{x}(m)}\right)=W^{-\frac{1}{2}}(x) \frac{r_{\lambda}(x, m)}{r_{\lambda}(m, m)} W^{\frac{1}{2}}(m) \tag{6.1}
\end{equation*}
$$

In order to prove this theorem we first recall a Lemma which is valid for stationary diffusions.
Lemma 6.1 Let $Y$ be a stationary diffusion starting from $x$ with transition density $q(t, x, y)$. Let $\tilde{q}_{\lambda}(x, y)=\int_{0}^{+\infty} \mathrm{e}^{-\lambda t} q(t, x, y) d t$. The hitting time $\tau_{x}(m)$ is defined as in Theorem 6.1. The following identity holds true

$$
\begin{equation*}
\mathbf{E}\left(\mathrm{e}^{-\lambda \tau_{x}(m)}\right)=\frac{\tilde{q}_{\lambda}(x, m)}{\tilde{q}_{\lambda}(m, m)} \tag{6.2}
\end{equation*}
$$

Proof of Lemma 6.1. Let $x<m<y$. Since $Y$ has continuous paths we can write

$$
\begin{align*}
q(t, x, y) & =\mathbf{P}\left(Y_{t}=y \mathbf{1}_{\left\{\tau_{x}(m)<t\right\}}\right)  \tag{6.3}\\
& =\int_{0}^{t} q(t-u, m, y) \mathbf{P}\left(\tau_{x}(m) \in d u\right) \tag{6.4}
\end{align*}
$$

The second equality holds thanks to the strong Markov property and stationarity of $Y$. Identity (6.2) follows easily.
Proof of Theorem 6.1 Since $\tilde{p}_{\lambda}(x, y)=\int_{0}^{+\infty} \mathrm{e}^{-\lambda t} p(t, x, y) d t$ we deduce from Theorem 5.1 that

$$
\begin{aligned}
\tilde{p}_{\lambda}(x, m)= & W^{-\frac{1}{2}}(x) \int_{0}^{+\infty} \mathrm{e}^{-t\left(\lambda-H_{B}+\frac{a}{2}\right)} d t W^{\frac{1}{2}}(m) \\
& =W^{-\frac{1}{2}}(x) r_{\lambda}(x, m) W^{\frac{1}{2}}(m)
\end{aligned}
$$

Taking the limit of these identities when $x \rightarrow m$ and using Lemma 6.1 we obtain identity (6.1). Q.E.D.

We now turn to the distribution function of hitting times.
Theorem 6.2 We keep the framework of Lemma 5.1. Let us define $\mathcal{H}_{+}(m)$ and $\mathcal{H}_{-}(m)$ respectively by $\mathcal{H}_{+}(m):=\lim _{\lambda \nearrow+\infty}\left(\mathcal{H}-\lambda \mathbf{1}_{]-\infty, m[ }\right)$ and $\mathcal{H}_{-}(m):=\lim _{\lambda \nearrow+\infty}\left(\mathcal{H}+\lambda \mathbf{1}_{] m,+\infty,[ }\right)$. Then

$$
\begin{array}{ll}
\forall x<m & \mathbf{P}\left(\tau_{x}(m)>t\right)=W^{-\frac{1}{2}}(x) \mathrm{e}^{t\left(\mathcal{H}_{+}(m)-\frac{a}{2}\right)} W^{\frac{1}{2}}(x) \\
\forall x>m & \mathbf{P}\left(\tau_{x}(m)>t\right)=W^{-\frac{1}{2}}(x) \mathrm{e}^{t\left(\mathcal{H}_{-}(m)-\frac{a}{2}\right)} W^{\frac{1}{2}}(x) \tag{6.6}
\end{array}
$$

Proof of Theorem 6.2. The following identity, which holds since $X$ has almost surely continuous paths, will enable us to apply Lemma 5.1:

$$
\begin{equation*}
\mathbf{P}\left(\tau_{x}(m)>t\right)=\lim _{\lambda \nearrow+\infty} \mathbf{P}\left(\mathrm{e}^{-\lambda \int_{0}^{t} \mathbf{1}_{]-\infty, m[ }\left(X_{s}\right) d s}\right) \tag{6.7}
\end{equation*}
$$

We then proceed as in the proof of Theorem 5.1 using Girsanov theorem and the fact that $\left.\left(H_{B}-\frac{a}{2}\right) W^{\frac{1}{2}}\right)=0$. We thus obtain

$$
\begin{equation*}
\mathbf{P}\left(\tau_{x}(m)>t\right)=\mathrm{e}^{-\frac{a}{2} t} W^{-\frac{1}{2}}(x) \lim _{\lambda \nearrow+\infty} \mathbf{E}\left(W^{\frac{1}{2}}\left(\bar{X}_{t}\right) \tilde{Z}_{t}\right) \tag{6.8}
\end{equation*}
$$

where $\tilde{Z}_{t}:=\mathrm{e}^{-\int_{0}^{t} \tilde{V}\left(\bar{X}_{s}\right) d s} \mathrm{e}^{\int_{0}^{t} \frac{B^{\prime}}{\sqrt{2 B}}\left(\bar{X}_{s}\right) d w_{s}-\frac{1}{2} \int_{0}^{t}\left|\frac{B^{\prime}}{\sqrt{2 B}}\right|^{2}\left(\bar{X}_{s}\right) d s}$ and $\tilde{V}(z):=V(x)+\lambda \mathbf{1}_{]-\infty, m[ }(z)$. Using Lemma 5.1 for the potential $\tilde{V}$, we obtain

$$
\begin{equation*}
\mathbf{P}\left(\tau_{x}(m)>t\right)=\mathrm{e}^{-\frac{a}{2} t} W^{-\frac{1}{2}}(x) \lim _{\lambda \nearrow+\infty} \int \mathrm{e}^{t\left(H_{B}-\lambda \mathbf{1}_{]-\infty, m!}\right)}(x, y) W^{\frac{1}{2}}(y) d y \tag{6.9}
\end{equation*}
$$

Q.E.D.

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