ON THE UNIFORM CONVEXITY OF SUBSETS OF BANACH SPACES

Kazutaka Eshita and Wataru Takahashi

Received April 6, 2004

Abstract. In this paper, we introduce the *uniform convex-like* property, which is a new method of scaling the convexity of subsets of Banach spaces. All bounded closed convex subsets of uniformly convex Banach spaces and all compact convex subsets of strictly convex Banach spaces are uniform convex-like. Using this concept, we prove a fixed point theorem in a Banach space. We also show the existence of ergodic retractions for nonexpansive mappings in a Banach space.

1 Introduction Let E be a real Banach space and C be a nonempty bounded closed convex subset of E. A mapping T of C into E is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. In 1965, Browder [4] and Göhde [11] proved that if E is uniformly convex then every nonexpansive mapping T of C into itself has a fixed point, while Kirk [15] proved that if E is reflexive and C has normal structure then T has a fixed point. We also know some results concerning the geometry of Banach spaces. Bae [2] and Maluta [18] proved that if E has uniform normal structure then E is reflexive and C has normal structure. Casini and Maluta [8] proved that if E has uniform normal structure then every uniformly γ-Lipschitz mapping of C into itself with $\gamma^2 \tilde{N}(E) < 1$ has a fixed point; see Section 2 for the definition of $\tilde{N}(E)$. Ishihara and Takahashi [14] proved that if C has uniform normal structure then C is weakly compact and C has normal structure. They also extended Casini and Maluta's fixed point theorem to uniformly γ-Lipschitz semigroups.

On the other hand, the first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [3]: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $z \in F(T)$. In this case, putting $z = Px$ for every $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\mathrm{co}}\{T^n x : n \geq 0\}$. Such a retraction was first called "an ergodic retraction" by Takahashi [20] and then has been studied by many authors; see, for example, [13, 16, 17, 21]. Bruck [5] also proved that if C is a bounded closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and T is a nonexpansive mapping of C into itself then the Cesàro means of T converge weakly to some $z \in F(T)$. For the purpose of proving the previous theorem, Bruck introduced the concept of type (γ) for mappings and obtained interesting results concerning type (γ). Later, Atsushiba and Takahashi [1] proved that if C is a compact convex subset of a strictly convex Banach space E and T is a nonexpansive

2000 *Mathematics Subject Classification.* 46B20, 46B50, 47A35, 47H09, 47H10.

Key words and phrases. Banach space, convex approximation property, ergodic retraction, fixed point theorem, nonexpansive, normal structure, strictly convex, type (γ), uniform normal structure, uniformly convex.

mapping of C into itself then the Cesàro means of T converge weakly to some $z \in F(T)$ by using an idea of Bruck [5].

In this paper, we introduce the notion of *uniformly convex-like (ucl)* subset in a Banach space. We show that every bounded closed convex subset of a uniformly convex Banach space is (ucl), and that every compact convex subset of a strictly convex Banach space is also. If a bounded closed convex subset C of a Banach space is (ucl), then C is weakly compact and has normal structure. Further, we show that every nonexpansive mapping of C into itself is of type (γ). To show the weak compactness of C, we also introduce the notion of *strong normal structure (sns)*. In addition, we give an extension of Casini and Maluta's fixed point theorem by using the concept of (sns). Further, we also try to show the existence of ergodic retractions of nonexpansive mappings in a Banach space by using the concept of (ucl) and the convex approximation property. This paper is organized as follows: In Section 2, we define some terminologies that we use in this paper. In Section 3, we define the notion of (ucl) subset in a Banach space and show some propositions. In Section 4, we define the notion of (sns) and prove the weak compactness of bounded closed convex subsets with (sns). In Section 5, we give a fixed point theorem for bounded closed convex subsets with (sns). In Section 6, we deal with type (γ) property of (ucl) subsets. In Sections 7 and 8, we obtain the existence of ergodic retractions for nonexpansive mappings on some (ucl) subsets in a Banach space.

2 Preliminaries Throughout this paper, E is a real Banach space with norm $\| \ \|$. We denote by E^* the dual of E. We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all reals. Let M be a nonempty subset of E. For $p \in \mathbb{N}$, let us define

$$\operatorname{co}_p M = \left\{ \sum_{i=1}^{p} \lambda_i x_i : \lambda_i \geq 0,\ \sum_{i=1}^{p} \lambda_i = 1,\ x_i \in M \right\}.$$

We also denote by $\operatorname{co} M$ the convex hull of M, i.e., $\operatorname{co} M = \bigcup_{p=1}^{\infty} \operatorname{co}_p M$, and by $\overline{\operatorname{co}}\, M$ the closure of $\operatorname{co} M$. For a nonempty subset C of E, let us define

$$I(C) = \{\|x - y\| : x, y \in C\}.$$

We denote by $d(C)$ the diameter of C, i.e., $d(C) = \sup I(C)$. If C is convex, $I(C)$ is a convex interval $[0, d(C))$ or $[0, d(C)]$. E is said to be *strictly convex* if $\|x\| = \|y\| = \|(x + y)/2\|$ implies $x = y$. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \varepsilon \right\}.$$

E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$. Let C be a nonempty convex subset of a Banach space E with $d(C) > 0$. Ishihara and Takahashi [14] defined a function $\delta_C : [0, 2] \to [0, 1]$ called the *modulus of uniform convexity on C* as follows:

$$\delta_C(\varepsilon) = \inf \left\{ 1 - \frac{1}{r} \left\| \frac{u + v}{2} - w \right\| \ :\ r > 0,\ u, v, w \in C, \|u - w\| \leq r,\ \|v - w\| \leq r,\ \|u - v\| \geq \varepsilon r \right\}.$$

A nonempty convex subset C of E with $d(C) > 0$ is said to be *uniformly convex* if $\delta_C(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$. Note that $\delta_C(\varepsilon) \geq \delta_E(\varepsilon) = \delta(\varepsilon)$ for every $\varepsilon \in [0, 2]$.

Let D be a bounded closed convex subset of E. Then we define

$$r(x, D) = \sup\{\|x - y\| : y \in D\}$$

for every $x \in D$ and

$$r(D) = \inf\{r(x, D) : x \in D\}.$$

A nonempty closed convex subset C of E is said to have *normal structure* if $r(D) < d(D)$ for every bounded closed convex subset D of C with $d(D) > 0$. A nonempty closed convex subset C of E is said to have *uniform normal structure* if there exists a constant $k < 1$ such that for every bounded nonempty closed convex subset D of C,

$$r(D) \leq kd(D).$$

Let C be a closed convex subset of E with $d(C) > 0$. As in Maluta[18], we define

$$\tilde{N}(C) = \sup\left\{\frac{r(D)}{d(D)} : D \subset C \text{ is bounded, closed, convex and } d(D) > 0\right\}.$$

Clearly, if $\tilde{N}(C) < 1$, then C has uniform normal structure. If E has uniform normal structure, then every nonempty closed convex subset C of E also has uniform normal structure. Further, if $d(C) > 0$, then $\tilde{N}(C) \leq \tilde{N}(E)$.

3 Uniformly convex-like subsets Let C be a nonempty bounded convex subset of a Banach space E. Then we define a function $\eta_C : I(C) \to [0, \infty)$ as follows:

$$\eta_C(t) = \inf\left\{(\|x - z\| \vee \|y - z\|) - \left\|\frac{x + y}{2} - z\right\| : x, y, z \in C,\ \|x - y\| \geq t\right\}$$

for each $t \in I(C)$, where $\|x - z\| \vee \|y - z\| = \max\{\|x - z\|, \|y - z\|\}$. A nonempty bounded convex subset C in a Banach space is said to be *uniformly convex-like* ((ucl) for short) if $\eta_C(t) > 0$ for each $t \in I(C) \setminus \{0\}$.

Proposition 3.1. *Let C be a bounded convex subset of a Banach space with $d(C) > 0$. If C is uniformly convex, then C is (ucl). Moreover, for each $t \in I(C)$,*

$$\frac{t}{2} \cdot \delta_C\left(\frac{t}{d(C)}\right) \leq \eta_C(t).$$

Proof. Let $t \in I(C)$ with $t > 0$. Choose $x, y, z \in C$ with $\|x - y\| \geq t$. Put $r = \|x - z\| \vee \|y - z\|$. Note that $t/2 \leq r \leq d(C)$. Then, we have $\|x - z\| \leq r$, $\|y - z\| \leq r$ and

$$\|x - y\| \geq t \geq \frac{t}{d(C)} \cdot r.$$

Therefore we have

$$\begin{aligned}(\|x - z\| \vee \|y - z\|) - \left\|\frac{x + y}{2} - z\right\| &= r \cdot \left(1 - \frac{1}{r}\left\|\frac{x + y}{2} - z\right\|\right) \\ &\geq \frac{t}{2} \cdot \delta_C\left(\frac{t}{d(C)}\right).\end{aligned}$$

Hence we have

$$\eta_C(t) \geq \frac{t}{2}\delta_C\left(\frac{t}{d(C)}\right) > 0.$$

It is obvious that $\eta_C(t) \geq (t/2)\delta_C(t/d(C))$ for each $t \in I(C)$. $\square$

Proposition 3.2. *Let C be a nonempty bounded convex subset of a uniformly convex Banach space E. Then C is (ucl).*

Proof. We may assume that $d(C) > 0$. Since E is uniformly convex, then C is also uniformly convex. By Proposition 3.1, then C is (ucl). □

Proposition 3.3. *Let C be a compact convex subset of a strictly convex Banach space E. Then C is (ucl).*

Proof. Let $t \in I(C) \setminus \{0\}$. Since C is compact, there exists $(u, v, w) \in C \times C \times C$ such that $\|u - v\| \geq t$ and that

$$(\|u - w\| \vee \|v - w\|) - \left\| \frac{u+v}{2} - w \right\| = \eta_C(t).$$

Suppose $\eta_C(t) = 0$. Then we have

$$\|u - w\| = \|v - w\| = \left\| \frac{u+v}{2} - w \right\|.$$

Since E is strictly convex, it follows that $u = v$. This contradicts that $\|u - v\| \geq t > 0$. Therefore we have $\eta_C(t) > 0$. □

4 Strong normal structure Let C be a nonempty closed convex subset of a Banach space E. Then, for each $t \geq 0$, we define $\nu_C(t)$ by

$$\nu_C(t) = \sup\{r(D) : D \subset C \text{ is nonempty, closed, convex and } d(D) \leq t\}.$$

Proposition 4.1. *Let C be a nonempty closed convex subset of a Banach space E. Then $\nu_C : [0, \infty) \to [0, \infty)$ satisfies the following:*

1. $\nu_C(t) \leq t$;
2. $t_1 \leq t_2 \implies \nu_C(t_1) \leq \nu_C(t_2)$;
3. $0 < t_1 \leq t_2 \implies \nu_C(t_1)/t_1 \geq \nu_C(t_2)/t_2$;
4. ν_C *is continuous.*

Proof. (1) and (2) are clear. We show (3). Take t_1, t_2 with $0 < t_1 \leq t_2$. Let D_2 be a nonempty closed convex subset of C with $d(D_2) \leq t_2$. Take a $z \in D_2$ and put $D_1 = (1 - (t_1/t_2))z + (t_1/t_2)D_2$. Then, we have

$$d(D_1) = (t_1/t_2)d(D_2) \leq t_1$$

and also

$$\nu_C(t_1) \geq r(D_1) = (t_1/t_2)r(D_2).$$

Therefore, we have $\nu_C(t_1) \geq (t_1/t_2)\nu_C(t_2)$ and obtain (3). Let us prove (4). From (1), ν_C is continuous at 0. Let $s, t > 0$. From (2) and (3), it follows that

$$\begin{aligned} \frac{s}{t}\nu_C(t) \leq \nu_C(s) \leq \nu_C(t), &\quad \text{if } s < t; \\ \nu_C(t) \leq \nu_C(s) \leq \frac{s}{t}\nu_C(t), &\quad \text{if } s > t. \end{aligned}$$

Hence, we have $\lim_{s \to t} \nu_C(s) = \nu_C(t)$ for each $t > 0$. Therefore, ν_C is continuous. This completes the proof. □

Let C be a closed convex subset of a Banach space E with $d(C) > 0$. Let $t > 0$. Then, for every closed convex subset D of C with $0 < d(D) \leq t$, we have

$$r(D) \leq \frac{r(D)}{d(D)} \cdot t \leq \tilde{N}(C) \cdot t.$$

So, we have that $\nu_C(t) \leq \tilde{N}(C) \cdot t$ for each $t \geq 0$. A nonempty closed convex subset C of a Banach space E is said to have *strong normal structure* ((sns), for short) if $\nu_C(t) < t$ for each $t > 0$. Clearly, if C has uniform normal structure, then C has (sns). Further, if C has (sns), then C has normal structure. In fact, let D be a bounded closed convex subset of C with $d(D) > 0$. Then we have $r(D) \leq \nu_C(d(D)) < d(D)$. Hence C has normal structure.

Theorem 4.2. *Let C be a nonempty bounded closed convex subset of a Banach space E. If C is (ucl), then C has (sns).*

Proof. Let $t \in (0, d(C)]$ and put $s = t/2$. We first prove that $\nu_C(t) \leq \max\{s, t - \eta_C(s)\}$. In fact, let D be a nonempty closed convex subset of C with $d(D) \leq t$. To complete the inequality, it is sufficient to show that $r(D) \leq \max\{s, t - \eta_C(s)\}$. We may assume that $d(D) \geq r(D) > s$. So, we can take some $x, y \in D$ with $\|x - y\| \geq s$. Put $w = (x + y)/2$. Let $z \in D$ be arbitrary. From the definition of $\eta_C(s)$, we have

$$\|w - z\| \leq (\|x - z\| \vee \|y - z\|) - \eta_C(s) \leq d(D) - \eta_C(s) \leq t - \eta_C(s).$$

Therefore, we have

$$r(D) \leq r(w, D) \leq t - \eta_C(s).$$

Further, since C is (ucl), we have $\eta_C(s) > 0$. So we get

$$\nu_C(t) \leq \max\{s, t - \eta_C(s)\} < t.$$

This completes the proof. □

Bae[2] and Maluta [18] proved that if a Banach space E satisfies $\tilde{N}(E) < 1$ then E is reflexive. Afterward, Ishihara and Takahashi [14] proved that every bounded closed convex subset C with $\tilde{N}(C) < 1$ is weakly compact. We shall show that if C has (sns), then C is weakly compact. For prove it, we need the following lemma.

Lemma 4.3. *Let C be a nonempty bounded closed convex subset of a Banach space E. If C has (sns), then there exists a function $b_C : [0, d(C)] \to [0, \infty)$ such that*

1. $t_1 \leq t_2 \implies b_C(t_1) \leq b_C(t_2)$;
2. $b_C(t_n) \to 0 \implies t_n \to 0$;
3. *if D is a nonempty closed convex subset of C, then there exists an $x \in D$ with $r(x, D) \leq d(D) - b_C(d(D))$.*

Proof. We define $b_C : [0, d(C)] \to [0, \infty)$ as follows:

$$b_C(t) = \frac{t - \nu_C(t)}{2} \quad \text{for each } t \in [0, d(C)].$$

To prove (1), let $t_1, t_2 \in [0, d(C)]$ with $t_1 \leq t_2$. Without loss of generality, we may assume $t_2 > 0$. From Proposition 4.1, it follows that $\nu_C(t_1) \geq (t_1/t_2)\nu_C(t_2)$. So, we have

$$\begin{aligned} b_C(t_1) &= \frac{1}{2}(t_1 - \nu_C(t_1)) \leq \frac{1}{2}\left(t_1 - \frac{t_1}{t_2} \cdot \nu_C(t_2)\right) \\ &= \frac{t_1}{t_2} \cdot \frac{1}{2}(t_2 - \nu_C(t_2)) = \frac{t_1}{t_2} \cdot b_C(t_2) \\ &\leq b_C(t_2). \end{aligned}$$

We show (2). Suppose $b_C(t_n) \to 0$ and $t_n \not\to 0$. Then, there exists a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that $t_{n_j} \geq \varepsilon > 0$. From (1) and the property of (sns) of C, we get

$$b_C(t_{n_j}) \geq b_C(\varepsilon) = \frac{\varepsilon - \nu_C(\varepsilon)}{2} > 0$$

This contradicts $b_C(t_n) \to 0$. So, we obtain (2). We show (3). Let D be a nonempty closed convex subset of C. If $d(D) = 0$, then $D = \{x\}$ for some $x \in C$ and therefore $r(x, D) = 0 = d(D) - b_C(d(D))$. We may assume that $d(D) > 0$. Since $\nu_C(d(D)) < d(D)$, we have

$$r(D) \leq \nu_C(d(D)) < \frac{\nu_C(d(D)) + d(D)}{2} = d(D) - b_C(d(D)).$$

So, there exists an $x \in D$ such that $r(x, D) < d(D) - b_C(d(D))$. This completes the proof. □

To prove Theorem 4.4, we shall use Šmulian's theorem: Let C be a nonempty closed convex subset of a Banach space E. Then C is weakly compact if and only if every decreasing sequence of nonempty closed convex subsets of C has a nonempty intersection; see [9, pp. 430–434] for more details.

Theorem 4.4. *Let C be a nonempty bounded closed convex subset of a Banach space E. If C has (sns), then C is weakly compact and has normal structure.*

Proof. Suppose C has (sns). We have already shown that if C has (sns) then C has normal structure. We shall show that C is weakly compact. Let $\{C_n\}_{n=1}^\infty$ be a sequence of nonempty closed convex subsets of C such that $C_1 \supset C_2 \supset C_3 \supset \cdots$. To complete the proof, by Šmulian's theorem, it is sufficient that we show $\bigcap_{n=1}^\infty C_n \neq \emptyset$. For each $m = 0, 1, 2, \ldots$, we construct a sequence $\{C_n^{(m)}\}_{n=1}^\infty$ of subsets of C and a sequence $\{x_n^{(m)}\}_{n=1}^\infty$ of C by the following steps: We give $\{C_n^{(0)}\}_{n=1}^\infty$ by $C_n^{(0)} = C_n$ for each n. If $\{C_n^{(m)}\}_{n=1}^\infty$ is given, by Lemma 4.3, there exists $x_n^{(m)} \in C_n^{(m)}$ such that

$$r(x_n^{(m)}, C_n^{(m)}) \leq d(C_n^{(m)}) - b_C(d(C_n^m))$$

for each n. Next we give $\{C_n^{(m+1)}\}_{n=0}^\infty$ by $C_n^{(m+1)} = \overline{\mathrm{co}}\{x_k^{(m)} : k \geq n\}$ for each n. Here b_C is as in Lemma 4.3. Note that $C_n \supset C_n^{(m)} \supset C_n^{(m+1)}$ and $C_n^{(m)} \supset C_{n+1}^{(m)}$ for each n and m. Put

$$a = \inf_{n,m} b_C(d(C_n^{(m)})) = \lim_{n,m\to\infty} b_C(d(C_n^{(m)})).$$

Then, we have $a = 0$. In fact, for each m, we have

$$\begin{aligned}
d(C_1^{(m+1)}) &= d(\overline{\text{co}}\{x_k^{(m)} : k \geq 1\}) = \sup_{i,j} \left\| x_i^{(m)} - x_j^{(m)} \right\| \\
&= \sup_i \sup_{j \geq i} \left\| x_i^{(m)} - x_j^{(m)} \right\| \leq \sup_i r(x_i^{(m)}, C_i^{(m)}) \\
&\leq \sup_i \left(d(C_i^{(m)}) - b_C(d(C_i^{(m)})) \right) \\
&\leq \sup_i d(C_i^{(m)}) - a \\
&= d(C_1^{(m)}) - a.
\end{aligned}$$

So, we have

$$ma \leq \sum_{k=0}^{m-1} \left(d(C_1^{(k)}) - d(C_1^{(k+1)}) \right) \leq d(C_1^{(0)}) < \infty.$$

Since m is arbitrary, we have that $a = 0$. Now, there exist increasing sequences $\{n_k\}$ and $\{m_k\}$ of integers such that $\lim_{k\to\infty} b_C(d(C_{n_k}^{(m_k)})) = 0$. By Lemma 4.3 (2), we have $\lim_{k\to\infty} d(C_{n_k}^{(m_k)}) = 0$. Then, by Cantor's theorem, we have

$$\bigcap_{n=1}^{\infty} C_n = \bigcap_{k=0}^{\infty} C_{n_k} \supset \bigcap_{k=0}^{\infty} C_{n_k}^{(m_k)} \neq \emptyset.$$

This completes the proof. □

Corollary 4.5 (Ishihara and Takahashi [14]). *Let C be a nonempty bounded closed convex subset of a Banach space E with $d(C) > 0$. If C has uniform normal structure, then C is weakly compact and has normal structure.*

Proof. Since C has uniform normal structure, we have C is (sns). Hence, by Theorem 4.4, C is weakly compact and has normal structure. □

The following is a direct consequence of Theorems 4.2 and 4.4.

Corollary 4.6. *Let C be a nonempty bounded closed convex subset of a Banach space E. If C is (ucl), then C is weakly compact and has normal structure.*

5 Fixed point theorem Throughout this section, C is a weakly compact convex subset of a Banach space E and T is a mapping of C into itself. Let $x \in C$. Then we define $K_m(x) = \overline{\text{co}}\{T^k x : k \geq m\}$ for each $m \geq 0$ and define $K(x) = \bigcap_{m\geq 0} K_m(x)$. Note that $K_m(x)$ is a nonempty closed convex subset of C for each $m \geq 0$, and hence $K(x)$ is also nonempty, closed and convex. We define, for every $x \in C$,

$$\begin{aligned}
d_x &= \inf_m d(K_m(x)) = \lim_{m\to\infty} d(K_m(x)), \\
g_x(y) &= \limsup_{n\to\infty} \|T^n x - y\| \quad \text{for every } y \in C, \text{ and} \\
A_x(D) &= \{z \in D : g_x(z) = \inf_{y\in D} g_x(y)\}
\end{aligned}$$

for every nonempty closed convex subset D of C. It is easy to see that $g_x : C \to \mathbb{R}$ is continuous and convex. Therefore, since D is weakly compact, $A_x(D)$ is nonempty, closed and convex.

The argument in this section mainly depends on Casini and Maluta [8].

Lemma 5.1. *If $z \in K(x)$, then $\|z-y\| \le g_x(y)$ for every $y \in C$.*

Proof. Let $m \ge 0$ be arbitrary. Since $z \in K_m(x)$, then

$$\|z-y\| \le \sup\{\|w-y\| : w \in K_m(x)\} = \sup_{k \ge m} \left\|T^k x - y\right\|.$$

Since $m \ge 0$ is arbitrary, we have the conclusion. □

Lemma 5.2. *Let $x \in C$. Then there exists $z \in K(x)$ such that $g_x(z) \le \nu_C(d_x)$.*

Proof. Since $A_x(K_m(x))$ is nonempty for each m, we can take a $z_m \in A_x(K_m(x))$. Then, we have

$$\begin{aligned} g_x(z_m) &= \limsup_{k\to\infty} \left\|T^k x - z_m\right\| \\ &\le \sup_{k\ge m} \left\|T^k x - z_m\right\| \\ &= r(z_m, K_m(x)) \\ &= r(K_m(x)) \\ &\le \nu_C(d(K_m(x))). \end{aligned}$$

Since C is weakly compact, there exists a subsequence $\{z_{m_j}\}$ of $\{z_m\}$ converging weakly to some $z \in C$. Since $z_m \in K_m(x)$, we have

$$z \in \bigcap_j K_{m_j}(x) = \bigcap_m K_m(x) = K(x).$$

So,

$$\begin{aligned} g_x(z) &\le \liminf_{j\to\infty} g_x(z_{m_j}) \\ &= \liminf_{j\to\infty} \inf\{g_x(y) : y \in K_{m_j}(x)\} \\ &= \lim_{m\to\infty} \inf\{g_x(y) : y \in K_m(x)\} \\ &\le \inf\{g_x(y) : y \in K(x)\} \\ &\le g_x(z). \end{aligned}$$

Therefore, noting that ν_C is continuous (Proposition 4.1), we have

$$\begin{aligned} g_x(z) &= \lim_{m\to\infty} \inf\{g_x(y) : y \in K_m(x)\} \\ &= \lim_{m\to\infty} g_x(z_m) \\ &\le \limsup_{m\to\infty} \nu_C(d(K_m(x))) \\ &= \nu_C(d_x). \end{aligned}$$

This completes the proof. □

Theorem 5.3. *Let C be a nonempty bounded closed convex subset of a Banach space E. Suppose C is (sns). Let φ be a continuous nondecreasing function of $[0, d(C)]$ into itself. Suppose there exists $a \in [0,1)$ such that $(\varphi \circ \nu_C \circ \varphi)(t) \le at$ for each $t \in [0, d(C)]$, and suppose that there exists $M \ge 0$ such that $\varphi(t) \le Mt$ for each $t \in [0, d(C)]$. If a mapping T of C into itself satisfies*

$$\|T^n x - T^n y\| \le \varphi(\|x-y\|) \quad \textit{for every } x, y \in C \textit{ and } n \ge 0,$$

then T has a fixed point.

Proof. Suppose $d_x = 0$ for some $x \in C$. Then, we have

$$d_x = \inf_m d(K_m(x)) = \inf_m \sup_{i,j \geq m} \left\| T^i x - T^j x \right\| = 0.$$

This means that $\{T^n x\}$ is a Cauchy sequence. Put $z = \lim_{n\to\infty} T^n x$. Then we have

$$z = \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^{n+1} x = T(\lim_{n\to\infty} T^n x) = Tz$$

and hence $z \in F(T)$. Now we may assume that $d_x > 0$ for any $x \in C$. We construct a sequence $\{x_n\}_{n=0}^{\infty}$ as follows: $x_0 \in C$ and $x_{n+1} \in K(x_n)$ such that $g_{x_n}(x_{n+1}) \leq \nu_C(d_{x_n})$ for each n. Put $r(y) = \sup_{k\geq 0} \left\| y - T^k y \right\|$ for every $y \in C$. Now, we have

$$\begin{aligned} g_{x_n}(x_{n+1}) \leq \nu_C(d_{x_n}) &= \lim_{m\to\infty} \nu_C(d(K_m(x_n))) \\ &= \lim_{m\to\infty} \nu_C \left(\sup_{i,j\geq m} \left\| T^i x_n - T^j x_n \right\| \right) \\ &\leq \nu_C \left(\varphi\Big(\sup_k \left\| x_n - T^k x_n \right\| \Big) \right) \\ &= \nu_C(\varphi(r(x_n))). \end{aligned}$$

For each N, we have

$$\begin{aligned} g_{x_n}(T^N x_{n+1}) &= \limsup_{k\to\infty} \left\| T^k x_n - T^N x_{n+1} \right\| = \limsup_{k\to\infty} \left\| T^N T^k x_n - T^N x_{n+1} \right\| \\ &\leq \limsup_{k\to\infty} \varphi\Big(\left\| T^k x_n - x_{n+1} \right\| \Big) = \varphi \left(\limsup_{k\to\infty} \left\| T^k x_n - x_{n+1} \right\| \right) \\ &= \varphi(g_{x_n}(x_{n+1})). \end{aligned}$$

So, since $x_{n+1} \in K(x_n)$, from Lemma 5.1, we have

$$\begin{aligned} r(x_{n+1}) = \sup_N \left\| x_{n+1} - T^N x_{n+1} \right\| &\leq \sup_N g_{x_n}(T^N x_{n+1}) \\ &\leq \varphi(g_{x_n}(x_{n+1})) \leq \varphi(\nu_C(\varphi(r(x_n)))) \\ &\leq a r(x_n). \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. In fact, we have, for each k,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \left\| x_{n+1} - T^k x_n \right\| + \left\| T^k x_n - x_n \right\| \\ &\leq \left\| x_{n+1} - T^k x_n \right\| + r(x_n). \end{aligned}$$

Putting $k \to \infty$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq g_{x_n}(x_{n+1}) + r(x_n) \\ &\leq \nu_C(\varphi(r(x_n))) + r(x_n) \\ &\leq (M+1) r(x_n) \\ &\leq (M+1) a^n r(x_0). \end{aligned}$$

This follows that $\{x_n\}$ is a Cauchy sequence. Put $z = \lim_{n\to\infty} x_n$. Then, we have

$$\|z - Tz\| = \lim_{n\to\infty} \|x_n - Tx_n\| \leq \lim_{n\to\infty} r(x_n) = 0.$$

Hence $z = Tz$. This completes the proof. □

The following result was proved by Casini and Maluta [8] in the case that E has uniform normal structure. Ishihara and Takahashi [14] obtained more general results for uniformly Lipschitz semigroups.

Corollary 5.4. *Let C be a nonempty bounded closed convex subset of a Banach space E. Suppose C has uniform normal structure. Let $\gamma > 0$ satisfy $\gamma^2 \tilde{N}(C) < 1$. Then every uniformly γ-Lipschitzian mappings on C has a fixed point.*

Proof. Let T be a uniformly γ-Lipschitzian mapping on C. Put $\varphi(t) = \gamma t$ for each $t \in [0, d(C)]$. Then, $\varphi : [0, d(C)] \to [0, d(C)]$ is continuous and nondecreasing, and satisfies

$$\|T^n x - T^n y\| \leq \gamma \|x - y\| = \varphi(\|x - y\|)$$

for every $x, y \in C$ and each n. Further, since $\nu_C(t) \leq \tilde{N}(C) \cdot t$ for each $t \geq 0$, we have

$$\begin{aligned}(\varphi \circ \nu_C \circ \varphi)(t) &\leq \gamma \cdot \nu_C(\varphi(t)) \leq \gamma \tilde{N}(C) \cdot \varphi(t) \\ &= \gamma^2 \tilde{N}(C) \cdot t\end{aligned}$$

By Theorem 5.3, putting $a = \gamma^2 \tilde{N}(C)$ and $M = \gamma$, T has a fixed point. □

6 Mappings of type (γ) Throughout this section, C is a nonempty bounded closed convex subset of a Banach space E and B is a nonempty convex subset of E. We denote by Γ the set of all strictly increasing, continuous and convex functions $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$. Let $\gamma \in \Gamma$. A mapping T of B into E is said to be *of type (γ)* [5] if for every $x, y \in B$ and $c \in [0, 1]$,

$$\gamma(\|cTx + (1 - c)Ty - T(cx + (1 - c)y)\|) \leq \|x - y\| - \|Tx - Ty\|.$$

Clearly if a mapping T of B into E is of type (γ) then T is nonexpansive. Bruck [5] proved that if E is uniformly convex and B is bounded closed and convex then there exists $\gamma \in \Gamma$ such that every nonexpansive mapping of B into E is of type (γ). This result is very important and useful to prove nonlinear ergodic theorems for nonexpansive mappings. In this section, we shall try to extend the result of Bruck.

Lemma 6.1. *$\eta_C(t) \leq t/2$ for each $t \in I(C)$.*

Proof. Let $t \in I(C)$. Then we can take some $x, y \in C$ such that $\|x - y\| = t$. Put $z = (x + y)/2$. Then we have $\|x - z\| \vee \|y - z\| = t/2$, and hence $\eta_C(t) \leq t/2$. □

Lemma 6.2. *For each $s, t \in I(C) \setminus \{0\}$ with $s \leq t$,*

$$\frac{\eta_C(s)}{s} \leq \frac{\eta_C(t)}{t} \leq \frac{1}{2}.$$

Proof. Take $x, y, z \in C$ with $\|x - y\| \geq t$. Put $c = s/t$, $u = cx + (1-c)z$ and $v = cy + (1-c)z$. Then, we have $u, v \in C$ and $\|u - v\| \geq s$. It follows that

$$\begin{aligned}\eta_C(s) &\leq (\|u - z\| \vee \|v - z\|) - \left\|\frac{u + v}{2} - z\right\| \\ &= c\left((\|x - z\| \vee \|y - z\|) - \left\|\frac{x + y}{2} - z\right\|\right).\end{aligned}$$

Hence we have $\eta_C(s) \leq c \cdot \eta_C(t)$. So, we obtain the conclusion. □

Lemma 6.3. *Let $u, v, w \in C$, $R \geq 0$ and $t \in I(C)$ satisfy $\|u - w\| \leq R$, $\|v - w\| \leq R$ and $\|u - v\| \geq t$. Then*

$$\begin{aligned}\|cu + (1-c)v - w\| &\leq R - 2\min\{c, 1-c\}\eta_C(t)\\ &\leq R - 2c(1-c)\eta_C(t)\end{aligned}$$

for each $c \in [0, 1]$.

Proof. Without loss of generality, we may assume $c \leq 1/2$. By the definition of $\eta_C(t)$, we have

$$\left\|\frac{u+v}{2} - w\right\| \leq R - \eta_C(t).$$

Then we have

$$\begin{aligned}\|cu + (1-c)v - w\| &= \left\|2c\,\frac{u+v}{2} + (1-2c)v - w\right\|\\ &\leq 2c\left\|\frac{u+v}{2} - w\right\| + (1-2c)\left\|v - w\right\|\\ &\leq 2c(R - \eta_C(t)) + (1-2c)R\\ &= R - 2c\eta_C(t)\\ &= R - 2\min\{c, 1-c\}\eta_C(t)\\ &\leq R - 2c(1-c)\eta_C(t).\end{aligned}$$

This completes the proof. □

Theorem 6.4. *Let C be a nonempty bounded closed convex subset of a Banach space E. Suppose C is (ucl). Then there exists $\gamma \in \Gamma$ such that for every nonempty convex subset B of E and for every nonexpansive mapping T of B into C, T is of type (γ).*

Proof. For every $s > 0$, we define

$$f(s) = \begin{cases}\eta_C(s)/s & \text{if } s \in I(C) \setminus \{0\};\\ 1/2 & \text{if } s \in (0, \infty) \setminus I(C).\end{cases}$$

From Lemma 6.2, the function $f : (0, \infty) \to (0, \infty)$ is nondecreasing. We define a function $\gamma : [0, \infty) \to [0, \infty)$ as

$$\gamma(t) = 2\int_0^t f(s)\, ds \quad \text{for each } t \geq 0.$$

Then we have $\gamma \in \Gamma$. Let B be a convex subset of E and let T be a nonexpansive mapping of B into C. Let $x, y \in B$ and $c \in (0, 1)$. Put

$$t_0 = \|cTx + (1-c)Ty - T(cx + (1-c)y)\|\,.$$

and put $u, v, w \in C$ as

$$\begin{aligned}u &= cTx + (1-c)Ty,\\ v &= T(cx + (1-c)y)\end{aligned}$$

and

$$w = cT(cx + (1-c)y) + (1-c)Ty.$$

Then we have

$$\begin{aligned} \|u - w\| &= c\,\|Tx - T(cx + (1-c)y)\| \le c(1-c)\,\|x - y\|\,, \\ \|v - w\| &= (1-c)\,\|T(cx + (1-c)y) - Ty\| \le c(1-c)\,\|x - y\|\,, \\ \|u - v\| &= \|cTx + (1-c)Ty - T(cx + (1-c)y)\| = t_0 \end{aligned}$$

and

$$\|(1-c)u + cv - w\| = c(1-c)\,\|Tx - Ty\|\,.$$

By Lemma 6.3, we have

$$c(1-c)\,\|Tx - Ty\| \le c(1-c)\,\|x - y\| - 2c(1-c)\eta_C(t_0)$$

and hence

$$\|Tx - Ty\| \le \|x - y\| - 2\eta_C(t_0).$$

From this, we have

$$\gamma(t_0) \le 2 \cdot t_0 f(t_0) = 2 \cdot t_0 \cdot \frac{\eta_C(t_0)}{t_0} = 2\eta_C(t_0) \le \|x - y\| - \|Tx - Ty\|\,.$$

This completes the proof. □

The following is Bruck's result [5, Lemma 1.1].

Corollary 6.5. *Let B be a nonempty bounded convex subset of a uniformly convex Banach space E. Then there exists $\gamma \in \Gamma$ such that every nonexpansive mapping T of B into E is of type (γ).*

Proof. Put $C = \{y \in E : \|y\| \le d(B)\}$. From Proposition 3.2 and Theorem 6.4, there exists $\gamma \in \Gamma$ such that every nonexpansive mapping of B into C is of type (γ). Let T be an arbitrary nonexpansive mapping of B into E. Take an $x_0 \in B$. Define a nonexpansive mapping $\tilde{T}$ of B into E as follows:

$$\tilde{T}(x) = T(x) - T(x_0) \quad \text{for every } x \in B.$$

It is clear that $\tilde{T}(x) \in C$ for every $x \in B$. Therefore $\tilde{T}$ is of type (γ). This implies that T is of type (γ). □

Bruck [5, Remark] also obtained the following result, which was completely proved by Atsushiba and Takahashi [1].

Corollary 6.6. *Let C be a compact convex subset of a strictly convex Banach space E. Then there exists $\gamma \in \Gamma$ such that every nonexpansive mapping T of C into itself is of type (γ).*

Proof. By Proposition 3.3 and Theorem 6.4 (putting $B = C$), we obtain the conclusion. □

7 The convex approximation property and means Let C be a convex subset of a Banach space E. Then, C is said to have the *convex approximation property* if for each $\varepsilon > 0$ there exists a positive integer p such that for each subset M of C,

$$\operatorname{co} M \subset \operatorname{co}_p M + B_\varepsilon,$$

where $B_\varepsilon = \{x \in E : \|x\| \le \varepsilon\}$; see also [6].

We denote by $\mathcal{N}(C)$ the set of all nonexpansive mappings of C into itself. For $\gamma \in \Gamma$, we also denote by $\mathcal{N}_\gamma(C)$ the set of all mappings of type (γ) of C into itself. Theorem 6.4 shows that if C is (ucl) then there exists $\gamma \in \Gamma$ such that $\mathcal{N}(C) = \mathcal{N}_\gamma(C)$.

Lemma 7.1. *Let C be a nonempty bounded closed convex subset of a Banach space E. Suppose C is (ucl) and has the convex approximation property. Then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $T \in \mathcal{N}(C)$,*

$$\overline{\operatorname{co}}\, F_\delta(T) \subset F_\varepsilon(T),$$

where $F_t(T) = \{x \in C : \|x - Tx\| \le t\}$ for $t > 0$.

Proof. We follow an idea in [6]. Since C is (ucl), by Theorem 6.4, there exists $\gamma \in \Gamma$ such that $\mathcal{N}(C) = \mathcal{N}_\gamma(C)$. By [5, Lemma 1.2], the inverse function σ of $t \mapsto \gamma^{-1}(2t) + t$ satisfies

$$\operatorname{co}_2 F_{\sigma(t)}(T) \subset F_t(T)$$

for each $t > 0$ and every $T \in \mathcal{N}_\gamma(C) = \mathcal{N}(C)$. Hence by induction, we have

$$\operatorname{co}_{2^n} F_{\sigma^n(t)}(T) = \operatorname{co}_2 \left(\operatorname{co}_{2^{n-1}} F_{\sigma^n(t)}(T)\right) \subset \operatorname{co}_2 F_{\sigma(t)}(T) \subset F_t(T)$$

for each $n \ge 1$, $t > 0$ and $T \in \mathcal{N}(C)$. Let $\varepsilon > 0$ be arbitrary. Since C has the convex approximation property, there exists a positive integer p such that

$$\operatorname{co} M \subset \operatorname{co}_p M + B_{\varepsilon/3}$$

for every $M \subset C$. Take a large n with $2^n \ge p$ and put $\delta = \sigma^n(\varepsilon/3)$. Let $T \in \mathcal{N}(C)$ be arbitrary. Then, we have

$$\operatorname{co} F_\delta(T) \subset \operatorname{co}_{2^n} F_\delta(T) + B_{\varepsilon/3} \subset F_{\varepsilon/3}(T) + B_{\varepsilon/3}.$$

Therefore we have $\operatorname{co} F_\delta(T) \subset F_\varepsilon(T)$. In fact, for every $z \in \operatorname{co} F_\delta(T)$, there exists $y \in F_{\varepsilon/3}(T)$ such that $\|z - y\| \le \varepsilon/3$. Since

$$\begin{aligned}\|z - Tz\| &= \|z - y\| + \|y - Ty\| + \|Ty - Tz\| \\ &\le 2\|z - y\| + \|y - Ty\| \\ &\le 2 \cdot \varepsilon/3 + \varepsilon/3 = \varepsilon,\end{aligned}$$

we have $z \in F_\varepsilon(T)$. Since $F_\varepsilon(T)$ is closed, we have $\overline{\operatorname{co}}\, F_\delta(T) \subset F_\varepsilon(T)$. □

The following lemma was proved by Bruck [5, Lemma 1.5].

Lemma 7.2. *Let C be a nonempty bounded closed convex subset of a Banach space E. Let $\gamma \in \Gamma$ and let $T \in \mathcal{N}_\gamma(C)$. Suppose sequences $\{y_i\}$ and $\{z_i\}$ of C and $\delta_n > 0$ satisfy that $(1/n)\sum_{i=0}^{n-1} \|y_{i+1} - Ty_i\| \le \delta_n$ and $(1/n)\sum_{i=0}^{n-1} \|z_{i+1} - Tz_i\| \le \delta_n$. Then, for each $\lambda \in [0, 1]$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} \|\lambda y_{i+1} + (1-\lambda)z_{i+1} - T(\lambda y_i + (1-\lambda)z_i)\| \le \gamma^{-1}(d(C)/n + 2\delta_n) + \delta_n.$$

To obtain Theorem 7.4, we need the following lemma.

Lemma 7.3. *Let C be a nonempty bounded closed convex subset of a Banach space E. Suppose C is (ucl). Let $\varepsilon > 0$. Then there exist $p \in \mathbb{N}$, $\delta > 0$ and $N \in \mathbb{N}$ such that for every $T \in \mathcal{N}(C)$ and every sequence $\{x_i\}$ of C with $\|x_{i+1} - Tx_i\| \leq \delta$ for each i,*

$$\frac{1}{n}\sum_{i=0}^{n-1}\left\|\overline{x_i^p} - T\overline{x_i^p}\right\| < \varepsilon \quad \textit{for each } n \geq N,$$

where $\overline{x_i^p} = (1/p)\sum_{j=0}^{p-1} x_{j+i}$.

Proof. Since C is (ucl), there exists $\gamma \in \Gamma$ such that $\mathcal{N}(C) = \mathcal{N}_\gamma(C)$. Choose $p \in \mathbb{N}$ with $d(C)/p < \varepsilon/2$. Put $\sigma(t) = \gamma^{-1}(2t) + t$ and choose $\delta > 0$ with $\sigma^{p-1}(\delta) < \varepsilon/2$. Put $\sigma_n(t) = \gamma^{-1}(d(C)/n + 2t) + t$. By the continuity of γ^{-1}, it follows that $\lim_{n\to\infty}\sigma_n(t) = \sigma(t)$. So, there exists $N \in \mathbb{N}$ such that $\sigma_n^{p-1}(\delta) < \varepsilon/2$ for each $n \geq N$. Let $T \in \mathcal{N}(C) = \mathcal{N}_\gamma(C)$ and let $\{x_i\}$ be a sequence C with $\|x_{i+1} - Tx_i\| \leq \delta$ for each i. Then, for each i,

$$\left\|\overline{x_{i+1}^p} - \overline{x_i^p}\right\| = \frac{1}{p}\|x_{p+i} - x_i\| \leq \frac{d(C)}{p} < \frac{\varepsilon}{2}.$$

We also have, for each $n \in \mathbb{N}$ and $q \in \mathbb{N}$ with $1 \leq q \leq p$,

$$\frac{1}{n}\sum_{i=0}^{n-1}\left\|\overline{x_{i+1}^q} - T\overline{x_i^q}\right\| \leq \sigma_n^{q-1}(\delta).$$

In fact, if $q = 1$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\left\|\overline{x_{i+1}^1} - T\overline{x_i^1}\right\| = \frac{1}{n}\sum_{i=0}^{n-1}\|x_{i+1} - Tx_i\| \leq \frac{1}{n}\sum_{i=0}^{n-1}\delta = \delta.$$

If $2 \leq q \leq p$, by induction, it follows from Lemma 7.2 that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1}\left\|\overline{x_{i+1}^q} - T\overline{x_i^q}\right\| \\
&\quad = \tfrac{1}{n}\textstyle\sum_{i=0}^{n-1}\left\|\left(1-\tfrac{1}{q}\right)\overline{x_{i+1}^{q-1}} + \tfrac{1}{q}x_{q+i} - T\left(\left(1-\tfrac{1}{q}\right)\overline{x_i^{q-1}} + \tfrac{1}{q}x_{q+i-1}\right)\right\| \\
&\quad \leq \sigma_n\left(\max\left\{\tfrac{1}{n}\textstyle\sum_{i=0}^{n-1}\left\|\overline{x_{i+1}^{q-1}} - T\overline{x_i^{q-1}}\right\|,\ \tfrac{1}{n}\textstyle\sum_{i=0}^{n-1}\|x_{q+i} - x_{q+i-1}\|\right\}\right) \\
&\quad \leq \sigma_n(\max\{\sigma_n^{q-2}(\delta),\ \delta\}) \\
&\quad = \sigma_n(\sigma_n^{q-2}(\delta)) = \sigma_n^{q-1}(\delta).
\end{aligned}$$

In particular, if $q = p$ and $n \geq N$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\left\|\overline{x_{i+1}^p} - T\overline{x_i^p}\right\| \leq \sigma_n^{p-1}(\delta) < \varepsilon/2.$$

Therefore, we obtain

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1}\left\|\overline{x_i^p} - T\overline{x_i^p}\right\| &\leq \frac{1}{n}\sum_{i=0}^{n-1}\left(\left\|\overline{x_{i+1}^p} - \overline{x_i^p}\right\| + \left\|\overline{x_{i+1}^p} - T\overline{x_i^p}\right\|\right) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

for each $n \geq N$. ∎

Theorem 7.4. *Let C be a nonempty bounded closed convex subset of a Banach space E. Suppose C is (ucl) and has the convex approximation property. Let $\varepsilon > 0$. Then there exist $p \in \mathbb{N}$, $\delta > 0$ and $N \in \mathbb{N}$ satisfying the following: For every nonexpansive mapping T of C into itself and every sequence $\{x_i\}$ of C with $\|x_{i+1} - Tx_i\| \leq \delta$ for each i,*

$$\frac{1}{n}\sum_{i=0}^{n-1} x_i \in F_\varepsilon(T) \quad \textit{for each } n \geq N.$$

Proof. By Lemma 7.1, there exists $\eta > 0$ such that

$$2\eta \cdot d(C) \leq \varepsilon/3 \text{ and } \overline{\mathrm{co}}\, F_\eta(T) \subset F_{\varepsilon/3}(T).$$

By Lemma 7.3, there exist $p \in \mathbb{N}$, $\delta > 0$ and $N \in \mathbb{N}$ such that for every $T \in \mathcal{N}(C)$ and every sequence $\{x_i\}$ of C with $\|x_{i+1} - Tx_i\| \leq \delta$ for each i,

$$\frac{1}{n}\sum_{i=0}^{n-1} \|w_i - Tw_i\| < \eta^2 \quad \text{for each } n \geq N,$$

where $w_i = \overline{x_i^p} = (1/p)\sum_{j=0}^{p-1} x_{j+i}$. We may also assume $p/N \leq \eta$. Put

$$A(n) = \{i \in \mathbb{N} : 0 \leq i \leq n-1 \text{ and } \|w_i - Tw_i\| \geq \eta\}$$

and

$$B(n) = \{i \in \mathbb{N} : 0 \leq i \leq n-1 \text{ and } \|w_i - Tw_i\| < \eta\}.$$

Fix $n \geq N$. Since $\sum_{i=0}^{n-1} \|w_i - Tw_i\| \leq n\eta^2$, we have $\sharp A(n) < n\eta$, where $\sharp$ denotes cardinality. On the other hand, by Corollary 4.6, C is weakly compact and has normal structure. So, by Kirk's fixed point theorem, T has a fixed point. Let $z \in F(T)$. Then, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} w_i &= \left(\frac{\sharp A(n)}{n} z + \frac{1}{n}\sum_{i \in B(n)} w_i\right) + \frac{1}{n}\sum_{i \in A(n)} (w_i - z)\\ &\in \overline{\mathrm{co}}\, F_\eta(T) + B_{\eta \cdot d(C)}.\end{aligned}$$

From

$$\frac{1}{n}\sum_{i=0}^{n-1} x_i = \frac{1}{n}\sum_{i=0}^{n-1} w_i + \frac{1}{np}\sum_{i=0}^{p-1}(p-1-i)(x_i - x_{i+n})$$

and

$$\left\|\frac{1}{np}\sum_{i=0}^{p-1}(p-1-i)(x_i - x_{i+n})\right\| \leq \frac{1}{np} \cdot p^2 \cdot d(C) \leq \frac{p}{N} \cdot d(C) \leq \eta \cdot d(C),$$

we have

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} x_i &\in \overline{\mathrm{co}}\, F_\eta(T) + B_{\eta \cdot d(C)} + B_{\eta \cdot d(C)}\\ &= \overline{\mathrm{co}}\, F_\eta(T) + B_{2\eta \cdot d(C)}\\ &\subset F_{\varepsilon/3}(T) + B_{\varepsilon/3}.\end{aligned}$$

As in the proof of Lemma 7.1, we have $F_{\varepsilon/3}(T) + B_{\varepsilon/3} \subset F_\varepsilon(T)$. This completes the proof. ∎

As a direct consequence of Theorem 7.4, we have the following result.

Corollary 7.5. *Let C be a nonempty bounded closed convex subset of a Banach space E and suppose C is (ucl) and has the convex approximation property. Then*

$$\lim_{n\to\infty} \sup \left\{ \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^i x - T\left(\frac{1}{n}\sum_{i=0}^{n-1} T^i x\right)\right\| : x \in C,\, T \in \mathcal{N}(C) \right\} = 0.$$

8 Existence of ergodic retraction We denote by l^∞ the Banach space of all bounded sequences of reals with supremum norm. $\mu \in (l^\infty)^*$ is said to be a mean on l^∞ if $\|\mu\| = \mu(1) = 1$. Let $\{u_n\}$ be a sequence of E such that $\overline{\mathrm{co}}\{u_n\}$ is weakly compact, and let μ be a mean on l^∞. Then, by [12], there exists a unique $u_\mu \in \overline{\mathrm{co}}\{u_n\}$ such that

$$\langle u_\mu, x^*\rangle = \mu_n \langle u_n, x^*\rangle \quad \text{for every } x^* \in E^*.$$

Here the right-hand side means the value of μ at $f = \{f(n)\} \in l^\infty$, where $f(n) = \langle u_n, x^*\rangle$ for each n. For more details, see [12, 22].

Theorem 8.1. *Let C be a nonempty bounded closed convex subset of a Banach space E. Suppose that C is (ucl) and has the convex approximation property. Let T be a nonexpansive mapping of C into itself. Then there exists a nonexpansive retraction P of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\mathrm{co}}\{T^n x : n \in \mathbb{N}\}$ for every $x \in C$.*

Proof. Since C is (ucl), by Corollary 4.6, C is weakly compact. Let μ be a Banach limit. Then, for every $x \in C$ there exists $Px \in \overline{\mathrm{co}}\{T^n x : n \in \mathbb{N}\}$ such that $\langle Px, x^*\rangle = \mu_n\langle T^n x, x^*\rangle$ for every $x^* \in E^*$. Since

$$\begin{aligned}\langle Px - Py, x^*\rangle &= \mu_n \langle T^n x - T^n y, x^*\rangle \\ &\le \mu_n(\|T^n x - T^n y\|\,\|x^*\|) \\ &\le \|x - y\|\,\|x^*\|\end{aligned}$$

for every $x^* \in E^*$, P is nonexpansive. Since

$$\begin{aligned}\langle Px, x^*\rangle &= \mu_n \langle T^n x, x^*\rangle \\ &= \mu_n\langle T^{n+1} x, x^*\rangle \\ &= \mu_n \langle T^n T x, x^*\rangle \\ &= \langle PTx, x^*\rangle,\end{aligned}$$

we have $P = PT$. For every $z \in F(T)$, we have

$$\langle Pz, x^*\rangle = \mu_n\langle T^n z, x^*\rangle = \langle z, x^*\rangle$$

and hence $Pz = z$. To finish the proof, we shall show $Px \in F(T)$ for every $x \in C$. Let $\varepsilon > 0$ be arbitrary. By Lemma 7.1, there exists $\delta > 0$ such that $\overline{\mathrm{co}}\, F_\delta(T) \subset F_\varepsilon(T)$. By Corollary 7.5, there exists $n \in \mathbb{N}$ such that $(1/n)\sum_{i=0}^{n-1} T^i y \in F_\delta(T)$ for every $y \in C$. Since

$$\begin{aligned}\langle Px, x^*\rangle &= \mu_k \langle T^k x, x^*\rangle \\ &= \frac{1}{n}\sum_{i=0}^{n-1} \mu_k \langle T^{i+k} x, x^*\rangle \\ &= \mu_k \left\langle \frac{1}{n}\sum_{i=0}^{n-1} T^i T^k x, x^*\right\rangle,\end{aligned}$$

we have

$$\begin{aligned} Px &\in \overline{\mathrm{co}}\left\{\frac{1}{n}\sum_{i=0}^{n-1} T^i T^k x : k \geq 0\right\} \\ &\subset \overline{\mathrm{co}}\, F_\delta(T) \subset F_\varepsilon(T). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $Px \in F(T)$. This completes the proof. □

References

[1] S. Atsushiba and W. Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domains*, Math. Japon. **52** (2000), 183–195.

[2] J. S. Bae, *Reflexivity of a Banach space with a uniformly normal structure*, Proc. Amer. Math. Soc. **90** (1984), 269–270.

[3] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), 1511–1514.

[4] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044.

[5] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), 107–116.

[6] R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. **38** (1981), 304–314.

[7] W. L. Bynum, *Normal structure coefficients for Banach spaces*, Pacific J. Math. **86** (1980), 427–436.

[8] E. Casini and E. Maluta, *Fixed points of uniformly Lipschitzian mappings in spaces with uniformly normal structure*, Nonlinear Anal. **9** (1985), 103–108.

[9] N. Dunford and J. T. Schwartz, Linear Operators. I. General Theory, Interscience Publishers, Inc., New York, 1958.

[10] K. Goebel and W. A. Kirk, Topics in metric fixed point theory, Cambridge University Press, Cambridge, 1990.

[11] D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251–258.

[12] N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269–1281.

[13] N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for an amenable semigroup of nonexpansive mappings in a Banach space*, Pacific J. Math. **112** (1984), 333–346.

[14] H. Ishihara and W. Takahashi, *Modulus of convexity, characteristic of convexity and fixed point theorems*, Kodai Math. J. **10** (1987), 197–208.

[15] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.

[16] A. T. Lau, N. Shioji and W. Takahashi, *Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces*, J. Funct. Anal. **161** (1999), 62–75.

[17] A. T. Lau and W. Takahashi, *Weak convergence and non-linear ergodic theorems for reversible semigroups of nonexpansive mappings*, Pacific J. Math. **126** (1987), 277–294.

[18] E. Maluta, *Uniformly normal structure and related coefficients*, Pacific J. Math. **111** (1984), 357–369.

[19] K. Nishiura, N. Shioji and W. Takahashi, *Nonlinear ergodic theorems for asymptotically nonexpansive semigroups in Banach spaces*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **10** (2003), 563–578.

[20] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[21] W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **97** (1986), 55–58.

[22] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[23] W. Takahashi and N. Tsukiyama, *Approximating fixed points of nonexpansive mappings with compact domains*, Comm. Appl. Nonlinear Anal. **7** (2000), 39–47.

Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, Oh-okayama, Meguro-ku,
Tokyo, 152-8552, Japan