ON PRIME AND SEMIPRIME IDEALS IN SUBTRACTION SEMIGROUPS

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Abstract. In this paper, we characterize prime and semiprime ideals in a subtraction semigroup. Among them for any subtraction semigroup, an ideal is semiprime if and only if it is intersection of all prime ideals containing it. Moreover, if an ideal of a subtraction semigroup is prime, then it is semiprime and strongly irreducible.

1. Introduction

B. M. Schein [4] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [5] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we characterize prime and semiprime ideals in a subtraction semigroup. Among them for any subtraction semigroup, an ideal is semiprime if and only if it is intersection of all prime ideals containing it. Moreover, if an ideal of a subtraction semigroup is prime, then it is semiprime and strongly irreducible.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

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A nonempty subset S of a subtraction algebra X is said to be a *subalgebra* of X if $x - y \in S$ whenever $x, y \in S$.

In a subtraction algebra, the following hold:
(S1) $x - 0 = x$ and $0 - x = 0$.
(S2) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(S3) $(x - y) - x = 0$.
(S4) $x - (x - (x - y)) = x - y$.
(S5) $(x - y) - (y - x) = x - y$.

By a *subtraction semigroup* we mean an algebra $(X; \cdot, -)$ with two binary operations "$-$" and "$\cdot$" that satisfies the following axioms: for any $x, y, z \in X$,
(SS1) $(X; \cdot)$ is a semigroup;
(SS2) $(X; -)$ is a subtraction algebra;
(SS3) $x(y - z) = xy - xz$ and $(x - y)z = xz - yz$.

Example 2.1. Let $X = \{0, 1\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	1
0	0	0
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

It is easy to check that X is a subtraction semigroup.

Lemma 2.2. ([2]) *Let X be a subtraction semigroup. Then $(X; \leq)$ is a poset, where $x \leq y \Leftrightarrow x - y = 0$ for any $x, y \in X$.*

Proposition 2.3. ([2]) *Let X be a subtraction semigroup. Then for any $x, y \in X$ $x \wedge y = x - (x - y)$ is the greatest lower bound of x and y.*

Lemma 2.4. *Let X be a subtraction semigroup. If $x \leq y$ for any $x, y \in X$, then we have*

$$sx \leq sy \text{ and } xs \leq ys.$$

Proof. Since $x \leq y$ implies $x - y = 0$, we obtain $sx - sy = s(x - y) = s0 = 0$ for any $s \in X$. Hence $sx \leq sy$. Similarly, $xs \leq ys$. □

Definition 2.5. ([2]) Let X be a subtraction semigroup. A subalgebra I of $(X, -)$ is called a *left ideal* of X if $XI \subseteq I$, a *right ideal* if $IX \subseteq I$, and an (*two-sided*) *ideal* if it is both a left and right ideal.

Example 2.6. Let $X = \{0, 1, 2, 3, 4, 5\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	0
5	0	0	5	0	0	5

It is easy to check that $(X; -, \cdot)$ is a subtraction semigroup. Let $I = \{0, 1, 3, 4\}$. Then I is an ideal of X.

Example 2.7. Let X be a subtraction semigroup and $a \in X$. Then

$$aX = \{ax \mid x \in X\}$$

is a right ideal of X.

Proof. Let $ax, ay \in aX$. Then $ax - ay = a(x-y) \in aX$, and so aX is a subalgebra of $(X, -)$. Let $ax \in aX$ and $z \in X$. Then $(ax)z = a(xz) \in aX$, which shows that $(aX)X \subseteq aX$. Therefore, aX is a right ideal. □

Lemma 2.8. *Let X be a subtraction semigroup. Then the following hold.*

(1) $x \wedge (x-y) = x - y$;
(2) $x \wedge (y-x) = 0$;
(3) $(x-y) \wedge (y-x) = 0$.
(4) $(x-z) \wedge (y-z) = (x-z) \wedge y$.

Proof. (1) For any $x, y \in X$, we have

$$\begin{aligned} x \wedge (x-y) &= x - (x - (x-y)) \\ &= x - y \quad \text{(from (S4))}. \end{aligned}$$

(2) For any $x, y \in X$, we have

$$\begin{aligned} x \wedge (y-x) &= x - (x - (y-x)) \\ &= x - x \quad \text{(from (S3))} \\ &= 0. \end{aligned}$$

(3) For any $x, y \in X$, we obtain

$$\begin{aligned} (x-y) \wedge (y-x) &= (x-y) - ((x-y) - (y-x)) \\ &= (x-y) - (x-y) \quad \text{(from (S5))} \\ &= 0. \end{aligned}$$

(4) For any x, y and $z \in X$, we obtain

$$\begin{aligned} (x-z) \wedge (y-z) &= (x-z) - ((x-z) - (y-z)) \\ &= (x-z) - ((x-y) - z) \quad \text{(from Lemma 2.2, [3])} \\ &= (x-z) - ((x-z) - y) \\ &= (x-z) \wedge y. \end{aligned}$$

□

The element 1 is called a *unity* in a subtraction semigroup X if $1x = x1 = x$ for all $x \in X$.

Definition 2.9. ([2]) A *strong subtraction semigroup* is a subtraction semigroup X that satisfies the following condition : for each $x, y \in X$,

$$x - y = x - xy.$$

If a strong subtraction semigroup X has a unity 1, then 1 is the greatest element in X since $x - 1 = x - x1 = x - x = 0$ for all $x \in X$.

Example 2.10. Let $X = \{0, a, b, 1\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

It is easy to check that $(X; -, \cdot)$ is a strong subtraction semigroup with unity 1.

Lemma 2.11. ([2]) *Let X be a strong subtraction semigroup with* 1*. Then we have*

$$x \wedge y = xy.$$

Let X be a strong subtraction semigroup and A be an ideal of X. For $x \in X$ such that $x \notin A$, we denote

$$A_x = \{y \mid y \wedge x \in A\}.$$

Lemma 2.12. *Let A be an ideal of a strong subtraction semigroup X with* 1.

Proof. Let $a, b \in A_x$. Then we have $a \wedge x \in A$ and $b \wedge x \in A$, and so $(a - b) \wedge x = (a \wedge x) - (b \wedge x) \in A$. Moreover, let $a \in A_x$ and $s \in X$. Then we have $a \wedge x = ax \in A$. So, we obtain $sa \wedge x = (sa)x = s(ax) \in A$. Similarly, $as \wedge x \in A$. This completes the proof. □

Theorem 2.13. ([2]) *Let $(X, -, \cdot)$ be a strong subtraction semigroup and I a subalgebra of $(X, -)$. Then the followings are equivalent :*

(1) *I is an ideal in $(X, -, \cdot)$,*
(2) *$y \in I$ and $x \leq y$ imply $x \in I$.*

Theorem 2.14. ([2]) *Let X be a strong subtraction semigroup with a unity* 1*. Then the following are equivalent :*

(1) *I is an ideal in X,*
(2) *$y \in I$ and $x \leq y$ imply $x \in I$.*

Let S be a subset of a subtraction semigroup X. The *ideal* of X generated by S is the intersection of all ideals in X containing S. Denote (S) the ideal generated by S. If $S = \{a\}$, then (S) is denoted by (a).

Theorem 2.15. ([2]) *If X is a strong subtraction semigroup, then the principal ideal generated by $a \in X$ is $(a] = \{x \in X \mid x \leq a\}$.*

If X is a strong subtraction semigroup with 1, then the principal ideal generated by a is $(a] = aX$.

Theorem 2.16. *Let X be a strong subtraction semigroup with* 1*. For an ideal $P(\neq X)$ of X, the following are equivalent:*

(1) *If A and B are ideals of X such that $AB \subseteq P$, then we have either $A \subseteq P$ or $B \subseteq P$.*
(2) *If $(a), (b)$ are principal ideals of X such that $(a)(b) \subseteq P$, then we have either $a \in P$ or $b \in P$.*
(3) *If $aXb \subseteq P$, then we have $a \in P$ or $b \in P$.*
(4) *If I_1 and I_2 are two right ideals of X such that $I_1 I_2 \subseteq P$, then we have either $I_1 \subseteq P$ or $I_2 \subseteq P$.*
(5) *If J_1 and J_2 are two left ideals of X such that $J_1 J_2 \subseteq P$, then we have either $J_1 \subseteq P$ or $J_2 \subseteq P$.*

Proof. (1)$\Longrightarrow$ (2). It is easy to prove, and so proof is omitted. (2)$\Longrightarrow$ (1). Suppose that A and B ideals of X such that $AB \subseteq P$ but $A \not\subseteq P$. Then there exists a $a \in A$ such that $a \notin P$. Now for any $b \in B$, we have $(a)(b) \subseteq AB \subseteq P$, whence $b \in P$. Thus we have $B \subseteq P$. (1)$\Longrightarrow$ (3). Let $aXb \subseteq P$. Then by (1), $(a)(b) \subseteq P$. So we get $a \in (a) \subseteq P$ or $b \in (b) \subseteq P$. (3)$\Longrightarrow$ (4). Let $I_1 I_2 \subseteq P$ and $I_1 \not\subseteq P$. Then there exists an element $a_1 \in I_1$ such that $a_1 \notin P$. So for every $a_2 \in I_2$, we have $a_1 X a_2 \subseteq I_1 I_2 \subseteq P$. Hence from (3), we obtain $a_2 \in P$, that is, $I_2 \subseteq P$. Similarly we have (3)$\Longrightarrow$ (5). □

3. Prime and Semiprime Ideals

In what follows, let X denote a subtraction semigroup unless otherwise specified.

Definition 3.1. ([2]) Let X and X' be subtraction semigroups. A mapping $f : X \to X'$ is called a *subtraction semigroup homomorphism* (briefly, *homomorphism*) if $f(x-y) = f(x) - f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$.

Lemma 3.2. *Let $f : X \to X'$ be a subtraction semigroup homomorphism. Then*
(1) $f(0) = 0$,
(2) $x \leq y$ *imply* $f(x) \leq f(y)$.
(3) $f(x \wedge y) = f(x) \wedge f(y)$.

Proof. (1). Suppose that x is an element of X. Then

$$f(0) = f(x - x) = f(x) - f(x) = 0$$

(2) Let $x \leq y$. Then we have $x - y = 0$. Thus we have

$$0 = f(x - y) = f(x) - f(y),$$

and so $f(x) \leq f(y)$.
(3) $f(x \wedge y) = f(x - (x - y)) = f(x) - (f(x) - f(y)) = f(x) \wedge f(y)$. □

Let X be a subtraction semigroup and $I, J \subseteq X$. Denote

$$I \wedge J = \{i \wedge j \mid i \in I, j \in J\}.$$

Lemma 3.3. *Let X and X' be subtraction semigroups and let $f : X \to X'$ be a subtraction semigroup homomorphism. Then*
(1) $(I \wedge J) \wedge K = I \wedge (J \wedge K)$ *for all* $I, J, K \subseteq X$,
(2) $f(I_1 \wedge I_2) = f(I_1) \wedge f(I_2)$ *for all* $I_1, I_2 \subseteq X$,
(3) $f^{-1}(J_1) \wedge f^{-1}(J_2) \subset f^{-1}(J_1 \wedge J_2)$ *for all* $J_1, J_2 \subset X'$.

Proof. (1) Since $(x\wedge y)\wedge z = x\wedge(y\wedge z)$, we have $(I\wedge J)\wedge K = I\wedge(J\wedge K)$ for all $I, J, K \subseteq X$.
(2) Since for any $x \in I_1$ and $y \in I_2, x\wedge y = x-(x-y)$, we obtain $f(x\wedge y) = f(x-(x-y)) = f(x) - (f(x) - f(y)) = f(x) \wedge f(y)$, and so $f(I_1 \wedge I_2) = f(I_1) \wedge f(I_2)$.
(3) Let $f^{-1}(j_1)\wedge f^{-1}(j_2) = x\wedge y$. Then we have $f(x\wedge y) = f(x)\wedge f(y) = j_1\wedge j_2 \in J_1\wedge J_2$, and so $x \wedge y \in f^{-1}(J_1 \wedge J_2)$. This completes the proof. □

Definition 3.4. Let X be a subtraction semigroup. A *prime ideal* of X is defined to be an ideal P such that $x \wedge y \in P$ then $x \in P$ or $y \in P$.

Example 3.5. In Example 2.10, define the operation "$\wedge$" are defined by

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Let $P = \{0, a\}$. Then P is a prime ideal of a subtraction semigroup X.

Let X be a strong subtraction semigroup with 1. We define the set

$$ann(a) = \{x \in X \mid x \wedge a = 0, a \in X\}$$

as the *annihilator* of a. It can be shown that $ann(a)$ is an ideal of X, and if $t \leq s$, then $ann(s) \subseteq ann(t)$ for any $s, t \in X$.

Proposition 3.6. *Suppose that A is an ideal of strong subtraction semigroup X with* 1. *Let I be an ideal that is maximal among all annihilators of non-zero elements of A. Then I is a prime ideal of X.*

Proof. Suppose $I = ann(x)$ for some $0 \neq x \in A$. Let $a, b \in X$ such that $a \wedge b \in I$ and $a \notin I$. Then $x \wedge a \neq 0$. Now $ann(x) \subseteq ann(x \wedge a)$. Since $x \in A, x \wedge a \leq x$ and by Theorem 2.13, we have $0 \neq x \wedge a \in A$. Since I is maximal among all annihilators of non-zero elements of A, we obtain that $ann(x \wedge a) = ann(x) = I$. Thus $a \wedge b \in I$ implies that $a \wedge b \wedge x = 0$ and hence $b \in ann(x \wedge a) = ann(x) = I$. This proves that I is a prime ideal of X. □

Let P be a prime ideal of strong subtraction semigroup X. Put

$$0_P = \{x \in X \mid x \wedge a = 0 \text{ for some } a \notin P\}.$$

0_P is evidently non-empty set because $0 \in 0_P$.

Proposition 3.7. *0_P is an ideal of strong subtraction semigroup X with* 1.

Proof. Let $x, y \in 0_P$. Then we have $x \wedge a = xa = 0$ and $y \wedge a = ya = 0$ for some $a, b \notin P$. Since P is prime ideal, $a \notin P, b \notin P$ implies $a \wedge b \notin P$. Hence we obtain

$$\begin{aligned}(x-y) \wedge (a \wedge b) &= (x-y)(a \wedge b) = (x-y)(ab)\\ &= x(ab) - y(ab) = (xa)b - y(ab)\\ &= 0b - y(ab) = 0 - y(ab)\\ &= 0,\end{aligned}$$

and so we get $x - y \in 0_P$. Let $x \in 0_P$ and $s \in X$. Then $x \wedge a = xa = 0$ for some $a \notin P$. Now

$$sx \wedge a = (sx)a = s(xa) = s0 = 0,$$

and so $sx \in 0_P$. Similarly $xs \in 0_P$. This completes the proof. □

Proposition 3.8. *Let P be an ideal of a subtraction semigroup X. Then the following statements are equivalent.*

(1) *P is prime.*
(2) *For any ideals I and J of $X, I \wedge J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.*
(3) *For any elements i and j in X, $i \notin P$ and $j \notin P$ implies $(i) \wedge (j) \nsubseteq P$.*

Proof. (1) $\Longrightarrow$ (2). Suppose that P is a prime ideal of X such that $A \wedge B \subset P$, where A and B are ideals of X. Assume that $A \not\subset P$ and $B \not\subset P$. Then there exist $x \in A \setminus P$ and $y \in B \setminus P$, and so $x \wedge y \in A \wedge B \subset P$. Since P is prime, it follows that $x \in P$ or $y \in P$, which is a contradiction.

(1)$\Longrightarrow$ (3). Let P be a prime, $i \notin P$ and $j \notin P$. Suppose that $(i) \wedge (j) \subseteq P$. Then $i \wedge j \in (i) \wedge (j) \in P$. Since P is prime, we have $i \in P$ or $j \in P$. This leads to a contradiction. Thus we have $(i) \wedge (j) \nsubseteq P$.

(3)$\Longrightarrow$ (1). Since $(i) \wedge (j) \subseteq P$ imply $i \in P$ or $j \in P$, the proof is trivial. □

The following results are easy to prove, and so proofs are omitted.

Proposition 3.9. *Let X be a subtraction semigroup. If I is an ideal of X and P a prime ideal of X, then $I \cap P$ is a prime ideal of I.*

Proposition 3.10. *Let X be a subtraction semigroup. If P is a prime ideal of X and a is an element of X such that $XaX \subseteq P$, then $a \in P$.*

Definition 3.11. Let X be a subtraction semigroup. A nonempty subset H of X is called an *m-system* of X if for any $a, b \in H$, there exist $a_1 \in (a)$ and $b_1 \in (b)$ such that $a_1 \wedge b_1 \in H$. The empty set is to be considered as an m-system of X.

Proposition 3.12. *Let X be a subtraction semigroup. An ideal $P(\neq X)$ of X is prime if and only if its complement P^c is an m-system.*

Proof. Assume that P is prime. Let $a \in X\backslash P$ and $b \in X\backslash P$. Then $(a) \wedge (b) \nsubseteq P$. So, there exist $a_1 \in (a)$ and $b_1 \in (b)$ such that $a_1 \wedge b_1 \notin P$, that is, $a_1 \wedge b_1 \in X\backslash P$. Thus $X\backslash P$ is an m-system.

Conversely, let $X\backslash P$ is an m-system and let $a \in X\backslash P$ and $b \in X\backslash P$. Then there exist $a_1 \in (a)$ and $b_1 \in (b)$ such that $a_1 \wedge b_1 \in X\backslash P$. Thus $(a) \wedge (b) \nsubseteq P$, and hence P is prime. □

Proposition 3.13. *Let $\{P_\alpha\}_{\alpha\in A}$ be family of prime ideals which are totally ordered by set inclusion. Then $\bigcap_{\alpha\in A} P_\alpha$ is prime*

Proof. Let I and J be ideals of X. If $I \wedge J \subseteq \bigcap_{\alpha\in A} P_\alpha$, then $I \wedge J \subseteq P_\alpha$, for all $\alpha \in A$. Assume that there exist $\alpha \in A$ such that $I \nsubseteq P_\alpha$. Then $J \subseteq P_\alpha$, and so $J \subseteq P_\beta$ for all $\beta \geq \alpha$. Suppose that there exist $\gamma < \alpha$ such that $J \subseteq P_\gamma$. Then $I \subseteq P_\gamma$ and so $I \subseteq P_\alpha$. This is impossible. Thus $J \subseteq P_\beta$ for any $\beta \in A$. Hence $\bigcap_{\alpha\in A} P_\alpha$, is prime. □

Proposition 3.14. *Let X be a strong subtraction semigroup with 1 and let P_1, P_2 be ideals of X such that $P_1 \cap P_2$ be a prime ideal of X. Then $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$*

Proof. Let $P_1 \nsubseteq P_2$ and $x \in P_2$. If $y \in P_1, y \notin P_2$, we obtain $x \wedge y = xy \in P_2X \subseteq P_2$ and $x \wedge y = xy \in XP_1 \subseteq P_1$. Then $x \wedge y \in P_1 \cap P_2$. Since $P_1 \cap P_2$ is prime, we have $x \in P_1 \cap P_2$ or $y \in P_1 \cap P_2$. Since $y \notin P_2$, we have $y \notin P_1 \cap P_2$. Hence $x \in P_1 \cap P_2$, and so $x \in P_1$. This completes the proof. □

Proposition 3.15. *Let P and Q be distinct prime ideals of subtraction semigroup X such that $P \subseteq Q$. Then there exist prime ideals P_1 and Q_1 such that*

$$P \subseteq P_1 \subseteq Q_1 \subseteq Q \tag{$*$}$$

and there is no other prime ideal that lies between P_1 and Q_1.

Proof. By the Zorn's lemma, we can insert a chain $\{P_i\}_{i\in\pi}$ of prime ideals between P and Q. Let $x \in Q$ but $x \notin P$. Define Q_1 as the intersection of P_i's containing x and P_1 as the union of P_i's not containing x. Since $\{P_i\}_{i\in\pi}$ is a chain of prime ideals, it is easy to see that P_1 and Q_1 are prime ideals and obviously the inclusion $(*)$ holds. Also, none of the P_i's lies properly between P_1 and Q_1. For if $x \in P_i$, then $Q_1 \subseteq P_i$ and if $x \notin P_i$ then $P_i \subseteq P_1$. This cannot happen by the maximality of the chain. Thus no ideal lies properly between P_1 and Q_1. This completes the proof. □

Definition 3.16. Let X be a subtraction semigroup. An ideal Q is said to be *semiprime* if for any ideal I of X, $I \wedge I \subseteq Q$ implies $I \subseteq Q$. A nonempty subset S is said an *s-system* if for every $s \in S$, there exists $s_1, s_2 \in (s)$ such that $s_1 \wedge s_2 \in S$.

Clearly every prime ideal is semiprime, and each m-system is an s-system.

Proposition 3.17. *Let X be a subtraction semigroup and Q an ideal of X. Then Q is semiprime if and only if $X\backslash Q$ is an s-system.*

Proof. Suppose that Q is semiprime of X. Let $a \in X\backslash Q$. Then we have $(a) \nsubseteq Q$ and so $(a) \wedge (a) \nsubseteq Q$. Thus there exists $a_1, a_2 \in (a)$ such that $a_1 \wedge a_2 \notin Q$. Hence $X\backslash Q$ is an s-system. Conversely, assume that $X\backslash Q$ is an s-system. Let I be an ideal with $I \wedge I \subseteq Q$. Suppose that $I \nsubseteq Q$. Then there exists $s \in I\backslash Q \subseteq X\backslash Q$. Since $X\backslash Q$ is an s-system, there exists $s_1, s_2 \in (s)$ such that $s_1 \wedge s_2 \in X\backslash Q$. Since $s_1 \wedge s_2 \in (s) \wedge (s) \subseteq I \wedge I$, we obtain $I \wedge I \nsubseteq Q$. This is impossible. So, $I \subseteq Q$ and Q is semiprime. □

Lemma 3.18. *Let S be a nonempty subset of a subtraction semigroup X. Then S is an s-system if and only if $S = \bigcup_{\alpha \in A} S_\alpha$, where S_α's are m-systems of X.*

Proof. Suppose that S is an s-system and $s_0 \in S$. Then there exist $s_0^1, s_0^2 \in (s_0)$ such that $s_1 = s_0^1 \wedge s_0^2 \in S$. Now for s_1, there exist $s_1^1, s_1^2 \in (s_1)$ such that $s_2 = s_1^1 \wedge s_1^2 \in S$. Continuing this process, we get a sequence $s_0, s_1, s_2, \cdots$. We claim that $M = \{s_0, s_1, s_2, \cdots\}$ is an m-system. Let $s_i, s_j \in M$. We may assume that $i < j$ without loss of generality. Then $(s_j) \subseteq (s_i)$. Take $s_j^1, s_j^2 \in (s_j)$. Then $s_j^1 \wedge s_j^2 = s_{j+1} \in M$. Thus M is an m-system. The converse is clear. □

Theorem 3.19. *Let Q be an ideal in a subtraction semigroup X. Then Q is semiprime if and only if Q is the intersection of all prime ideals $P_\alpha(\alpha \in A)$ containing Q.*

Proof. Suppose that Q is semiprime and let $S = X \backslash Q$. Then S is an s-system. By Lemma 3.18, $S = \bigcup_{\beta \in B} S_\beta$ for some m-system S_β. Since for each $\beta \in B, S_\beta \subseteq S, P_\beta = X \backslash S_\beta$ is prime containing Q and so $Q \subseteq \bigcap_{\alpha \in A} P_\alpha \subseteq \bigcap_{\beta \in B} P_\beta = \bigcap_{\beta \in B} (X \backslash S_\beta) = X \backslash \bigcup_{\beta \in B} S_\beta = X \backslash S = Q$. Thus Q is the intersection of P_α. Conversely, let I be an ideal in X with $I \wedge I \subseteq Q$. then $I \wedge I \subseteq P_\alpha$ for all $\alpha \in A$. Since P_α is prime, we have $I \subseteq P_\alpha$ for all $\alpha \in A$ and so $I \subseteq Q$. thus Q is semiprime. □

Definition 3.20. An ideal I of a subtraction semigroup X is said to be *irreducible* if for any ideals H, K in $X, I = H \cap K$ implies $I = H$ or $I = K$. An ideal is said to be *strongly irreducible* if $H \cap K \subseteq I$ implies $H \subseteq I$ or $K \subseteq I$.

Theorem 3.21. *If an ideal P of a strong subtraction semigroup X with 1 is prime, then it is semiprime and strongly irreducible.*

Proof. If P is prime, then it is semiprime. Moreover, if H, K are ideals of X such that $H \cap K \subseteq P$, then we have $H \wedge K = HK \subseteq H \cap K \subseteq P$. Since P is prime, we obtain $H \subseteq P$ or $K \subseteq P$, and so P is strongly irreducible. □

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