ON THE ALUTHGE TRANSFORMATIONS OF ∞-HYPONORMAL OPERATORS

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Abstract. A bounded linear operator T is called ∞-hyponormal if T is p-hyponormal for every $p > 0$. In this paper ∞-hyponormality of the Aluthge transformations of ∞-hyponormal operators is investigated. It is shown that the Aluthge transformation of an ∞-hyponormal operator is not necessarily ∞-hyponormal. It is also shown that the (generalized) Aluthge transformation of an ∞-hyponormal operator T is ∞-hyponormal provided $|T||T^*| = |T^*||T|$. Moreover we give an example of an ∞-hyponormal operator T whose Aluthge transformation $\tilde{T}$ is ∞-hyponormal but $|T||T^*| \neq |T^*||T|$.

1 Introduction A bounded linear operator T on a Hilbert space is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ $(p > 0)$. (The notion of p-hyponormal operators for $p \in \mathbf{N}$ was introduced first by Fujii and Nakatsu[6]. A 1-hyponormal operator is nothing but a hyponormal operator.) Concerning p-hyponormal operators, many interesting results have been obtained (e.g., [1], [4], [5], [7]–[10], [14]). The unilateral shift is a simple example being p-hyponormal for every $p > 0$. In the hope of getting fruitful results, the authors [11], [12] investigated operators that are p-hyponormal for every $p > 0$ and they called them ∞-hyponormal. Let us recall the definition of ∞-hyponormal operators.

Definition. A bounded linear operator T on a Hilbert space is called ∞-hyponormal if it is p-hyponormal for every $p > 0$. By the Löwner-Heinz theorem, T is ∞-hyponormal if and only if $(T^*T)^n \geq (TT^*)^n$ for every $n \in \mathbf{N}$, or $|T|^n \geq |T^*|^n$ for every $n \in \mathbf{N}$.

Concerning ∞-hyponormal operators, the authors [11], [12] proved that the outer boundary of the spectrum of a pure ∞-hyponormal operator is the circle with the radius $||T||$, and showed the existence of non-trivial invariant subspaces for any ∞-hyponormal operator. (Recall that an operator T is said to be pure if T has no reducing subspace on which it is normal.) For other facts(e.g., algebraic properties of the set of ∞-hyponormal operators), see [11].

On the other hand, Aluthge[1] introduced the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ for a p-hyponormal operator T, where $T = U|T|$ is the polar decomposition of T. Further Furuta[7] introduced the operator $\tilde{T}_{s,t} = |T|^s U|T|^t$ for a p-hyponormal operator T and $s, t > 0$. The operators $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $\tilde{T}_{s,t} = |T|^s U|T|^t$ are called the Aluthge transformation and the generalized Aluthge transformation of T, respectively. These operators have nice properties as follows.

Theorem A[1]. *For a p-hyponormal* $(0 < p \leq 1)$ *operator with the polar decomposition* $T = U|T|$*, Aluthge transformation* $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *of* T *is* $\min\{p + \frac{1}{2}, 1\}$*-hyponormal.*

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Theorem B[8], [9], [14]. *Let T be a p-hyponormal $(0 < p \leq 1)$ operator with the polar decomposition $T = U|T|$. Then the following assertions hold.*

(1) If $s, t > 0$, $\max\{s, t\} \geq p$ hold, then $\tilde{T}_{s,t} = |T|^s U |T|^t$ is $\frac{p+\min\{s,t\}}{s+t}$-hyponormal.

(2) If $s, t > 0$, $\max\{s, t\} \leq p$ hold, then $\tilde{T}_{s,t} = |T|^s U |T|^t$ is hyponormal.

Roughly speaking, these theorems say that the Aluthge transformations or generalized Aluthge transformations "improve" the degree of hyponormality, although restricted in the range of $0 < p \leq 1$. So, it is natural to ask whether the Aluthge transformations of ∞-hyponormal operators preserve the ∞-hyponormality or not.

In section 2, we answer the question in the negative by giving an example of an ∞-hyponormal operator for which the Aluthge transformation is not ∞-hyponormal.

However, for many ∞-hyponormal operators (e.g., the unilateral shift, a unilateral weighted shift operator with increasing weight sequence, a bilateral weighted shift operator with increasing weight sequence and a quasinormal operator), their Aluthge transformations are also ∞-hyponormal. The common properties of these ∞-hyponormal operators T are that T is hyponormal and $|T||T^*| = |T^*||T|$. (In [2], [3] and [6], operators satisfying $|T||T^*| = |T^*||T|$ were investigated and many interesting results were obtained. Especially Fujii and Nakatsu[6] noted that a hyponormal operator T satisfying $|T||T^*| = |T^*||T|$ is ∞-hyponormal.)

In section 3, we first show that the Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ and more generally the generalized Aluthge transformation $\tilde{T}_{s,t} = |T|^s U |T|^t$ of an ∞-hyponormal operator T are also ∞-hyponormal provided $|T||T^*| = |T^*||T|$. Next we show that the equality $|T|\,|T^*| = |T^*|\,|T|$ is not a necessary condition for the Aluthge transformation $\tilde{T}$ of an ∞-hyponormal operator T to be ∞-hyponormal.

2 Example of an ∞-hyponormal operator whose Aluthge transformation is not ∞-hyponormal We show that the Aluthge transformations of ∞-hyponormal operators are not necessarily ∞-hyponormal by giving an example. To do so, we need the following lemma, where 2×2 matrices are considered as bounded linear operators on the Hilbert space $\mathbf{C}^2$.

Lemma 2.1 *Let A and B are positive 2×2 matrices and let p_1, p_2 $(p_1 \leq p_2)$ and q_1, q_2 $(q_1 \leq q_2)$ be eigenvalues of A and B, respectively.*

Then $A^n \geq B^n$ holds for all $n \in \mathbf{N}$ if and only if the following assertion (i) or (ii) holds.

(i) $q_1 \leq q_2 \leq p_1 \leq p_2$

(ii) $q_1 \leq p_1 < q_2 \leq p_2$ and $\mathcal{N}(B - q_1) = \mathcal{N}(A - p_1)$ hold, where $\mathcal{N}(T)$ is the null space of T.

To prove this lemma, we use the following theorem[13]. For a proof, see [11], [13].

Theorem C [13]. *Let A, B be positive operators on a Hilbert space with spectral resolutions $A = \int \lambda dP(\lambda)$, $B = \int \lambda dQ(\lambda)$, respectively. Then $A^n \geq B^n$ holds for every $n \in \mathbf{N}$ if and only if $P(\lambda) \leq Q(\lambda)$ holds for every $\lambda \geq 0$.*

Now we give a proof of Lemma 2.1.

Proof of Lemma 2.1. Suppose that $A^n \geq B^n$ holds for every $n \in \mathbf{N}$. First we show that $q_1 \leq p_1$ and $q_2 \leq p_2$. Suppose that $q_1 > p_1$ or $q_2 > p_2$ holds. Then $Q(p_1) = 0 < P(p_1)$ or $Q(p_2) < I = P(p_2)$, where $A = \int \lambda dP(\lambda)$, $B = \int \lambda dQ(\lambda)$ are spectral resolutions, respectively. ($Q(\lambda) < P(\lambda)$ means $Q(\lambda) \leq P(\lambda)$ and $Q(\lambda) \neq P(\lambda)$.) This contradicts Theorem C. Hence $q_1 \leq p_1$ and $q_2 \leq p_2$ hold.

If $q_2 \le p_1$, then (i) $q_1 \le q_2 \le p_1 \le p_2$ holds. If $q_2 > p_1$, then $q_1 \le p_1 < q_2 \le p_2$ and so $\mathcal{N}(B - q_1) = \mathcal{R}(Q(q_1)) = \mathcal{R}(Q(p_1)) \supseteq \mathcal{R}(P(p_1)) = \mathcal{N}(A - p_1)$ by Theorem C. ($\mathcal{R}(T)$ denotes the range of T.) Since $\mathcal{N}(B - q_1)$ and $\mathcal{N}(A - p_1)$ are not equal to (0) nor $\mathbf{C}^2$, we obtain $\mathcal{N}(B - q_1) = \mathcal{N}(A - p_1)$. Hence (ii) holds.

Conversely, suppose that (i) holds. Then $P(\lambda) = 0 \le Q(\lambda)$ $(0 \le \lambda < p_1)$, $P(\lambda) \le I = Q(\lambda)$ $(q_2 \le \lambda)$. Hence $P(\lambda) \le Q(\lambda)$ for every $\lambda \ge 0$. By Theorem C, $A^n \ge B^n$ holds for every $n \in \mathbf{N}$.

Next suppose that (ii) holds. Since $\mathcal{R}(Q(q_1)) = \mathcal{N}(B - q_1) = \mathcal{N}(A - p_1) = \mathcal{R}(P(p_1))$ holds, we obtain $P(\lambda) = 0 \le Q(\lambda)$ $(\lambda < p_1)$, $P(\lambda) = Q(\lambda)$ $(p_1 \le \lambda < q_2)$ and $P(\lambda) \le I = Q(\lambda)$ $(q_2 \le \lambda)$. Hence $P(\lambda) \le Q(\lambda)$ for every $\lambda \ge 0$. By Theorem C, $A^n \ge B^n$ holds for every $n \in \mathbf{N}$. $\square$

Remark. If the assertion (i) holds in the above Lemma 2.1, then $A \ge p_1 I \ge q_2 I \ge B$ holds. From this inequality, we can easily obtain the inequality $A^n \ge p_1^n I \ge q_2^n I \ge B^n$ for every $n \in \mathbf{N}$ without using Theorem C.

Now we give an example of an ∞-hyponormal operator for which the Aluthge transformation is not ∞-hyponormal.

Example 2.2 Let

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

and set

$$T = \begin{pmatrix} 0 & & & & & \\ A_1 & 0 & & & & \\ & A_2 & 0 & & & \\ & & A_2 & 0 & & \\ & & & A_2 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

on $\bigoplus_{n=1}^{\infty} H_n$, where $H_n = \mathbf{C}^2$ for $n \in \mathbf{N}$.

First we show that T is ∞-hyponormal. A calculation yields

$$|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & A_2 & & \\ & & & A_2 & \\ & & & & \ddots \end{pmatrix},$$

$$|T^*| = (TT^*)^{\frac{1}{2}} = \begin{pmatrix} 0 & & & & \\ & A_1 & & & \\ & & A_2 & & \\ & & & A_2 & \\ & & & & \ddots \end{pmatrix}.$$

The eigenvalues of A_1 and A_2 are 0,2 and 2,4, respectively. This implies $A_1^n \le A_2^n$ for every $n \in \mathbf{N}$ because of Lemma 2.1 (i). Hence $|T|^n \ge |T^*|^n$ for any $n \in \mathbf{N}$, and so T is ∞-hyponormal.

Next we show that the Aluthge transformation $\tilde{T}$ is not ∞-hyponormal. The polar decomposition of T is $T = V|T|$, where

$$V = \begin{pmatrix} 0 & & & & \\ I & 0 & & & \\ & I & 0 & & \\ & & I & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

By a calculation, we obtain

$$\tilde{T}^*\tilde{T} = (|T|^{\frac{1}{2}}V|T|^{\frac{1}{2}})^*(|T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}) = \begin{pmatrix} A_1^{\frac{1}{2}}A_2A_1^{\frac{1}{2}} & & & & \\ & A_2^2 & & & \\ & & A_2^2 & & \\ & & & A_2^2 & \\ & & & & \ddots \end{pmatrix},$$

$$\tilde{T}\tilde{T}^* = (|T|^{\frac{1}{2}}V|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}V|T|^{\frac{1}{2}})^* = \begin{pmatrix} 0 & & & & \\ & A_2^{\frac{1}{2}}A_1A_2^{\frac{1}{2}} & & & \\ & & A_2^2 & & \\ & & & A_2^2 & \\ & & & & \ddots \end{pmatrix}.$$

Hence $\tilde{T}$ is ∞-hyponormal if and only if $A_2^{2n} \geq (A_2^{\frac{1}{2}}A_1A_2^{\frac{1}{2}})^n$ holds for any $n \in \mathbf{N}$. A calculation yields

$$A_2^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix},$$

$$A_2^{\frac{1}{2}}A_1A_2^{\frac{1}{2}} = \begin{pmatrix} 3+2\sqrt{2} & 1 \\ 1 & 3-2\sqrt{2} \end{pmatrix},$$

since

$$A_2^{\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 2-\sqrt{2} & 2+\sqrt{2} \end{pmatrix}.$$

The eigenvalues of A_2^2 and $A_1^{\frac{1}{2}}A_2A_1^{\frac{1}{2}}$ are $4, 16$ and $0, 6$, respectively. Since

$$\mathcal{N}(A_2^2 - 4) = \left\{ t\begin{pmatrix} 1 \\ -1 \end{pmatrix}; t \in \mathbf{C} \right\} \neq \mathcal{N}(A_1^{\frac{1}{2}}A_2A_1^{\frac{1}{2}}) = \left\{ t\begin{pmatrix} -1 \\ 3+2\sqrt{2} \end{pmatrix}; t \in \mathbf{C} \right\},$$

there exists an $n \in \mathbf{N}$ such that $A_2^n \not\geq A_1^{\frac{1}{2}}A_2A_1^{\frac{1}{2}}$ by Lemma 2.1 (ii). Hence $\tilde{T}$ is not ∞-hyponormal.

3 A sufficient condition for the generalized Aluthge transformations of ∞-hyponormal operators to be also ∞-hyponormal. We give a sufficient condition for the generalized Aluthge transformations of ∞-hyponormal operators to be also ∞-hyponormal.

Theorem 3.1 *Let T be an ∞-hyponormal operator with the polar decomposition $T = U|T|$. If $|T||T^*| = |T||T^*|$ holds, then $\tilde{T}_{s,t} = |T|^sU|T|^t$ is ∞-hyponormal for every $s, t \geq 0$.*

Proof. Let $T = U|T|$ be the polar decomposition of T, and let $\tilde{T}_{s,t} = |T|^sU|T|^t$ be the generalized Aluthge transformation of T. If $\tilde{T}_{s,t} = |T|^sU|T|^t$ is ∞-hyponormal for any $s,t > 0$, then $|\tilde{T}_{\frac{1}{k},t}|^p \geq |\tilde{T}^*_{\frac{1}{k},t}|^p$, $|\tilde{T}_{s,\frac{1}{k}}|^p \geq |\tilde{T}^*_{s,\frac{1}{k}}|^p$ and $|\tilde{T}_{\frac{1}{k},\frac{1}{k}}|^p \geq |\tilde{T}^*_{\frac{1}{k},\frac{1}{k}}|^p$ for $k \in \mathbf{N}$, $p \geq 0$. By taking the limit as $k \to \infty$, we see that $\tilde{T}_{s,t} = |T|^sU|T|^t$ is also ∞-hyponormal for $s = 0$ or $t = 0$. Hence we may assume that $s,t > 0$.

It is known that U^*U and UU^* are the orthogonal projection onto $\overline{\mathcal{R}(|T|)} = \overline{\mathcal{R}(|T|^p)} = \mathcal{N}(|T|^p)^\perp$ and $\overline{\mathcal{R}(T)} = \mathcal{N}(T^*)^\perp = \mathcal{N}(|T^*|)^\perp = \mathcal{N}(|T^*|^p)^\perp$, respectively, and so $|T|^pU^*U = |T|^p$, $|T^*|^pUU^* = |T^*|^p$ and $U^*U|T|^p = |T|$, $UU^*|T^*|^p = |T^*|^p$ hold for any $p > 0$. ($\overline{\mathcal{R}(T)}$ denotes the closure of the range of T.) Moreover it is also known that $|T^*|^p = U|T|^pU^*$ and $U^*|T^*|^pU = |T|^p$ hold for any $p > 0$. Hence

$$\begin{aligned} |\tilde{T}_{s,t}|^2 &= \tilde{T}^*_{s,t}\tilde{T}_{s,t} = (|T|^tU^*|T|^s)(|T|^sU|T|^t) = (U^*U)|T|^tU^*|T|^{2s}U|T|^t(U^*U) \\ &= U^*(U|T|^tU^*)|T|^{2s}(U|T|^tU^*)U = U^*|T^*|^t|T|^{2s}|T^*|^tU \end{aligned}$$

and

$$|\tilde{T}^*_{s,t}|^2 = \tilde{T}_{s,t}\tilde{T}^*_{s,t} = (|T|^sU|T|^t)(|T|^tU^*|T|^s) = |T|^s(U|T|^{2t}U^*)|T|^s = |T|^s|T^*|^{2t}|T|^s$$

hold. From the commutativity of $|T|$ and $|T^*|$, we obtain

$$\begin{aligned} |\tilde{T}_{s,t}|^{2n} &= U^*|T^*|^{nt}|T|^{2ns}|T^*|^{nt}U = U^*(|T^*|^{nt}UU^*)|T|^{2ns}(UU^*|T^*|^{nt})U \\ &= (U^*|T^*|^{nt}U)U^*|T|^{2ns}U(U^*|T^*|^{nt}U) = |T|^{nt}U^*|T|^{2ns}U|T|^{nt} \end{aligned}$$

and

$$|\tilde{T}^*_{s,t}|^{2n} = |T|^{ns}|T^*|^{2nt}|T|^{ns}$$

for any $n \in \mathbf{N}$. Hence $\tilde{T}_{s,t}$ is ∞-hyponormal if and only if

$$|T|^{nt}U^*|T|^{2ns}U|T|^{nt} \geq |T|^{ns}|T^*|^{2nt}|T|^{ns}$$

holds for any $n \in \mathbf{N}$. From the ∞-hyponormality of T, $|T|^p \geq |T^*|^p$ holds, and so $|T|^p \geq U|T|^pU^*$ holds for any $p > 0$. So we obtain $U^*|T|^pU \geq |T|^p$ for any $p > 0$. Hence

$$|T|^{nt}U^*|T|^{2ns}U|T|^{nt} \geq |T|^{nt}|T|^{2ns}|T|^{nt} = |T|^{ns}|T|^{2nt}|T|^{ns} \geq |T|^{ns}|T^*|^{2nt}|T|^{ns}$$

holds for any $n \in \mathbf{N}$. Hence $\tilde{T}_{s,t}$ is ∞-hyponormal. $\square$

Next we give an example of an ∞-hyponormal operator T whose Aluthge transformation $\tilde{T}$ is also ∞-hyponormal but $|T||T^*| \neq |T^*||T|$.

In [11, p. 365] the authors constructed an ∞-hyponormal operator T satisfying $|T||T^*| \neq |T^*||T|$. To examine the ∞-hyponormality of the Aluthge transformation of T, we need to reconstruct T into some operator matrix which is unitarily equivalent to T. The operator T in the following Example 3.2 is such an operator matrix.

Example 3.2 Let $\{\lambda_n\}_{n=0}^\infty$ be a bounded increasing sequence of positive numbers with $\lambda_0 = \lambda_1$ and $\lambda_k < \lambda_{k+1}$ for every $k \geq 1$, and let

$$U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_{n+1} \end{pmatrix}$$

for $n \geq 0$ and set

$$U = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ \ddots & 0 & & & & & & \\ & U_0 & 0 & & & & & \\ & & U_0 & 0 & & & & \\ \hline & & & U_0 & 0 & & & \\ & & & & U_0 & 0 & & \\ & & & & & U_0 & 0 & \ddots \\ & & & & & & \ddots & \ddots \end{array}\right),$$

$$A = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ & A_0 & & & & & & \\ & & A_0 & & & & & \\ & & & A_0 & & & & \\ \hline & & & & A_1 & & & \\ & & & & & A_2 & & \\ & & & & & & A_3 & \\ & & & & & & & \ddots \end{array}\right).$$

Matrices U, A are considered as bounded operators on $\bigoplus_{n=-\infty}^{\infty} H_n$, where $H_n = \mathbf{C}^2$ for every $n \in \mathbf{N}$.

Consider the operator $T = UA$. Then $T = UA$ is the polar decomposition of T since U is unitary and hence $T^*T = AU^*UA = A^2$. Note that $|T| = A$, $|T^*| = (TT^*)^{\frac{1}{2}} = (UA^2U^*)^{\frac{1}{2}} = UAU^*$ hold. By a calculation, we obtain

$$|T^*| = UAU^* = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ & B_0 & & & & & & \\ & & B_0 & & & & & \\ & & & B_0 & & & & \\ \hline & & & & B_0 & & & \\ & & & & & B_1 & & \\ & & & & & & B_2 & \\ & & & & & & & \ddots \end{array}\right),$$

where

$$B_n = U_0 A_n U_0^* = \frac{1}{2}\begin{pmatrix} \lambda_n + \lambda_{n+1} & \lambda_n - \lambda_{n+1} \\ \lambda_n - \lambda_{n+1} & \lambda_n + \lambda_{n+1} \end{pmatrix}.$$

(Note that $B_0 = A_0$.)

First we show that T is ∞-hyponormal. A simple calculation yields

$$A_n^k = \begin{pmatrix} \lambda_n^k & 0 \\ 0 & \lambda_{n+1}^k \end{pmatrix},$$

$$B_n^k = U_0 A_n^k U_0^* = \frac{1}{2}\begin{pmatrix} \lambda_n^k + \lambda_{n+1}^k & \lambda_n^k - \lambda_{n+1}^k \\ \lambda_n^k - \lambda_{n+1}^k & \lambda_n^k + \lambda_{n+1}^k \end{pmatrix},$$

and so

$$A_{n+1}^k - B_n^k = \frac{1}{2}\begin{pmatrix} \lambda_{n+1}^k - \lambda_n^k & \lambda_{n+1}^k - \lambda_n^k \\ \lambda_{n+1}^k - \lambda_n^k & 2\lambda_{n+2}^k - \lambda_{n+1}^k - \lambda_n^k \end{pmatrix} \geq 0$$

for $n, k \geq 0$. The last assertion follows from $\mathrm{tr}(A_{n+1}^k - B_n^k) = (\lambda_{n+2}^k - \lambda_n^k) \geq 0$ and $\det(A_{n+1}^k - B_n^k) = \frac{1}{2}(\lambda_{n+2}^k - \lambda_{n+1}^k)(\lambda_{n+1}^k - \lambda_n^k) \geq 0$, where $\mathrm{tr}(A_{n+1}^k - B_n^k)$ and $\det(A_{n+1}^k - B_n^k)$ are the trace of $(A_{n+1}^k - B_n^k)$ and the determinant of $(A_{n+1}^k - B_n^k)$, respectively. Therefore $|T|^k \geq |T^*|^k$ holds for any $k \geq 0$, and hence T is ∞-hyponormal.

Next we show that $|T||T^*| \neq |T^*||T|$. By a calculation, we obtain

$$A_{n+1}B_n = \frac{1}{2}\begin{pmatrix} \lambda_{n+1}(\lambda_n + \lambda_{n+1}) & \lambda_{n+1}(\lambda_n - \lambda_{n+1}) \\ \lambda_{n+2}(\lambda_n - \lambda_{n+1}) & \lambda_{n+2}(\lambda_n + \lambda_{n+1}) \end{pmatrix},$$

$$B_nA_{n+1} = \frac{1}{2}\begin{pmatrix} \lambda_{n+1}(\lambda_n + \lambda_{n+1}) & \lambda_{n+2}(\lambda_n - \lambda_{n+1}) \\ \lambda_{n+1}(\lambda_n - \lambda_{n+1}) & \lambda_{n+2}(\lambda_n + \lambda_{n+1}) \end{pmatrix}.$$

From the assumption, $\lambda_{n+1} \neq \lambda_{n+2}$, $\lambda_{n+1} - \lambda_n > 0$ and so $B_nA_{n+1} \neq A_{n+1}B_n$ hold for every $n \geq 1$. Hence $|T||T^*| \neq |T^*||T|$ holds.

Finally we show that the Aluthge transformation $\tilde{T}$ of T is also ∞-hyponormal. By a calculation, we obtain the following equalities for $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = A^{\frac{1}{2}}UA^{\frac{1}{2}}$.

$$\tilde{T}^*\tilde{T} = A^{\frac{1}{2}}U^*AUA^{\frac{1}{2}} = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ & A_0^2 & & & & & & \\ & & A_0^2 & & & & & \\ & & & C_0 & & & & \\ \hline & & & & C_1 & & & \\ & & & & & C_2 & & \\ & & & & & & C_3 & \\ & & & & & & & \ddots \end{array}\right),$$

$$\tilde{T}\tilde{T}^* = A^{\frac{1}{2}}UAU^*A^{\frac{1}{2}} = A^{\frac{1}{2}}|T^*|A^{\frac{1}{2}} = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ & A_0^2 & & & & & & \\ & & A_0^2 & & & & & \\ & & & D_0 & & & & \\ \hline & & & & D_1 & & & \\ & & & & & D_2 & & \\ & & & & & & D_3 & \\ & & & & & & & \ddots \end{array}\right),$$

where $C_n = A_n^{\frac{1}{2}}U_0^*A_{n+1}U_0A_n^{\frac{1}{2}}$, $D_n = A_n^{\frac{1}{2}}B_{n-1}A_n^{\frac{1}{2}}$ for $n \geq 0$ under the convention that $B_{-1} := B_0 = A_0$. Hence $\tilde{T}$ is ∞-hyponormal if and only if $C_n^k \geq D_n^k$ for every $n, k \geq 0$. Let $p_1^{(n)}, p_2^{(n)}$ $(p_1^{(n)} \leq p_2^{(n)})$ be eigenvalues of $2C_n$, and let $q_1^{(n)}, q_2^{(n)}$ $(q_1^{(n)} \leq q_2^{(n)})$ be eigenvalues of $2D_n$. If $q_2^{(n)} \leq p_1^{(n)}$ holds for any $n \in \mathbf{N}$, then $(2D_n)^k \leq (2C_n)^k$ and so $D_n^k \leq C_n^k$ for every $k \geq 0$ by Lemma 2.1, and hence $\tilde{T}$ is ∞-hyponormal. Therefore it suffices to show that $q_2^{(n)} \leq p_1^{(n)}$ holds for every $n \geq 0$.

By a calculation,

$$C_n = \frac{1}{2}\begin{pmatrix} \lambda_n(\lambda_{n+1} + \lambda_{n+2}) & \sqrt{\lambda_n\lambda_{n+1}}(\lambda_{n+1} - \lambda_{n+2}) \\ \sqrt{\lambda_n\lambda_{n+1}}(\lambda_{n+1} - \lambda_{n+2}) & \lambda_{n+1}(\lambda_{n+1} + \lambda_{n+2}) \end{pmatrix},$$

$$D_0 = A_0^{\frac{1}{2}} A_0 A_0^{\frac{1}{2}} = A_0^2,$$
$$D_m = \frac{1}{2} \begin{pmatrix} \lambda_m(\lambda_{m-1} + \lambda_m) & \sqrt{\lambda_m \lambda_{m+1}}(\lambda_{m-1} - \lambda_m) \\ \sqrt{\lambda_m \lambda_{m+1}}(\lambda_{m-1} - \lambda_m) & \lambda_{m+1}(\lambda_{m-1} + \lambda_m) \end{pmatrix}$$

for $n \geq 0, m \geq 1$. Solving the characteristic equation of matrices $2C_0$, $2D_0$, we obtain $q_1^{(0)} = q_2^{(0)} = p_1^{(0)} = 2\lambda_1^2 \leq p_2^{(0)} = 2\lambda_1\lambda_2$. Next let us fix an arbitrary n with $n \geq 1$. Then

$$\mathrm{tr}(2C_n) = (\lambda_n + \lambda_{n+1})(\lambda_{n+1} + \lambda_{n+2}), \quad \det(2C_n) = 4\lambda_n \lambda_{n+1}^2 \lambda_{n+2},$$
$$\mathrm{tr}(2D_n) = (\lambda_n + \lambda_{n+1})(\lambda_{n-1} + \lambda_n), \quad \det(2D_n) = 4\lambda_{n-1}\lambda_n^2 \lambda_{n+1}$$

and

$$p_1^{(n)} = \frac{\mathrm{tr}(2C_n) - \sqrt{\mathrm{tr}(2C_n)^2 - 4\det(2C_n)}}{2},$$
$$q_2^{(n)} = \frac{\mathrm{tr}(2D_n) + \sqrt{\mathrm{tr}(2D_n)^2 - 4\det(2D_n)}}{2}.$$

From these expressions and the fact that $\mathrm{tr}(2C_n) \geq \mathrm{tr}(2D_n)$, we can see that $q_2^{(n)} \leq p_1^{(n)}$ is equivalent to

$$\left(\sqrt{\mathrm{tr}(2D_n)^2 - 4\det(2D_n)} + \sqrt{\mathrm{tr}(2C_n)^2 - 4\det(2C_n)}\right)^2 \leq (\mathrm{tr}(2C_n) - \mathrm{tr}(2D_n))^2.$$

So $q_2^{(n)} \leq p_1^{(n)}$ holds if and only if

$$\begin{aligned} (1) \qquad & 2\sqrt{\mathrm{tr}(2C_n)^2 - 4\det(2C_n)}\sqrt{\mathrm{tr}(2D_n)^2 - 4\det(2D_n)} \\ \leq \quad & (\mathrm{tr}(2C_n) - \mathrm{tr}(2D_n))^2 - \left(\mathrm{tr}(2C_n)^2 - 4\det(2C_n)\right) - \left(\mathrm{tr}(2D_n)^2 - 4\det(2D_n)\right). \end{aligned}$$

Hereafter, we denote the left-hand side and the right-hand side of (1) by (lhs) and (rhs), respectively. Moreover we introduce new variables a, x, y and z by setting

$$\lambda_{n-1} = a, \quad \lambda_n = a + x, \quad \lambda_{n+1} = a + x + y, \quad \lambda_{n+2} = a + x + y + z.$$

Since $\{\lambda_n\}_{n=0}^{\infty}$ is an increasing sequence of positive numbers, $a, x, y, z \geq 0$. By using new variables, we can rewrite (rhs) as

(rhs) $= 16a^2xy + 32ax^2y + 16x^3y + 8a^2y^2 + 36axy^2 + 28x^2y^2 + 8ay^3 + 12xy^3 + 8a^2xz + 16ax^2z + 8x^3z + 16a^2yz + 40axyz + 24x^2yz + 12ay^2z + 14xy^2z$,

which shows that (rhs) ≥ 0. Therefore $q_2^{(n)} \leq p_1^{(n)}$ holds if and only if $(\mathrm{lhs})^2 \leq (\mathrm{rhs})^2$. By using new variables, we can write

$(\mathrm{rhs})^2 - (\mathrm{lhs})^2 = 192a^4x^2y^2 + 768a^3x^3y^2 + 1152a^2x^4y^2 + 768ax^5y^2 + 192x^6y^2 + 256a^4xy^3 + 1472a^3x^2y^3 + 2880a^2x^3y^3 + 2368ax^4y^3 + 704x^5y^3 + 640a^3xy^4 + 2240a^2x^2y^4 + 2560ax^3y^4 + 960x^4y^4 + 512a^2xy^5 + 1088ax^2y^5 + 576x^3y^5 + 128axy^6 + 128x^2y^6 + 256a^4x^2yz + 1024a^3x^3yz + 1536a^2x^4yz + 1024ax^5yz + 256x^6yz + 640a^4xy^2z + 3072a^3x^2y^2z + 5376a^2x^3y^2z + 4096ax^4y^2z + 1152x^5y^2z + 256a^4y^3z + 2304a^3xy^3z + 5696a^2x^2y^3z + 5504ax^3y^3z + 1856x^4y^3z + 384a^3y^4z + 1984a^2xy^4z + 2880ax^2y^4z + 1280x^3y^4z + 128a^2y^5z + 448axy^5z + 320x^2y^5z + 256a^4xyz^2 + 1024a^3x^2yz^2 + 1536a^2x^3yz^2 + 1024ax^4yz^2 + 256x^5yz^2 + 192a^4y^2z^2 + 1280a^3xy^2z^2 + 2688a^2x^2y^2z^2 + 2304ax^3y^2z^2 + 704x^4y^2z^2 + 320a^3y^3z^2 + 1280a^2xy^3z^2 + 1600ax^2y^3z^2 + 640x^3y^3z^2 + 128a^2y^4z^2 + 320axy^4z^2 + 192x^2y^4z^2$.

Hence the inequality $(\mathrm{lhs})^2 \leq (\mathrm{rhs})^2$ holds and hence $q_2^{(n)} \leq p_1^{(n)}$ holds for every $n \geq 0$ and so $\tilde{T}$ is ∞-hyponormal.

From the above Example 3.2, we see that the equality $|T||T^*| = |T^*||T|$ is not necessary for the Aluthge transformation $\tilde{T}$ of an ∞-hyponormal operator T to be ∞-hyponormal. Hence there remains the problem of finding a necessary and sufficient condition for the Aluthge transformation of an ∞-hyponormal operator to be also ∞-hyponormal.

References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13** (1990), 307–315.

[2] S. L. Campbell, *Linear operators for which* T^*T *and* TT^* *commute*, Proc. Amer. Math. Soc., **34** (1972), 177–180.

[3] S. L. Campbell, *Linear operators for which* T^*T *and* TT^* *commute (II)* , Pacific J. Math., **53** (1974), 355–361.

[4] M. Cho and M. Itoh, *Putnam's inequality for p-hyponormal operators*, Proc. Amer. Math. Soc., **123** (1995), 2435–2440.

[5] B. P. Duggal, *On p-hyponormal contractions*, Proc. Amer. Math. Soc., **123** (1995), 81–86.

[6] M. Fujii and Y. Nakatsu, *On subclasses of hyponormal operators*, Proc. Japan Acad., **51** (1975), 243–246.

[7] T. Furuta, *Generalized Aluthge transformation on p-hyponormal operators*, Proc. Amer. Math. Soc., **124** (1996), 3071–3075.

[8] T. Furuta and M. Yanagida, *Further extensions of Aluthge transformation on p-hyponormal operators*, Integral Equations Operator Theory, **29** (1997), 122–125.

[9] T. Huruya, *A note on p-hyponormal operators*, Proc. Amer. Math. Soc., **125** (1997), 3617–3624.

[10] C.-S. Lin, *Unifying approach to the study of p-hyponormal operators*, Math. Inequal. Appl., **2** (1999), 579–584.

[11] S. Miyajima and I. Saito, *∞-hyponormal operators and their spectral properties*, Acta Sci. Math. (Szeged), **67** (2001), 357–371.

[12] S. Miyajima and I. Saito, *A remark on the spectra of ∞-hyponormal operators*, Acta Sci. Math. (Szeged), **69** (2003), 863–869.

[13] M. H. Olson, *The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice*, Proc. Amer. Math. Soc., **188** (1974), 419–428.

[14] T. Yoshino, *The p-hyponormality of Aluthge transform*, Interdiscip. Inform. Sci., **3** (1997), 91–93.

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