T-FUZZY IDEALS OF INCLINE ALGEBRAS

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Abstract. In this paper, we introduce the concept of T-fuzzy ideals of incline algebras and some related properties are investigated.

1.Introduction

In[17], L.A.Zadeh introduced the notion of fuzzy sets and fuzzy set operations. Since then the fuzzy set theory developed by L.A.Zadeh and others has evoked great interest among researchers working in different branches of mathematics. Cao et al. [5] introduced the notion of incline algebras in their book. Kim and Roush[11] studied algebraic structures of inclines, and they with Markowsky [13] discussed the representation of inclines, and Ahn [2] investigated permanent over inclines. Moreover, Ahn and Kim [3]introduced the notion of positive implicative incline and studied some relations between R(L)-maps and positive implicative and Ahn etc. [3] constructed the quotient incline and discussed prime and maximal ideals in incline algebras. In [9], Y.B.Jun considered the fuzzification of subinclines (ideals)in inclines and investigated some related properties. In this paper, we introduce the concept of T-fuzzy ideals of incline algebras and some related properties are investigated.

2.Preliminaries

Inclines are a generalization of Boolean and fuzzy algebras, and a special type of a semi-ring, and give a way to combine algebras and ordered structures to express the degree of intensity of binary relations.

An incline algebra is a set $\mathcal{H}$ with two binary operations denoted by " $+$ " and " $*$ " satisfies the following axioms for all $x, y, z \in \mathcal{H}$:

(i) $x + y = y + x$
(ii) $x + (y + z) = (x + y) + z$
(iii) $x * (y * z) = (x * y) * z$
(iv) $x * (y + z) = x * y + x * z$
(v) $(y + z) * x = y * x + z * x$
(vi) $x + x = x$
(vii) $x + (x * y) = x$
(viii) $y + (x * y) = y$

For convenience, we pronounce " $+$ "(resp. " $*$ ") as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice(as semi-ring) if and only if $x * x = x$ for all $x \in \mathcal{H}$. Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in \mathcal{H}$. A subincline of an incline $\mathcal{H}$ is a subset $\mathcal{M}$ of $\mathcal{H}$ closed under addition and multiplication . An ideal in an incline $\mathcal{H}$ is a subincline $\mathcal{M} \subseteq \mathcal{H}$ such that if $x \in \mathcal{M}$ and $y \leq x$ then $y \in \mathcal{M}$. By a homomorphism of incline $\mathcal{H}$ into an incline $\mathcal{I}$ such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in \mathcal{H}$.

3. T-fuzzy ideals

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In what follows, $F(\mathcal{H})$ denotes the set of all fuzzy subsets in $\mathcal{H}$, i.e., maps from $\mathcal{H}$ into $([0,1],\vee,\wedge)$,where $[0,1]$ is the set of reals between 0 and 1 and $x\vee y = max\{x,y\}, x\wedge y = min\{x,y\}$. A fuzzy set μ is called a fuzzy ideal of an incline algebra $\mathcal{H}$ if it satisfies: (i) $\mu(x+y)\wedge\mu(x*y)\geq\mu(x)\wedge\mu(y)$ for all $x,y\in\mathcal{H}$; (ii) $\mu(x)\geq\mu(y)$ whenever $x\leq y$.

Definition 3.1 ([1]) By a t-norm T, we mean a function $T:[0,1]\times[0,1]\rightarrow[0,1]$ satisfying the following conditions:

(T1) $T(x,1)=x$;

(T2) $T(x,y)\leq T(x,z)$ if $y\leq z$;

(T3) $T(x,y)=T(y,x)$;

(T4) $T(x,T(y,z))=T(T(x,y),z)$ for all $x,y,z\in[0,1]$.

For a t-norm T on $[0,1]$, we denote it by $\Delta_T=\{\alpha\in[0,1]|T(\alpha,\alpha)=\alpha\}$.

Note 3.2 Every t-norm T has the following property: $T(\alpha,\beta)\leq min\{\alpha,\beta\}$.

Definition 3.3 Let T be a t-norm. Then we say that the fuzzy set μ in $\mathcal{H}$ satisfies the imaginable property if $Im\mu\subseteq\Delta_T$.

Definition 3.4 $\mu\in F(\mathcal{H})$ is called a fuzzy ideal of $\mathcal{H}$ with respect to a t-norm T(briefly, T-fuzzy ideal of $\mathcal{H}$) if it satisfies the following conditions:

(i) $\mu(x+y)\wedge\mu(x*y)\geq T(\mu(x),\mu(y))$;

(ii) $\mu(x)\geq\mu(y)$ whenever $x\leq y$ for all $x,y\in\mathcal{H}$.

Example 3.5 Note that for any $x\in\mathcal{H}$, the set $\mathcal{M}=\{a|a\leq x\}$ is an ideal of $\mathcal{H}$ (see [5,Example 1.1.5]). Define $\mu\in F(\mathcal{H})$ by

$$\mu(x)=\begin{cases}0.3 & \text{if } x\in\mathcal{M}\\ 0.8 & \text{otherwise}\end{cases}$$

for all $x\in\mathcal{H}$. Let $T_m:[0,1]\times[0,1]\rightarrow[0,1]$ be a function defined by $T_m(\alpha,\beta)=max\{\alpha+\beta-1\}$ for all $\alpha,\beta\in[0,1]$. It's easy to check that μ is a T-fuzzy ideal of $\mathcal{H}$.

For $\alpha\in[0,1]$, the set $\mu_\alpha=\{x\in\mathcal{H}|\mu(x)\geq\alpha\}$ is called a level subset of μ.

Theorem 3.6 (i) Let T be a t-norm. Then every imaginable T-fuzzy ideal of $\mathcal{H}$ is a fuzzy ideal of $\mathcal{H}$;

(ii) If μ is an imaginable T-fuzzy ideal of $\mathcal{H}$, then every non-empty level subset μ_α of μ is also an ideal of $\mathcal{H}$.

Proof (i) Let μ be an imaginable T-fuzzy ideal of $\mathcal{H}$, then $\mu(x+y)\wedge\mu(x*y)\geq T(\mu(x),\mu(y))$ for all $x,y\in\mathcal{H}$. Since μ is imaginable, we have $\mu(x)\wedge\mu(y)=T(\mu(x)\wedge\mu(y),\mu(x)\wedge\mu(y))\leq T(\mu(x),\mu(y))\leq\mu(x)\wedge\mu(y)$. Hence, it follows that $\mu(x+y)\wedge\mu(x*y)\geq\mu(x)\wedge\mu(y)$ for all $x,y\in\mathcal{H}$. (ii) Let $x,y\in\mu_\alpha$. Then we have $\mu(x+y)\wedge\mu(x*y)\geq T(\mu(x),\mu(y))\geq T(\alpha,\alpha)=\alpha$, which implies that $\mu(x+y)\geq\alpha$ and $\mu(x*y)\geq\alpha$, i.e, $x+y\in\mu_\alpha$ and $x*y\in\mu_\alpha$. Let $y\leq x$. Then $\mu(y)\geq\mu(x)\geq\alpha$, and so $y\in\mu_\alpha$. This proves that μ_α is an ideal of $\mathcal{H}$.□

Let $f:\mathcal{H}\rightarrow\mathcal{H}'$ be a mappping of incline algebras. For a fuzzy set μ in $\mathcal{H}$, the image of μ under f, denoted by $f^{-1}(\mu)$, is denoted by $f^{-1}(\mu)(x)=\mu(f(x))$ for all $x\in\mathcal{H}$.

Theorem 3.7 Let $f:\mathcal{H}\rightarrow\mathcal{H}'$ be a homomorphism of incline algebras. If μ is a T-fuzzy ideal of $\mathcal{H}'$, then $f^{-1}(\mu)$ is a T-fuzzy ideal of $\mathcal{H}$.

Proof For any $x,y\in\mathcal{H}$, we have $f^{-1}(\mu)(x+y)=\mu(f(x+y))=\mu(f(x)+f(y))\geq T(\mu(f(x)),\mu(f(y)))=T(f^{-1}(\mu)(x),f^{-1}(\mu)(y)); f^{-1}(\mu)(x*y)=\mu(f(x*y))=\mu(f(x)*f(y))\geq T(\mu(f(x)),\mu(f(y)))=T(f^{-1}(\mu)(x),f^{-1}(\mu)(y))$. Hence $f^{-1}(\mu)(x+y)\wedge f^{-1}(\mu)(x*y)\geq T(f^{-1}(\mu)(x),f^{-1}(\mu)(y))$ for all $x,y\in\mathcal{H}$. (ii) Let $x\leq y$ in $\mathcal{H}$, we have $x+y=y$, and so $f(x+y)=f(y)=f(x+y)$, that is, $f(x)\leq f(y)$. Hence $f^{-1}(\mu)(x)=\mu(f(x))\geq\mu(f(y))=f^{-1}(\mu)(y)$. Therefore $f^{-1}(\mu)$ is a T-fuzzy ideal of $\mathcal{H}$. This completes the proof .□

Let μ be a fuzzy set in an incline algebra $\mathcal{H}$ and f a mapping defined on $\mathcal{H}$. Then we call the fuzzy set μ^f in $f(\mathcal{H})$ defined by $\mu^f(y) = sup_{x\in f^{-1}(y)}\mu(x)$ for all $y \in f(\mathcal{H})$ the image of μ under f. A fuzzy set μ in $\mathcal{H}$ is said to have the sup property if for every subset $T \subseteq \mathcal{H}$, there exists $t_0 \in T$ such that $\mu(t_0) = sup_{t\in T}\mu(t)$.

Theorem 3.8 An onto homomorphic image of a fuzzy ideal with the sup property is a fuzzy ideal.

Proof Let $f : \mathcal{H} \to \mathcal{H}'$ be an onto homomorphism of incline algebras and let μ be a fuzzy ideal of $\mathcal{H}$ with the sup property . Given $u, v \in \mathcal{H}'$, we let $x_0 \in f^{-1}(u)$ and $y_0 \in f^{-1}(v)$ be such that $\mu(x_0) = sup_{t\in f^{-1}(u)}\mu(t), \mu(y_0) = sup_{t\in f^{-1}(v)}\mu(t)$, respectively. Then we can deduce that

(i) $\mu^f(u + v) = sup_{z\in f^{-1}(u+v)}\mu(z) \geq min\{\mu(x_0), \mu(y_0)\}$
$= min\{sup_{z\in f^{-1}(u)}\mu(z), sup_{z\in f^{-1}(v)}\mu(z)\} = min\{\mu^f(u), \mu^f(v)\}; \mu^f(u * v)$
$= sup_{z\in f^{-1}(u*v)}\mu(z) \geq min\{\mu(x_0), \mu(y_0)\} = min\{sup_{z\in f^{-1}(u)}\mu(z), sup_{z\in f^{-1}(v)}\mu(z)\}$
$= min\{\mu^f(u), \mu^f(v)\}$. Hence $\mu^f(u + v) \wedge \mu^f(u * v) \geq min\{\mu^f(u), \mu^f(v)\}$.

(ii) Let $x' \leq y'$ in $\mathcal{H}'$ as above, then we have $\mu^f(x') = sup_{t\in f^{-1}(x')}\mu(t)$
$\geq sup_{t\in f^{-1}(y')}\mu(t) = \mu^f(y')$. Therefore, μ^f is a fuzzy ideal of $\mathcal{H}'$.□

The above theorem can be further strengthened . We first give the following definition:

Definition 3.9 A t-norm T on $[0, 1]$ is called a continuous t-norm if T is a continuous function from $[0, 1] \times [0, 1] \to [0, 1]$ with respect to the usual topology.

We observe that the function " min " is always a continuous t-norm .

Theorem 3.10 Let T be a continuous t-norm and $f : \mathcal{H} \to \mathcal{H}'$ an onto homomorphism of incline algebras. If μ is a T-fuzzy ideal of $\mathcal{H}$, then μ^f is a T-fuzzy ideal of $\mathcal{H}'$.

Proof (i) Let $A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1 + y_2)$, where $y_1, y_2 \in \mathcal{H}'$. Consider the set

$A_1 + A_2 = \{x \in M|x = a_1 + a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2\}$

If $x \in A_1 + A_2$, then $x = x_1 + x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that we have $f(x) = f(x_1 + x_2) = f(x_1) + f(x_2) = y_1 + y_2$, that is , $x \in f^{-1}(y_1 + y_2) = A_{12}$. Thus $A_1 + A_2 \subset A_{12}$. It follows that

$\mu^f(y_1 + y_2) = sup\{\mu(x)|x \in f^{-1}(y_1 + y_2)\} = sup\{\mu(x)|x \in A_{12}\} \geq sup\{\mu(x)|x \in A_1 + A_2\} \geq sup\{\mu(x_1 + x_2)|x_1 \in A_1, x_2 \in A_2\}$
$\geq sup\{T(\mu(x_1), \mu(x_2))|x_1 \in A_1, x_2 \in A_2\}$

Since T is continuous , for every $\epsilon > 0$, we see that if $sup\{\mu(x_1)|x \in A_1\} - x_1^* \leq \delta$ and $sup\{\mu(x_2)|x \in A_2\} - x_2^* \leq \delta$, then

$T(sup\{\mu(x_1)|x_1 \in A_1\}, sup\{\mu(x_2)|x_2 \in A_2\}) - T(x_1^*, x_2^*) \leq \epsilon$.

Choose $a_1 \in A_1$ and $a_2 \in A_2$, such that $sup\{\mu(x_1)|x_1 \in A_1\} - \mu(a_1) \leq \delta$ and $sup\{\mu(x_2)|x_2 \in A_2\} - \mu(a_2) \leq \delta$. Then we have

$T(sup\{\mu(x_1)|x_1 \in A_1\}, sup\{\mu(x_2)|x_2 \in A_2\}) - T(\mu(a_1), \mu(a_2)) \leq \epsilon$.

Consequently , we have $\mu^f(y_1 + y_2) \geq sup\{T(\mu(x_1), \mu(x_2))|x_1 \in A_1, x_2 \in A_2\}) \geq T(sup\{\mu(x_1)|x_1 \in A_1\}, sup\{\mu(x_2)|x_2 \in A_2\}) = T(\mu^f(y_1), \mu^f(y_2))$

Similarly, we can show that $\mu^f(y_1 * y_2) \geq \mu^f(y_1 + y_2) \geq T(\mu^f(y_1), \mu^f(y_2))$. Therefore $\mu^f(y_1 * y_2) \wedge \mu^f(y_1 + y_2) \geq T(\mu^f(y_1), \mu^f(y_2))$

(ii) Let $x' \leq y'$ in $\mathcal{H}'$ as above, then we have $\mu^f(x') = sup_{t\in f^{-1}(x')}\mu(t) \geq sup_{t\in f^{-1}(y')}\mu(t) = \mu^f(y')$.This shows that μ^f is a T-fuzzy ideal of $\mathcal{H}'$. □

Lemma 3.11 ([1]) For all $\alpha, \beta, \gamma, \delta \in, [0, 1]$, $T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta))$

Theorem 3.12 Let T be a $t-$norm and let $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ be the direct product of incline algebras $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\mu_1(resp.\mu_2)$ is a $T-$fuzzy ideal of $\mathcal{H}_1(resp.\mathcal{H}_2)$,then $\mu = \mu_1 \times \mu_2$ is a $T-$fuzzy ideal of $\mathcal{H}$ defined by $\mu(x) = \mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$ for all $x = (x_1, x_2) \in \mathcal{H}$.

Proof (i) Let $x = (x_1, x_2), y = (y_1, y_2)$. Then we have $\mu(x+y) = \mu((x_1, x_2)+(y_1, y_2)) = \mu(x_1+y_1, x_2+y_2) = T(\mu_1(x_1+y_1), \mu_2(x_2+y_2)) \geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2))) = T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2))) = T(\mu(x_1, x_2), \mu(y_1, y_2)) = T(\mu(x), \mu(y))$; $\mu(x * y) = \mu((x_1, x_2) * (y_1, y_2)) = \mu(x_1 * y_1, x_2 * y_2) = T(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2)) \geq$
$T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2))) = T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2)))$
$= T(\mu(x_1, x_2), \mu(y_1, y_2)) = T(\mu(x), \mu(y))$. Hence $\mu(x + y) \wedge \mu(x * y) \geq T(\mu(x), \mu(y))$.

(ii) Let $x = (x_1, x_2), y = (y_1, y_2)$. If $x \leq y$, then $x + y = y$, that is, $(x_1, x_2) + (y_1, y_2) = (y_1, y_2)$, and so $(x_1 + y_1, x_2 + y_2) = (y_1, y_2)$. Hence $x_1 + y_1 = y_1$ and $x_2 + y_2 = y_2$. This shows that $x_1 \leq y_1$ and $x_2 \leq y_2$, and so $\mu_1(x_1) \geq \mu_1(y_1)$ and $\mu_2(x_2) \geq \mu_2(y_2)$. It follows that $\mu(x) = \mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu(x_1), \mu_2(x_2)) \geq T(\mu(y_1), \mu_2(y_2)) = (\mu_1 \times \mu_2)(y_1, y_2) = \mu(y_1, y_2) = \mu(y)$. Therefore $\mu = \mu_1 \times \mu_2$ is a $T-$fuzzy ideal of $\mathcal{H}$.□

We will generalize the idea to the product of n $T-$fuzzy ideals, We first need to generalize the domain of T to $\prod_{i=1}^{n}[0, 1]$ as follows:

Definition 3.13([1]) The function $T_n : \prod_{i=1}^{n}[0, 1] \to [0, 1]$ is defined by
$T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))$
for all $1 \leq i \leq n$,where $n \geq 2, T_2 = T$,and $T_1 = id$(identity)

Lemma 3.14 ([1]) For every $\alpha_i, \beta_i \in [0, 1]$,where $1 \leq i \leq n$ and $n \geq 2$,
$T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \ldots, T(\alpha_n, \beta_n)) = T(T_n(\alpha_1, \alpha_2, \ldots, \alpha_n), T_n(\beta - 1, \beta_2, \ldots \beta_n))$

Theorem 3.15 Let $\{\mathcal{H}_i\}_{i=1}^{n}$ be the finite collection of incline algebras and $\mathcal{H} = \prod_{i=1}^{n} \mathcal{H}_i$ the direct product incline algebras of $\{\mathcal{H}_i\}$. Let μ_i be a $T-$fuzzy ideal of $\mathcal{H}_i$, where $1 \leq i \leq n$.Then $\mu = \prod_{i=1}^{n} \mu_i$ defined by
$\mu(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \mu_i(x_1, x_2, \ldots, x_n) = T_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))$
is a $T-$fuzzy ideal of $\mathcal{H}$.

Proof It is Similar to Theorem 3.12.□

Definition 3.16 Let μ and ν be fuzzy sets in $\mathcal{H}$. then the product of μ and ν, written $[\mu \cdot \nu]_T$, is defined by $[\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))$ for all $x \in \mathcal{H}$.

Theorem 3.17 Let μ and ν be $T-$fuzzy ideals of incline algebra $\mathcal{H}$. If T^* is a $t-$norm which dominates T, that is, $T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T(T^*(\alpha, \gamma), T^*(\beta, \delta))$ for all $\alpha, \beta, \gamma, \delta \in [0, 1]$,then T^*-product of μ and $\nu, [\mu \cdot \nu]_T*$ is a $T-$fuzzy ideal of $\mathcal{H}$.

Proof Let $x, y \in \mathcal{H}$, then we have

(i) $[\mu \cdot \nu]_T * (x + y) = T^*(\mu(x + y), \nu(x + y)) \geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \geq T(T^*(\mu(x), \nu(x)), T^*(\mu(y), \nu(y))) = T([\mu \cdot \nu]_T * (x), [\mu \cdot \nu]_T * (y))$; $[\mu \cdot \nu]_T * (x * y) = T^*(\mu(x * y), \nu(x * y)) \geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \geq T(T^*(\mu(x), \nu(x)), T^*(\mu(y), \nu(y))) = T([\mu \cdot \nu]_T * (x), [\mu \cdot \nu]_T * (y))$. Hence $[\mu \cdot \nu]_T * (x + y) \wedge [\mu \cdot \nu]_T * (x * y) \geq T([\mu \cdot \nu]_T * (x), [\mu \cdot \nu]_T * (y))$.

(ii) Let $x \leq y$ in $\mathcal{H}$, then $\mu(x) \geq \mu(y)$ and $\nu(x) \geq \nu(y)$ since μ and ν are T-fuzzy ideals of $\mathcal{H}$. Hence $[\mu \cdot \nu]_T * (x) = T^*(\mu(x), \nu(x)) \geq T^*(\mu(y), \nu(y)) = [\mu \cdot \nu]_T * (y)$. Therefore, $[\mu \cdot \nu]_T*$ is a $T-$fuzzy ideal of $\mathcal{H}$.□

Let $f : \mathcal{H} \to \mathcal{H}'$ be an onto homomorphism of incline algebras. Let T and T^* be $t-$norms such that T^* dominates T. If μ and ν are $T-$fuzzy ideal of $\mathcal{H}'$, then the T^*-product of μ and $\nu, [\mu \cdot \nu]_T*$ is a $T-$fuzzy ideal of $\mathcal{H}'$. Since every onto homomorphic inverse image of a $T-$fuzzy ideal is a $T-$fuzzy ideal, the inverse images $f^{-1}(\mu), f^{-1}(\nu)$, and $f^{-1}([\mu \cdot \nu]_T*)$ are $T-$fuzzy ideals of $\mathcal{H}$. The next theorem provides that the relation between $f^{-1}([\mu \cdot \nu])_T*)$ and T^*-product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_T*$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 3.18 Let $f : \mathcal{H} \to \mathcal{H}'$ be an onto homomorphism of incline algebras. Let T^* be a $t-$norm such that T^* dominates T. Let μ and ν be $T-$fuzzy ideals of $\mathcal{H}'$. If $[\mu \cdot \nu]_T*$ is the T^*-product of μ and ν, and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_T*$ is the T^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then
$f^{-1}([\mu \cdot \nu]_T*) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_T*$

Proof Let $x \in \mathcal{H}$, then we have $f^{-1}([\mu \cdot \nu]_T*)(x) = T^*(\mu(f(x)), \nu(f(x)))$
$= T^*(f^{-1}(\mu(x)), f^{-1}(\nu(x))) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_T * (x)$. □

4. Chain conditions

Definition 4.1 An incline $\mathcal{H}$ is said to satisfy the ascending (resp.descending) chain condition (briefly,ACC (resp. DCC)) if for every ascending (resp. descending) sequence $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots (resp.\mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \cdots)$ of ideals of $\mathcal{H}$, there exists a natural number n such that $\mu_n = A_k$ for all $n \geqslant k$. If $\mathcal{H}$ satisfies ACC, we say that $\mathcal{H}$ is a Notherian incline algebras.

Theorem 4.2 Let $\mathcal{H}$ be an incline algebra satisfying DCC and μ be an imaginable $T-$fuzzy ideal of $\mathcal{H}$. If a sequence of elements of $Im\mu$ is strictly increasing, then $Im\mu$ has finite number of values.

Proof Let $\{t_n\}$ be a strictly increasing sequence of elements of $Im\mu$, then $0 \leq t_1 < t_2 \cdots \leqslant 1$. Define $\mu_r = \{x \in \mathcal{H} | \mu(x) \geq t_r\}, r = 1, 2, 3 \cdots$. Then μ_r is an ideal by Theorem 3.6. Let $x \in \mu_r$, then $\mu(x) \geqslant t_r > t_{r-1}$, which implies that $x \in \mu_{r-1}$. Hence $\mu_r \subseteq \mu_{r-1}$. Since $t_{r-1} \in Im\mu$, there exists $x_{r-1} \in \mathcal{H}$, such that $\mu(x_{r-1}) = t_{r-1}$. It follows that $x_{r-1} \in \mu_{r-1}$, but $x_{r-1} \notin \mu_r$. Thus $\mu_r \subset \mu_{r-1}$, and so we obtain a strictly descending sequence $\mu_1 \supset \mu_2 \supset \mu_3 \supset \cdots$ of ideals of $\mathcal{H}$ which is not terminating. This contradicts the assumption . Hence $Im\mu$ has finite number of values.□

Now we consider the converse of Theorem 4.2.

Theorem 4.3 Let $\mathcal{H}$ be an incline algebra. If every T-fuzzy ideals of $\mathcal{H}$ has finite number of values, then $\mathcal{H}$ satisfies DCC.

Proof Suppose $\mathcal{H}$ does not satisfy DCC, then there exists a strictly descending chain $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots$ of ideals of $\mathcal{H}$.

Define $\mu \in F(\mathcal{H})$ by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in \mathcal{M}_n - \mathcal{M}_{n+1} \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} \mathcal{M}_n \end{cases}$$

where $\mathcal{M}_0$ stands for $\mathcal{H}$.

We prove that μ is T-fuzzy ideal of $\mathcal{H}$. Assume that $x \in \mathcal{M}_n - \mathcal{M}_{n+1}$ and $y \in \mathcal{M}_k - \mathcal{M}_{k+1}$ for $n = 0, 1, 2 \cdots ; k = 0, 1, 2 \cdots$. Without loss of generality, we may assume that $n \leq k$. Then clearly $x + y \in \mathcal{M}_n$ and $x * y \in \mathcal{M}_n$. Thus $\mu(x+y) \wedge \mu(x*y) \geq \frac{n}{n+1} = T(\mu(x), \mu(y))$. If $x, y \in \bigcap_{n=0}^{\infty} \mathcal{M}_n$, then $x + y, x * y \in \bigcap_{n=0}^{\infty} \mathcal{M}_n$. Thus $\mu(x+y) \wedge \mu(x*y) = T(\mu(x), \mu(y))$. If $x \notin \bigcap_{n=0}^{\infty} \mathcal{M}_n$ and $y \in \bigcap_{n=0}^{\infty} \mathcal{M}_n$, then there exists $k \in \mathbb{N}$, such that $x \in \mathcal{M}_k - \mathcal{M}_{k+1}$. It follows that $\mu(x+y) \wedge \mu(x*y) \geq \frac{k}{k+1} = \mu(x) \wedge \mu(y) \geq T(\mu(x), \mu(y))$. Now, let $y \leq x$. If $x \in \mathcal{M}_n - \mathcal{M}_{n+1}$, then $y \in \mathcal{M}_n$ since $\mathcal{M}$ is an ideal of $\mathcal{H}$, and so $\mu(x) = \frac{n}{n+1} = \mu(y)$. If $x \in \bigcap_{n=0}^{\infty} \mathcal{M}_n$, then $y \in \mathcal{M}_n$ since $\mathcal{M}$ is an ideal of $\mathcal{H}$, and so $\mu(x) = 1 = \mu(y)$. Therefore μ is a T-fuzzy ideal of $\mathcal{H}$. Consequently we find that μ is a T-fuzzy ideal and μ has infinite number of different values. This is a contradiction and the proof is complete.□

Theorem 4.4 Let $\mathcal{H}$ be an incline algebra and $\alpha \in [0, 1]$ be such that $T(\alpha, \alpha) = \alpha$, then the following are equivalent:

(i) $\mathcal{H}$ is a Notherian incline algebra;

(ii) The set of values of any T-fuzzy ideal in $\mathcal{H}$ is a well-ordered subset of $[0, 1]$.

Proof (i)⇒(ii) Let μ be a T-fuzzy ideal of $\mathcal{H}$. Assume that the set of values of μ is not a well-ordered subset of $[0, 1]$, then there exists a strictly infinite decreasing sequence $\{t_n\}$ such that $\mu(x_n) = t_n$. Let $\mathcal{M} = \{x \in \mathcal{H} | \mu(x) \geq t_n\}$. Then $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \cdots$ is a strictly infinite ascending chain of ideals of $\mathcal{H}$, a contradiction.

(ii)⇒(i) Assume that there exists a strictly infinite ascending chain:

$(*)$ $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \cdots$

of ideals of $\mathcal{H}$. Let $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$. Then clearly $\mathcal{M}$ is an ideal of $\mathcal{H}$. Define $\mu \in F(\mathcal{H})$ by

$$\mu(x) = \begin{cases} 0 & \text{if } x \in \mathcal{M}_n \\ \frac{1}{k} & where\ k = min\{n \in \mathbb{N} | x \in \mathcal{M}_n\} \end{cases}$$

We claim that μ is a T-fuzzy ideal of $\mathcal{H}$. For any $x, y \in \mathcal{H}$, if any one of x and y does not belong to $\mathcal{M}_n$, then clearly $\mu(x+y) \wedge \mu(x*y) \geq 0 = \mu(x) \wedge \mu(y) \geq T(\mu(x), \mu(y))$. If $x, y \in \mathcal{M}_n - \mathcal{M}_{n+1} \geq \frac{1}{n} = T(\mu(x), \mu(y))$. If $x \in \mathcal{M}_n$ and $y \in \mathcal{M}_n - \mathcal{M}_m (or, y \in \mathcal{M}_n$ and $y \in \mathcal{M}_n - \mathcal{M}_m)$, then $\mu(x+y) \wedge \mu(x*y) \geq \frac{1}{n} \geq \frac{1}{m+1} \geq T(\mu(x), \mu(y))$. Now, let $y \leq x$. If $x \notin \mathcal{M}_n$, then $y \notin \mathcal{M}_n$, and so $\mu(x) = \mu(y)$. If $x \in \mathcal{M}_n$, then $y \in \mathcal{M}_n$, and so $\mu(x) = \mu(y)$. Hence μ is a T-fuzzy ideal of $\mathcal{H}$. Since the chain (*) is not terminating, μ has strictly infinite ascending sequence of values. This contradicts that the value set of any T-fuzzy ideal is welled-ordered. This completes the proof.□

5. T-fuzzy characteristic ideals

Definition 5.1 If μ is a T-fuzzy ideal of $\mathcal{H}$ and δ is a mapping from $\mathcal{H}$ into itself. We define a mapping $\mu^\delta : \mathcal{H} \to [0,1]$ by $\mu^\delta(x) = \mu(\delta(x))$ for all $x \in \mathcal{H}$.

Theorem 5.2 If μ is a T-fuzzy ideal of δ is an endomorphism of $\mathcal{H}$, then μ^δ is a T-fuzzy ideal of $\mathcal{H}$.

Proof For every $x, y \in \mathcal{H}$, we have (i) $\mu^\delta(x+y) \wedge \mu^\delta(x*y) = \mu(\delta(x+y)) \wedge \mu(\delta(x*y))$ $= \mu(\delta(x)+\delta(y)) \wedge \mu(\delta(x)*\delta(y)) \geq T(\mu(\delta(x)), \mu(\delta(y))) = T(\mu^\delta(x), \mu^\delta(y))$. (ii) Let $x, y \in \mathcal{H}$ be such that $x \leq y$. Then we have $x + y = y$ and so $\delta(y) = \delta(x+y) = \delta(x) + \delta(y)$. Thus $\delta(x) \leq \delta(y)$. It follows that $\mu^\delta(x) = \mu(\delta(x)) \geq \mu^\delta(y) = \mu^\delta(y)$. Thus μ^δ is a T-fuzzy ideal of $\mathcal{H}$.□

Definition 5.3 An ideal $\mathcal{M}$ of $\mathcal{H}$ is said to be characteristic if $\delta(\mathcal{M}) = \mathcal{M}$ for all $\delta \in Aut(\mathcal{H})$, where $Aut(\mathcal{H})$ is set of all automorphisms of $\mathcal{H}$.

A T-fuzzy ideal μ of $\mathcal{H}$ is said to be T-fuzzy characteristic if $\mu(\delta(x)) = \mu(x)$ for all $x \in \mathcal{H}$ and $\delta \in Aut(\mathcal{H})$.

Lemma 5.4 Let T be a t-norm such that $T(\alpha, \alpha) = \alpha$ and $\mathcal{M}$ be an ideal of $\mathcal{H}$. Define $\mu \in F(\mathcal{H})$ by

$$\mu(x) = \begin{cases} t_0 & \text{if } x \in \mathcal{M} \\ t_1 & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{H}$, where $t_0, t_1 \in [0,1], t_0 > t_1$. Then μ is a T-fuzzy ideal of $\mathcal{H}$.

Proof Let $x, y \in \mathcal{H}$. If any one of x, y does not belong to $\mathcal{M}$, then clearly

$\mu(x+y) \wedge \mu(x*y) \geq t_1 \geq \mu(x) \wedge \mu(y) \geq T(\mu(x), \mu(y))$. If $x, y \in \mathcal{M}$, then $x+y, x*y \in \mathcal{M}$. It follows that $\mu(x+y) \wedge \mu(x*y) \geq t_1 = T(t_1, t_1) = T(\mu(x), \mu(y))$. Now, let $y \leq x$. If $x \in \mathcal{M}$, then $y \in \mathcal{M}$ since $\mathcal{M}$ is an ideal of $\mathcal{H}$. Hence $\mu(x) = \mu(y)$. If $x \notin \mathcal{M}$, then $\mu(x) = t_1 \leq \mu(y)$. Then μ is a T-fuzzy ideal of $\mathcal{H}$.□

Theorem 5.5 Let $\mu \in F(\mathcal{H})$ be an imaginable T-fuzzy characteristic ideal of $\mathcal{H}$. Then each level ideal of $\mathcal{H}$ is a characteristic ideal of $\mathcal{H}$.

Proof Let $\mu \in F(\mathcal{H})$ be an imaginable T-fuzzy characteristic ideal of $\mathcal{H}$. Then $\mu_t, t \in Im\mu$, is an ideal of $\mathcal{H}$. It suffices to show that $\delta(\mu_t) = \mu_t$ for $t \in Im\mu$. Let $\delta \in Aut(\mathcal{H})$ and $x \in \mu_t$. Since μ is a T-fuzzy characteristic, we have $\delta(\mu(x)) = \mu(x) \geq t$. It follows that $\delta(x) \in \mu_t$ and hence $\delta(\mu_t) \subseteq \mu_t$. Conversely, let $y \in \mu_t$ and $x \in \mathcal{H}$ be such that $\delta(x) = y$. Then $\mu(x) = \mu(\delta(x)) = \mu(y) \leq t$, whence $x \in \mu_t$. This implies that $y = \delta(x) \in \delta(\mu_t)$, so that $\mu_t \subseteq \delta(\mu_t)$. Hence $\mu_t, t \in Im\mu$, is characteristic ideal of $\mathcal{H}$.□

Lemma 5.6 Let μ be an imaginable T-fuzzy ideal of $\mathcal{H}$ and let $x \in \mathcal{H}$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.

Theorem 5.7 Let μ be an imaginable T-fuzzy ideal of $\mathcal{H}$. If each level ideal of μ is characteristic, then μ is an imaginable T-fuzzy ideal of $\mathcal{H}$.

Proof Let $\mu \in F(\mathcal{H})$ be an imaginable T-fuzzy ideal, and let $x \in \mathcal{H}$, $\delta \in Aut(\mathcal{H})$ and $\mu(x) = t$. By Lemma 5.6, $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. From the hypothesis, it follows that $\delta(\mu_t) = \mu_t$. Thus $\delta(x) \in \delta(\mu_t) = \mu_t$ and so $\mu(\delta(x)) \geq t$. Let $\mu(\delta(x)) = s$ and assume that $s > t$. Then $\delta(x) \in \mu_s = \delta(\mu_s)$. Since δ is one-to-one, it follows that $x \in \mu_s$. This is a contradiction. Hence $\mu(\delta(x)) = t = \mu(x)$, showing that μ is a T-fuzzy characteristic ideal of $\mathcal{H}$.□

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