DUAL POSITIVE IMPLICATIVE HYPER K-IDEALS OF TYPE 4

L. Torkzadeh and M.M Zahedi

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Abstract.In this note first we define the notion of *dual positive implicative hyper K-ideal of type 4*, where for simplicity is written by $DPIHKI-T4$. Then we give some related results. Finally we determine all of hyper K-algebras of order 3, which have $D_1 = \{1\}$, $D_2 = \{1, 2\}$ or $D_3 = \{0, 1\}$ as a $DPIHKI-T4$.

1 Introduction The hyperalgebraic structure theory was introduced by F. Marty [6] in 1934. Imai and Iseki [6] in 1966 introduced the notion of a BCK-algebra. Borzooei, Jun and Zahedi et.al. [2,3,9] applied the hyperstructure to BCK-algebras and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. In [1], the authors have defined 8 types of positive implicative hyper K-ideals. Recently in [8] we introduced the notion of dual positive implicative hyper K-ideal of type 3 and then we characterized them. Now in this note first we define the notion of dual positive implicative hyper K-ideal of type 4, then we obtain some related results which have been mentioned in the abstract. We will define and study the other types of dual positive implicative hyper K-idaels in the next papers.

2 Preliminaries

Definition 2.1. [2] Let H be a nonempty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$. Then H is called a hyper K-algebra if it contains a constant "0" and satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y,\ y < x \Rightarrow x = y$
(HK5) $0 < x$,
for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{\substack{a\in A\\ b\in B}} a \circ b$ of H.

Theorem 2.2. [2] Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H the following hold:

(i) $x \circ y < z \Leftrightarrow x \circ z < y$, (ii) $(x \circ z) \circ (x \circ y) < y \circ z$,
(iii) $x \circ (x \circ y) < y$, (iv) $x \circ y < x$,
(v) $A \subseteq B$ implies $A < B$, (vi) $x \in x \circ 0$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$, (viii) $(A \circ C) \circ (B \circ C) < A \circ B$,
(ix) $A \circ B < C \Leftrightarrow A \circ C < B$, (x) $A \circ B < A$.

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Definition 2.3. [2] Let $(H, \circ, 0)$ be a hyper K-algebra. If there exists an element $1 \in H$ such that $x < 1$ for all $x \in H$, then H is called a bounded hyper K-algebra and 1 is said to be the unit of H.
In a bounded hyper K-algebra, we denote $1 \circ x$ by Nx.

Definition 2.4. [8] Let H be a bounded hyper K-algebra. Then a non-empty subset D of H is called *a dual positive implicative hyper K-ideal type 3 (DPIHKI-T3)* if it satisfies:
(i)$1 \in D$
(ii)$N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$ imply $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.5. [8] Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D = \{0, 1\}$ in H. Then D is a $DPIHKI - T3$ if and only if $2 \notin 1 \circ 2$ and $2 \notin 1 \circ 1$.

3 Dual positive implicative hyper K-ideals of type 4

From now on H is a bounded hyper K-algebra with unit 1.

Definition 3.1. A non-empty subset D of H is called *a dual positive implicative hyper K-ideal type* 4 *($DPIHKI - T4$)* if it satisfies:
(i)$1 \in D$
(ii) $N((Nx \circ Ny) \circ Nz) \subseteq D$ and $N(Ny \circ Nz) < D$ imply that $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Example 3.2. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

Furthermore $D_1 = \{1\}$, $D_2 = \{1, 2\}$ and $D_3 = \{0, 1\}$ are $DPIHKI - T4$.

Example 3.3. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

Also $D_1 = \{1\}$ and $D_3 = \{0, 1\}$ are $DPIHKI-T4$, but $D_2 = \{1, 2\}$ is not a $DPIHKI-T4$.

From now on we let D is a non-empty subset of a bounded hyper K-algebra H with unit 1, and $1 \in D$.

Theorem 3.4. A non-empty subset D of H is a $DPIHKI - T4$ if and only if
$N((Nx \circ Ny) \circ Nz) \subseteq D$ implies that $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Proof. Let D be a $DPIHKI-T4$ and $N((Nx\circ Ny)\circ Nz)\subseteq D$. Then by Definition 2.1 and Theorem 2.2(x) we conclude that $N(Ny\circ Nz)<D$; $\forall y,z\in H$. So by hypothesis we get that $N(Nx\circ Nz)\subseteq D$, $\forall x,z\in H$.
The proof of the converse is trivial.

Theorem 3.5. If D is a $DPIHKI-T3$ of H, then D is a $DPIHKI-T4$.
Proof. Straightforward.

Theorem 3.6. Let $D\subseteq H$ and $0\notin D$. If $1\in 1\circ x$; $\forall x\in H$, then D is a $DPIHKI-T4$.

Proof. By hypothesis we get that $1\in 1\circ 1\subseteq (1\circ 1)\circ 1\subseteq ((1\circ x)\circ(1\circ y))\circ(1\circ z)$; $\forall x,y,z\in H$. So $0\in 1\circ 1\subseteq 1\circ(((1\circ x)\circ(1\circ y))\circ(1\circ z))$, $\forall x,y,z\in H$. Since $0\notin D$, so $N((Nx\circ Ny)\circ Nz)\nsubseteq D$,$\forall x,y,z\in H$. Therefore by Theorem 3.4. we conclude that D is a $DPIHKI-T4$.

Note that $D_1=\{1\}$ and $D_2=\{1,2\}$ in Example 3.2 satisfy the conditions of Theorem 3.6.

Theorem 3.7. In H we have $1\circ 0=\{1\}$.

Proof. By Theorem 2.2(vi) we have $1\in 1\circ 0$, now we prove that $1\circ 0=\{1\}$. On the contrary let $1\circ 0\neq\{1\}$. Then there exists $1\neq x\in 1\circ 0$. By $(HK2)$ we have $0\in x\circ x\subseteq(1\circ 0)\circ x=(1\circ x)\circ 0$. So there exists $t\in 1\circ x$ such that $0\in t\circ 0$, hence $t<0$. By $(HK4)$ and $(HK5)$ we get that $t=0$. Thus $0\in 1\circ x$, so $1<x$. Since 1 is the unit of H, by $(HK4)$ we conclude that $x=1$, which is a contradiction.

Theorem 3.8. Let $1\circ 1=\{0\}$. If $0\notin D$, then D is not a $DPIHKI-T4$.

Proof. By $(HK2)$, Theorem 3.7 and hypothesis we have $0\circ 0=(1\circ 1)\circ 0=(1\circ 0)\circ 1=1\circ 1=0$. So we get that $1=1\circ 0=1\circ(0\circ 0)=1\circ((1\circ 1)\circ(1\circ 1))=1\circ(((1\circ 0)\circ(1\circ 0))\circ(1\circ 1))\subseteq D$, that is $N((N0\circ N0)\circ N1)\subseteq D$. Also $0=1\circ 1=1\circ(1\circ 0)=1\circ((1\circ 0)\circ(1\circ 1))$. Now since $0\notin D$, we have $N(N0\circ N1)\nsubseteq D$. Thus D is not a $DPIHKI-T4$.

Example 3.9. Let $H=\{0,1,2\}$.Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

And $D_1=\{1\}$ and $D_2=\{1,2\}$ are not $DPIHKI-T4$, by Theorem 3.8.

Theorem 3.10. Let $NNx=x$, $\forall x\in H$ and $D\subseteq H$. If $0\notin D$, then D is not a $DPIHKI-T4$.

Proof. We prove that $1\circ 1=\{0\}$. On the contrary let $1\circ 1\neq\{0\}$. Then there exists $0\neq x\in 1\circ 1$. By hypothesis we get that $1\circ x\subseteq 1\circ(1\circ 1)=NN1=1$, so $1\circ x=1$. Since $NNx=x$, hence $x=1\circ(1\circ x)=1\circ 1$. Therefore $0\in 1\circ 1=x$, that is $x=0$ which is a contradiction. Thus $1\circ 1=\{0\}$. So by Theorem 3.8 we conclude that D is not a $DPIHKI - T4$.

Example 3.11. Let $H = \{0, 1, 2\}$.Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

Then we can see that $NNx = x$. Also by Theorem 3.10 $D_1 = \{1\}$ and $D_2 = \{1, 2\}$ are not $DPIHKI - T4$.

Lemma 3.12. Let $NNx = x$. Then $1 \circ 1 = \{0\}$ and $0 \circ 0 = \{0\}$.

Proof. See the proofs of Theorems 3.10 and 3.8.

Theorem 3.13. Let $1 \neq x \in H$ and $x \notin D$ or $0 \notin D$. If $1 \circ 1 = \{0, x\}$ and $1 \circ x = \{1\}$, then D is not a $DPIHKI - T4$.

Proof. By $(HK2)$ we have $(1 \circ 1) \circ x = (1 \circ x) \circ 1 = 1 \circ 1 = \{0, x\}$ and $(1 \circ 1) \circ 0 = (1 \circ 0) \circ 1 = 1 \circ 1 = \{0, x\}$. So by hypothesis we get that $0 \circ x$, $x \circ x$ and $0 \circ 0 \subseteq \{0, x\}$, also $x \circ 0 = \{x\}$. Hence we have $1 \circ (((1 \circ 0) \circ (1 \circ 0)) \circ (1 \circ 1)) \subseteq 1 \circ \{0, x\} = \{1\} \subseteq D$ and $\{0, x\} = 1 \circ 1 \subseteq 1 \circ ((1 \circ 0) \circ (1 \circ 1))$. Since 0 or $x \notin D$, hence $N(N0 \circ N1) \nsubseteq D$. Therefore D is not a $DPIHKI - T4$.

Example 3.14. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0,2\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

And $D_1 = \{1\}$, $D_2 = \{1, 2\}$ and $D_3 = \{0, 1\}$ are not $DPIHKI - T4$, by Theorem 3.13.

4 Dual positive implicative hyper K-algebras of type 4 of order 3

In the sequel we let $H = \{0, 1, 2\}$ be a bounded hyper K-algebra of order 3 with unit 1 and $D_1 = \{1\}$, $D_2 = \{1, 2\}$ and $D_3 = \{0, 1\}$ be subsets of H.

Lemma 4.1. Let $NNx = x$; $\forall x \in H$. Then D_3 is a $DPIHKI - T4$ if and only if $2 \in (2 \circ 2) \bigcap (2 \circ 1)$.

Proof. Theorem 3.7 and Lemma 3.12 imply that $1 \circ 0 = \{1\}$ and $1 \circ 1 = \{0\}$. Now by a simple argument we get that $1 \circ 2 = \{2\}$. By $(HK2)$ we have $2 \circ 0 = (1 \circ 2) \circ 0 = (1 \circ 0) \circ 2 = 1 \circ 2 = \{2\}$. Let D_3 be a $DPIHKI - T4$ we prove that $2 \in (2 \circ 2) \bigcap (2 \circ 1)$. On the contrary let $2 \notin (2 \circ 2) \bigcap (2 \circ 1)$.

If $2 \notin 2 \circ 2$, then $1 \circ (((1 \circ 2) \circ (1 \circ 2)) \circ (1 \circ 1)) = 1 \circ ((2 \circ 2) \circ 0) \subseteq 1 \circ \{0, 1\} = \{0, 1\} = D_3$ and $2 = 1 \circ 2 = 1 \circ (2 \circ 0) = 1 \circ ((1 \circ 2) \circ (1 \circ 1))$. Since $2 \notin D_3$, we get that $N(N2 \circ N1) \nsubseteq D_3$. Thus D_3 is not a $DPIHKI - T4$, which is a contradiction. So $2 \in 2 \circ 2$.
If $2 \notin 2 \circ 1$, Then $1 \circ (((1 \circ 2) \circ (1 \circ 0)) \circ (1 \circ 1)) = 1 \circ ((2 \circ 1) \circ 0) \subseteq 1 \circ \{0, 1\} = \{0, 1\} = D_3$ and $2 = 1 \circ ((1 \circ 2) \circ (1 \circ 1))$. Since $2 \notin D_3$, we get that $N(N2 \circ N1) \nsubseteq D_3$. Therefore D_3 is not a $DPIHKI - T4$, which is a contradiction. Thus $2 \in (2 \circ 1)$.
Conversely let $2 \in (2 \circ 2) \bigcap (2 \circ 1)$ we prove that D_3 is a $DPIHKI - T4$. By hypothesis and $(HK2)$ we have $0 \circ 2 = (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 2 \circ 1$. Since $2 \in 2 \circ 1$, then $2 \in 0 \circ 2$.
Now by some manipulations we can check that :
(i) $1 \circ ((1 \circ 0) \circ (1 \circ 0))$, $1 \circ ((1 \circ 0) \circ (1 \circ 1))$ and $1 \circ ((1 \circ 1) \circ (1 \circ 1))$ are subsets of D_3.
(ii) $1 \circ ((1 \circ 0) \circ (1 \circ 2))$, $1 \circ ((1 \circ 1) \circ (1 \circ 2))$, $1 \circ ((1 \circ 2) \circ (1 \circ 0))$, $1 \circ ((1 \circ 2) \circ (1 \circ 1))$ and $1 \circ ((1 \circ 2) \circ (1 \circ 2))$ are not subsets of D_3.
So in the case of (i), by Theorem 3.4 we are done. And in the case of (ii), by some calculations we see that $N((Nx \circ Ny) \circ Nz) \nsubseteq D_3$. So in this case also the conditions of Theorem 3.4 hold.
Now consider $1 \circ ((1 \circ 1) \circ (1 \circ 0)) = 1 \circ (0 \circ 1)$. If $0 \circ 1 \subseteq \{0, 1\}$, then $1 \circ ((1 \circ 1) \circ (1 \circ 0) \subseteq D_3$ and we are done. If $2 \in 0 \circ 1$, then $1 \circ ((1 \circ 1) \circ (1 \circ 0)) \nsubseteq D_3$, since $2 \in 1 \circ ((1 \circ 1) \circ (1 \circ 0))$. So we can see that $N((N1 \circ Ny) \circ N0) \nsubseteq D_3$, for all $y \in H$. Thus D_3 is a $DPIHKI - T4$.

Lemma 4.2. Let $1 \circ 1 \subseteq \{0, 1\}$ and $1 \circ 2 = \{1\}$. Then D_3 is a $DPIHKI - T4$.

Proof. Since $1 \circ 1 \subseteq \{0, 1\}$ and $1 \circ 2 = \{1\}$, then by Theorem 2.5 D_3 is a $DPIHKI - T3$. Hence D_3 is a $DPIHKI - T4$ by Theorem 3.5.

Lemma 4.3. Let $1 \circ 1 = \{0\}$ and $1 \circ 2 = \{1, 2\}$. Then D_3 is a $DPIHKI - T4$ if and only if $2 \in (2 \circ 2) \bigcap (2 \circ 1)$.
Proof. The proof is similar to the proof of Lemma 4.1.

Theorem 4.4. Let $1 \circ 1 = \{0\}$. Then the following statements hold:
(i) D_1 and D_2 are not $DPIHKI - T4$.
(ii) If $2 \in 1 \circ 2$, then D_3 is a $DPIHKI - T4$ if and only if $2 \in (2 \circ 2) \bigcap (2 \circ 1)$.
(iii) If $1 \circ 2 = \{1\}$, then D_3 is a $DPIHKI - T4$.

Proof. (i) Follows from Theorem 3.8.
(ii) Consider two cases: $1 \circ 2 = \{2\}$ or $1 \circ 2 = \{1, 2\}$. In the first case, the proof of Lemma 4.1 shows that D_3 is a $DPIHKI - T4$ if and only if $2 \in (2 \circ 2) \bigcap (2 \circ 1)$. In the second case, the proof follows from Lemma 4.3.
(iii) The proof follows from Lemma 4.2.

Now we give some examples about the above Theorem.
Example 4.5. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

H_4	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

H_5	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

H_6	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$. Moreover:
(a) In H_1, H_2 and H_6, D_3 is a $DPIHKI-T4$, by Theorem 4.4 (ii),(iii), While D_1 and D_2 are not $DPIHKI-T4$, by Theorem 4.4 (i).
(b) In H_3, H_4 and H_5, D_1, D_2 and D_3 are not $DPIHKI-T4$, by Theorem 4.4 (i),(ii).

Lemma 4.6. Let $1\circ 1=\{0,1\}$ and $1\circ 2=\{1,2\}$. Then D_3 is a $DPIHKI-T4$.

Proof. By $(HK2)$ we have $\{1,2\}=1\circ 2\subseteq(1\circ 1)\circ 2=(1\circ 2)\circ 1=\{1,2\}\circ 1=\{0,1\}\bigcup(2\circ 1)$, thus $2\in 2\circ 1$. Since $1\in 1\circ x$, $\forall x\in H$ and $2\in 1\circ 2$, then $1\circ(((1\circ x)\circ(1\circ y))\circ(1\circ 2))\not\subseteq D_3$. We easily see that $1\circ((1\circ 0)\circ(1\circ 0))$ and $1\circ((1\circ 0)\circ(1\circ 1))$ are subsets of D_3. Since $1\in 1\circ x$, $\forall x\in H$ and $2\in 2\circ 1$, hence $1\circ(((1\circ x)\circ(1\circ 2))\circ(1\circ y))\not\subseteq D_3$. If $2\notin 0\circ 1$, then $1\circ((1\circ 1)\circ(1\circ 1))$ and $1\circ((1\circ 1)\circ(1\circ 0))$ are subsets of D_3. Let $2\in 0\circ 1$, then $N((Nx\circ Ny)\circ Nz)\not\subseteq D_3$,$\forall x,y,z\in H$. So D_3 is a $DPIHKI-T4$.

Lemma 4.7. Let $1\circ 1=\{0,1\}$ and $1\circ 2=\{2\}$. Then the following statements hold:
(i) D_1 is a $DPIHKI-T4$ if and only if $2\circ 2\neq\{0\}$.
(ii) D_2 is a $DPIHKI-T4$ if and only if $1\in 2\circ 1$.
(iii) D_3 is a $DPIHKI-T4$ if and only if $2\in 2\circ 2$.

Proof. (i) Let D_1 is a $DPIHKI-T4$ we prove that $2\circ 2\neq\{0\}$. On the contrary let $2\circ 2=\{0\}$. Then
$1\circ(((1\circ 0)\circ(1\circ 2))\circ(1\circ 2))=1\circ((1\circ 2)\circ 2)=1\circ(2\circ 2)=1\circ 0=\{1\}=D_1$ and $1\circ((1\circ 0)\circ(1\circ 2))=1\circ(1\circ 2)=1\circ 2=2$. Since $2\notin D_1$, hence D_1 is not a $DPIHKI-T4$, which is a contradiction. Thus $2\circ 2\neq\{0\}$.
Conversely let $2\circ 2\neq\{0\}$. We prove that D_1 is a $DPIHKI-T4$. By $(HK2)$ we have $\{2\}=1\circ 2\subseteq(1\circ 1)\circ 2=(1\circ 2)\circ 1=2\circ 1$, thus $2\in 2\circ 1$. From $2\in 2\circ 1$ and $2\circ 2\neq\{0\}$ and some manipulations we get that $N((Nx\circ Ny)\circ Nz)\not\subseteq D_1$; $\forall x,y,z\in H$. So there is nothing to prove, in other words D_1 is a $DPIHKI-T4$.
(ii) Let D_2 is a $DPIHKI-T4$ we prove that $1\in 2\circ 1$. On the contrary let $1\notin 2\circ 1$. Then $1\circ(((1\circ 0)\circ(1\circ 2))\circ(1\circ 0))=1\circ((1\circ 2)\circ 1)=1\circ(2\circ 1)\subseteq 1\circ\{0,2\}=\{1,2\}=D_2$ and $0\in 1\circ((1\circ 0)\circ(1\circ 0))$. Since $0\notin D_2$, then D_2 is not a $DPIHKI-T4$, which is a contradiction. Thus $1\in 2\circ 1$. Conversely let $1\in 2\circ 1$. By $(HK2)$ we have $(1\circ 1)\circ 2=(1\circ 2)\circ 1=2\circ 1$, so $(0\circ 2)\bigcup\{2\}=2\circ 1$. Hence $1\in 0\circ 2$. Also we have $1\circ((1\circ 0)\circ(1\circ 2))=\{2\}\subseteq D_2$. Now since $1\in 1\circ 1$ we can see that $N((Nx\circ Ny)\circ Nz)\not\subseteq D_2$, $\forall x,y,z\in\{0,1\}$. Since $1\in 0\circ 2$, then $N(Nx\circ Ny)\circ Nz)\not\subseteq D_2$, for all $x\in\{0,1\}$ and $y,z\in\{0,1,2\}$. If $2\circ 2\neq\{0\}$, then by hypothesis we can check that $N((N2\circ Ny)\circ Nz)\not\subseteq D_2$, $\forall y,z\in\{0,1,2\}$. If $2\circ 2=\{0\}$, then by $(HK2)$ we have $0\circ 1=(2\circ 2)\circ 1=(2\circ 1)\circ 2=\{0,1,2\}$. Hence $N((N2\circ Ny)\circ Nz)\not\subseteq D_2$;

$\forall y, z \in \{0, 1, 2\}$. Therefore D_2 is a $DPIHKI - T4$.
(iii) Let D_3 is a $DPIHKI - T4$ we prove that $2 \in 2 \circ 2$. On the contrary let $2 \notin 2 \circ 2$. Then $2 \circ 2 \subseteq \{0, 1\}$, hence $1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 2) = 1 \circ ((1 \circ 2) \circ 2) = 1 \circ (2 \circ 2) \subseteq 1 \circ \{0, 1\} = \{0, 1\} = D_3$. But $1 \circ ((1 \circ 0) \circ (1 \circ 2)) \not\subseteq D_3$, since $2 \in 1 \circ ((1 \circ 0) \circ (1 \circ 2))$ and $2 \notin D_3$, we conclude that D_3 is not a $DPIHKI - T4$, which is a contradiction. Thus $2 \in 2 \circ 2$. Conversely the proof is similar to (i) and (ii).

Theorem 4.8. Let $1 \circ 1 = \{0, 1\}$. Then the following statements hold :
(i) If $1 \in 1 \circ 2$, then D_1, D_2 and D_3 are $DPIHKI - T4$.
(ii) If $1 \circ 2 = \{2\}$, then:
(a) D_1 is a $DPIHKI - T4$ if and only if $2 \circ 2 \neq \{0\}$.
(b) D_2 is a $DPIHKI - T4$ if and only if $1 \in 2 \circ 1$.
(c) D_3 is a $DPIHKI - T4$ if and only if $2 \in 2 \circ 2$.

Proof. (i) Follows from Theorem 3.6 and Lemmas 4.2 and 4.6.
(ii) Follows from Lemma 4.7.

Now we give some examples about the above theorem.
Example 4.9. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

H_4	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

H_5	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

H_6	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0, 1, 2\}$. Moreover:
(a) In H_1, H_2 and H_3, D_1,D_2 and D_3 are $DPIHKI - T4$, by Theorem 4.8(i).
(b) In H_4, D_1 and D_2 are $DPIHKI-T4$, by Theorem 4.8(ii), while D_3 is not a $DPIHKI-T4$, by Theorem 4.8(ii).
(c) In H_5, D_1, D_3 are $DPIHKI-T4$, by Theorem 4.8(ii), while D_2 is not a $DPIHKI-T4$, by Theorem 4.8(ii).
(d) In H_6, D_1, D_2 and D_3 are not $DPIHKI - T4$, by Theorem 4.8(ii).

Theorem 4.10. Let $1 \circ 1 = \{0, 2\}$ and $1 \circ 2 = \{2\}$. Then $D_1(D_3)$ is a $DPIHKI - T4$ if and only if $2 \circ 2 \neq \{0\}$.

Proof. We give the proof of D_1, the proof of D_3 is similar to D_1. Let D_1 is a $DPIHKI-T4$ we prove that $2 \circ 2 \neq \{0\}$. On the contrary let $2 \circ 2 = \{0\}$. By hypothesis we have

$1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 2)) = 1 \circ ((1 \circ 2) \circ 2) = 1 \circ (2 \circ 2) = 1 \circ 0 = \{1\} \subseteq D_1$ and $2 = 1 \circ 2 = 1 \circ (1 \circ 2) = 1 \circ ((1 \circ 0) \circ (1 \circ 2))$. Since $2 \notin D_1$, so D_1 is not a $DPIHKI - T4$, which is a contradiction. Thus $2 \circ 2 \neq \{0\}$. Conversely let $2 \circ 2 \neq \{0\}$ we prove that D_1 is a $DPIHKI - T4$. By $(HK3)$ we have $2 \circ 1 = (1 \circ 2) \circ 1 = (1 \circ 1) \circ 2 = \{0, 2\} \circ 2 = (0 \circ 2) \bigcup (2 \circ 2)$. Since $2 \circ 2 \neq \{0\}$ implies that $2 \circ 1 \neq \{0\}$. Since $2 \in (1 \circ 2) \bigcap (1 \circ 1)$, $2 \circ 1$ and $2 \circ 2 \neq \{0\}$, then by some calculations we can get that $N((Nx \circ Ny) \circ Nz) \nsubseteq D_1$. Hence D_1 is a $DPIHKI - T4$.

Now we give some examples about the above theorem.
Example 4.11. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0, 1, 2\}$. Furthermore:
(a) In H_1, H_2, D_1 and D_3 are $DPIHKI - T4$.
(b) In H_3, D_1 and D_3 are not $DPIHKI - T4$.

Theorem 4.12. Let $1 \circ 1 = \{0, 2\}$ and $1 \circ 2 = \{2\}$. Then the following statements hold :
(i) If $2 \circ 2 \subseteq \{0, 2\}$, then D_2 is not a $DPIHKI - T4$.
(ii) If $2 \circ 2 = \{0, 1, 2\}$, then D_2 is a $DPIHKI - T4$.
(iii) If $2 \circ 2 = \{0, 1\}$, then D_2 is a $DPIHKI - 4$ if and only if $1 \in 0 \circ 2$.

Proof.(i) Let $2 \circ 2 \subseteq \{0, 2\}$. We have $1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 1)) \subseteq 1 \circ \{0, 2\} = \{1, 2\} = D_2$ and $0 \in 1 \circ 1 \subseteq 1 \circ ((1 \circ 0) \circ (1 \circ 1))$. Since $0 \notin D_2$, so D_2 is not a $DPIHKI - T4$.
(ii) Let $2 \circ 2 = \{0, 1, 2\}$. In the proof of Theorem 4.10 we obtained that $2 \circ 1 = (2 \circ 2) \bigcup (0 \circ 2)$, so $2 \circ 1 = \{0, 1, 2\}$. Since $2 \in (1 \circ 1) \bigcap (1 \circ 2)$ and $\{1, 2\} \subseteq (2 \circ 1) \bigcap (2 \circ 2)$ then by some manipulations we will see that $N((Nx \circ Ny) \circ Nz) \nsubseteq D_2$; $\forall x, y, z \in H$. Therefore D_2 is a $DPIHKI - T4$.
(iii) Let $2 \circ 2 = \{0, 1\}$ and D_2 is a $DPIHKI - 4$ we prove that $1 \in 0 \circ 2$. On the contrary let $1 \notin 0 \circ 2$. Then we have $1 \circ (((1 \circ 2) \circ (1 \circ 2)) \circ (1 \circ 2)) = 1 \circ ((2 \circ 2) \circ 2) = 1 \circ (\{0, 1\} \circ 2) = 1 \circ \{0, 2\} = \{1, 2\} = D_2$. But $0 \in 1 \circ 1 \subseteq 1 \circ (2 \circ 2) = 1 \circ ((1 \circ 2) \circ (1 \circ 2))$, so $N(N2 \circ N2) \nsubseteq D_2$. Hence D_2 is not a $DPIHKI - T4$, which is a contradiction. Thus $1 \in 0 \circ 2$. Conversely if $1 \in 0 \circ 2$, then by the proof of Theorem 4.10 we have $2 \circ 1 = (2 \circ 2) \bigcup (0 \circ 2)$, hence $1 \in 2 \circ 1$. By $(HK3)$ we have $0 \circ 2 \subseteq (2 \circ 1) \circ 2 = (2 \circ 2) \circ 1 = \{0, 2\} \bigcup (0 \circ 1)$. Since $1 \in 0 \circ 2$, then $1 \in 0 \circ 1$. Now $0 \in 2 \circ 1$ and hypothesis imply that $N((Nx \circ Ny) \circ Nz) \nsubseteq D_2$; $\forall x, y, z \in H$. Therefore D_2 is a $DPIHKI - T4$.

Now we give some examples about the above theorem.
Example 4.13. Consider the following tables :

H_1	0	1	2
0	$\{0,2\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

H_3	0	1	2
0	$\{0,2\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_4	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

H_5	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

H_6	0	1	2
0	$\{0,2\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$. Also:
(a) In H_3 and H_4 , D_2 is a $DPIHKI-T4$, by Theorem 4.12 (ii),(iii).
(b) in H_1, H_2, H_5 and H_6, D_2 is not a $DPIHKI-T4$, by Theorem 4.12 (i),(iii) .

Theorem 4.14. Let $1\circ 1=\{0,2\}$ and $1\circ 2=\{1,2\}$. Then $D_1(D_3)$ is a $DPIHKI-T4$ if and only if $0\circ 1\neq\{0\}$ or $2\circ 1\neq\{0\}$.

Proof. We prove theorem for D_1, the proof of D_3 is the same as D_1. Let D_1 is a $DPIHKI-T4$ we prove that $0\circ 1\neq\{0\}$ or $2\circ 1\neq\{0\}$. On the contrary let $0\circ 1=\{0\}$ and $2\circ 1=\{0\}$. Then we have $1\circ(((1\circ 0)\circ(1\circ 0))\circ(1\circ 0))=1\circ(\{0,2\}\circ 1)=1\circ 0=\{1\}\subseteq D_1$ and $1\circ((1\circ 0)\circ(1\circ 0))=\{1,2\}\not\subseteq D_1$. Hence D_1 is not a $DPIHKI-T4$, which is a contradiction. So $0\circ 1\neq\{0\}$ or $2\circ 1\neq\{0\}$. Conversely let $0\circ 1\neq\{0\}$ or $2\circ 1\neq\{0\}$. Then by $(HK3)$ we have $(0\circ 2)\bigcup(2\circ 2)=(1\circ 1)\circ 2=(1\circ 2)\circ 1=\{1,2\}\circ 1=\{0,2\}\bigcup(2\circ 1)$. So $2\in 0\circ 2$ or $2\in 2\circ 2$. Now by some calculations we can get that $N((Nx\circ Ny)\circ Nz)\not\subseteq D_1;\ \forall x,y,z\in H$. Therefore D_1 is a $DPIHKI-T4$.

Now we give some examples about the above theorem.
Example 4.15. Consider the following tables :

H_1	0	1	2
0	$\{0,2\}$	$\{0,1\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$, Furthermore:

(a) In H_1 and H_2, D_1 and D_3 are $DPIHKI-T4$.
(b) In H_3, D_1 and D_3 are not $DPIHKI-T4$.

Theorem 4.16. Let $1\circ 1=\{0,2\}$ and $1\circ 2=\{1,2\}$. Then the following statements hold:
(i) If $2\circ 2=\{0,1\}$, then D_2 is a $DPIHKI-T4$.
(ii) If $2\circ 2\subseteq\{0,2\}$, then D_2 is a $DPIHKI-T4$ if and only if $1\in 0\circ 2$.
(iii) If $2\circ 2=\{0,1,2\}$, then D_2 is a $DPIHKI-T4$ if and only if $2\circ 1\neq\{0,1\}$ or $0\circ 1\neq\{0\}$.

Proof. (i) Let $2\circ 2=\{0,1\}$. Now similar to the proof of (conversely of) Theorem 4.14 we have $(0\circ 2)\bigcup(2\circ 2)=\{0,2\}\bigcup(2\circ 1)$, so $1\in 2\circ 2$ implies that $1\in 2\circ 1$. Therefore $\{1,2\}=1\circ 2\subseteq(2\circ 1)\circ 2=(2\circ 2)\circ 1=\{0,1\}\circ 1=(0\circ 1)\bigcup\{0,2\}$, which implies that $1\in 0\circ 1$. By hypothesis and some calculations we can get that $N((Nx\circ Ny)\circ Nz)\not\subseteq D_2$; $\forall x,y,z\in H$. Therefore D_2 is a $DPIHKI-T4$.
(ii) Let $2\circ 2\subseteq\{0,2\}$ and D_2 is a $DPIHKI-T4$ we prove that $1\in 0\circ 2$. On the contrary let $1\notin 0\circ 2$. Then by $(HK2)$ we have $2\circ 0\subseteq(1\circ 1)\circ 0=(1\circ 0)\circ 1=1\circ 1=\{0,2\}$. Since $0\notin 2\circ 0$, thus $2\circ 0=\{2\}$. Therefore we get that $1\circ(((1\circ 0)\circ(1\circ 0))\circ(1\circ 1))=1\circ((1\circ 1)\circ(1\circ 1))=1\circ(\{0,2\}\circ\{0,2\})=1\circ\{0,2\}=\{1,2\}=D_2$. Since $0\in 1\circ 1\subseteq 1\circ((1\circ 0)\circ(1\circ 1))$, then $N(N0\circ N1)\not\subseteq D_2$. So D_2 is not a $DPIHKI-T4$, which is a contradiction. Thus $1\in 0\circ 2$. Conversely let $2\circ 2=\{0\}$ and $1\in 0\circ 2$. By $(HK2)$ we have $1\in 0\circ 2\subseteq(2\circ 1)\circ 2=(2\circ 2)\circ 1=0\circ 1$. By some manipulations we can get that $N((Nx\circ Ny)\circ Nz)\not\subseteq D_2$; $\forall x,y,z\in H$. Hence D_2 is a $DPIHKI-T4$. Now let $2\circ 2=\{0,2\}$ and $1\in 0\circ 2$. By the proof of (i) we have $1\in 2\circ 1$. So $\{1,2\}=1\circ 2\subseteq(2\circ 1)\circ 2=(2\circ 2)\circ 1=\{0,2\}\circ 1=(0\circ 1)\bigcup(2\circ 1)$. Thus $2\in(0\circ 1)\bigcup(2\circ 1)$. By some calculations we conclude that $N((Nx\circ Ny)\circ Nz\not\subseteq D_2$; $\forall x,y,z\in H$. That is D_2 a $DPIHKI-T4$.
(iii) The proof is similar to (ii), by some suitable modifications.

Now we give some examples about the above theorem.
Example 4.17. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

H_4	0	1	2
0	$\{0,2\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_5	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_6	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

H_7	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

H_8	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$. Also:
(a) In H_1, H_2, H_3, H_5 and H_8, D_2 is a $DPIHKI-T4$, by Theorem (i), (ii), (iii) and (iii), respectively.
(b) In H_4, H_6 and H_7, D_2 is not a $DPIHKI-T4$, by Theorem 4.16 (ii), (iii) and (ii), respectively.

Lemma 4.18. Let $1\circ 1=\{0,2\}$ and $1\circ 2=\{1\}$. Then D_1, D_2 and D_3 are not $DPIHKI-T4$.

Proof. The proof follows from Theorem 3.13.

Now we give some examples about the above lemma.
Example 4.19. The following tables show the hyper K-algebra structures on $\{0,1,2\}$ such that D_1, D_2 and D_3 are not $DPIHKI-T4$.

H_1	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0,2\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

Theorem 4.20. Let $1\circ 1=\{0,1,2\}$ and $1\circ 2=\{2\}$. Then the following statements hold:
(i) D_2 is a $DPIHKI-T4$ if and only if $1\in 0\circ 2$ or $2\circ 2=\{0,1,2\}$.
(ii) $D_1(D_3)$ is a $DPIHKI-T4$ if and only if $2\circ 2\neq\{0\}$.

Proof. Note that from hypothesis and $(HK2)$ we conclude that

$$(0\circ 2)\bigcup(1\circ 2)\bigcup(2\circ 2)=(1\circ 1)\circ 2=(1\circ 2)\circ 1=2\circ 1. \qquad (1)$$

(i) Let D_2 is a $DPIHKI-T4$ we prove that $1\in 0\circ 2$ or $2\circ 2=\{0,1,2\}$. On the contrary Let $1\notin 0\circ 2$ and $2\circ 2\neq\{0,1,2\}$. Consider two cases: (a) $1\in 2\circ 2$, (b) $1\notin 2\circ 2$.
Case(a): Let $1\in 2\circ 2$. Then by hypothesis we get that $2\circ 2=\{0,1\}$. So we have $1\circ(((1\circ 2)\circ(1\circ 2))\circ(1\circ 2))=1\circ((2\circ 2)\circ 2)=1\circ(\{0,1\}\circ 2)=1\circ\{0,2\}=\{1,2\}=D_2$ and $1\circ((1\circ 2)\circ(1\circ 2))=1\circ\{0,1\}=\{0,1,2\}$. Since $0\notin D_2$, then $N(N2\circ N2)\nsubseteq D_2$. So D_2 is not a $DPIHKI-T4$, which is a contradiction. Thus $1\in 0\circ 2$ or $2\circ 2=\{0,1,2\}$.
Case(b): Let $1\notin 2\circ 2$, by (1) we conclude that $1\notin 2\circ 1$. Hence we get that $1\circ(((1\circ 0)\circ(1\circ 2))\circ(1\circ 0))=1\circ((1\circ 2)\circ 1)=1\circ(2\circ 1)\subseteq 1\circ\{0,2\}=\{1,2\}=D_2$. Also $1\circ((1\circ 0)\circ(1\circ 0))=1\circ(1\circ 1)=1\circ\{0,1,2\}=\{0,1,2\}$. Since $0\notin D_2$,so $N(N0\circ N0)\nsubseteq D_2$. Thus D_2 is not a $DPIHKI-T4$, which is a contradiction. So $1\in 0\circ 2$ or $2\circ 2=\{0,1,2\}$. Conversely Let $1\in 0\circ 2$ or $2\circ 2=\{0,1,2\}$. Then by (1) we get that $1\in 2\circ 1$ and $1\circ((1\circ 0)\circ(1\circ 2))\subseteq D_2$. By hypothesis and some manipulations we can see that $N((Nx\circ Ny)\circ Nz)\nsubseteq D_2$; $\forall x,y,z\in H$. Therefore D_2 is a $DPIHKI-T4$.
(ii) The proof is not difficult and nearly similar to (i).

Now we give some examples about the above theorem.
Example 4.21. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0,2\}$	$\{0,1\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_4	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_5	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

H_6	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$. Moreover:
(a) In H_1 and H_4, D_1 and D_3 are $DPIHKI-T4$, by Theorem 4.20 (ii), while D_2 is not a $DPIHI-T4$, by Theorem 4.20 (i).
(b) In H_2 and H_3, D_2, D_1 and D_3 are $DPIHKI-T4$, by Theorem 4.20 (i) and (ii), respectively.
(c) In H_5, D_2 is a $DPIHKI-T4$, by Theorem 4.20 (i), while D_1 and D_3 are not $DPIHKI-T4$, by Theorem 4.20 (ii).
(d) In H_6, D_2, D_1 and D_3 are not $DPIHKI-T4$, by Theorem 4.20 (i) and (ii), respectively.

Theorem 4.22. Let $1\circ 1=\{0,1,2\}$ and $1\in 1\circ 2$. Then D_1, D_2 and D_3 are $DPIHKI-T4$.

Proof. By Theorem 3.6 we have D_1 and D_2 are $DPIHKI-T4$. We now prove that D_3 is a $DPIHKI-T4$. By hypothesis we have $1\in 1\circ x$; $\forall x\in H$, hence $1\in((1\circ x)\circ(1\circ y))\circ(1\circ z)$; $\forall x,y,z\in H$. Thus we get that $2\in 1\circ 1\subseteq 1\circ(((1\circ x)\circ(1\circ y))\circ(1\circ z))$; $\forall x,y,z\in H$. Since $2\notin D_3$, then $N((Nx\circ Ny)\circ Nz)\not\subseteq D_3$; $\forall x,y,z\in H$. Therefore D_3 is a $DPIHKI-T4$.

Now we give some examples about the above theorem.
Example 4.23. The following tables show the hyper K-algebra structures on $\{0,1,2\}$ such that D_1, D_2 and D_3 are $DPIHKI-T4$.

H_1	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

Remark 4.24. Note that Examples 4.5, 4.9, 4.11, 4.13, 4.17 and 4.19. show that the conditions of Theorems 4.4, 4.8, 4.12, 4.16 and 4.18, are necessary and we can not omit or reduce these conditions.

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Dept. of Math., Shahid Bahonar University of Kerman, Kerman, Iran
e-mail: zahedi_mm@mail.uk.ac.ir
email: ltorkzadeh@yahoo.com