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ALGEBRAIC CLOSURES IN CERTAIN ELEMENTARY CLASSES

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Abstract. The main purpose of this paper is to give some of the necessary and sufficient conditions for the existence of algebraic closures in a certain kind of elementary classes.

This study is essentially built on the foundation of the paper: Zur Theorie der algebraischen Erweiterungen by the late Prof. K. Shoda (Cf. [3], [4]). However our notion of algebraic extensions will be very generalized from his notion. But in the variety with the fundamental conditions (I), (IV) and (a), (b) in his paper, our notion and his notion are equivalent (Cf. [3; Satz 6]).

In this paper, we shall introduce new notions of algebraic extensions, algebraic closedness, and algebraic closures in a certain kind of elementary classes, which are generalizations of the usual ones in the theory of commutative fields. And we shall investigate their properties. Especially, we shall state some of the necessary and sufficient conditions for the existence of algebraic closures (Theorem 4.3). Our theory (§1 $\sim$ §5) can be applied not only to the class of commutative fields but also to every injectively complete universal class, for example, the variety of Abelian groups, the variety of Boolean algebras, the variety of semilattices, the class of partially ordered sets, the class of totally ordered sets, etc. The main parts (§1 $\sim$ §4) of our theory can be also applied to every residually small variety with the (somewhat weak) amalgamation property.

§0. **Terminology and notation.**

The usual terminology and notation in the model theory will be used without explanation.

Throughout this paper, we assume that L is a first order language with equality. For structures $\mathfrak{A}, \mathfrak{B}$ for L, $\mathfrak{A} \subseteq \mathfrak{B}$ denotes that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. If a is an element of the domain of a structure $\mathfrak{A}$, we simply say that a is an element of $\mathfrak{A}$ or a is (an element) in $\mathfrak{A}$ and denote it by $a \in \mathfrak{A}$.

Let $\mathfrak{A}, \mathfrak{B}$ be structures for L such that $\mathfrak{A} \subseteq \mathfrak{B}$, and let $b_0, \cdots, b_m \in \mathfrak{B}$. We denote by $\mathfrak{A}(b_0, \cdots, b_m)$ the substructure of $\mathfrak{B}$ generated by $\mathfrak{A}$ and $b_0, \cdots, b_m$.

Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be structures for L such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$. A homomorphism ϕ of $\mathfrak{B}$ onto or into $\mathfrak{C}$ is called an $\mathfrak{A}$-homomorphism, if $\phi(a) = a$ for all elements a in $\mathfrak{A}$. An $\mathfrak{A}$-isomorphism and an $\mathfrak{A}$-embedding are defined similarly.

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Let $\{k_0, k_1, \cdots\}$ be a set of new constant symbols. Then we denote by $L(\{k_0, k_1, \cdots\})$ or simply $L(k_0, k_1, \cdots)$ the first order language formed from L by adding the new constant symbols $k_0, k_1, \cdots$.

Let $\mathfrak{A}$ be a structure for L. We denote by $\{\bar{a} \mid a \in \mathfrak{A}\}$ a set of new constant symbols indexed by all elements in $\mathfrak{A}$. The language $L(\{\bar{a} \mid a \in \mathfrak{A}\})$ is called the diagram language of $\mathfrak{A}$, and simply denoted by $L(\mathfrak{A})$. We denote by $\bar{\mathfrak{A}}$ the expansion of $\mathfrak{A}$ to $L(\mathfrak{A})$ by interpreting each $\bar{a}$ by a. We denote by $D(\mathfrak{A})$ the diagram of $\mathfrak{A}$ i.e., the set of all atomic or negated atomic sentences of $L(\mathfrak{A})$ which hold in $\bar{\mathfrak{A}}$. And we denote by $D^+(\mathfrak{A})$ the positive diagram of $\mathfrak{A}$ i.e., the set of all atomic sentences of $L(\mathfrak{A})$ which hold in $\bar{\mathfrak{A}}$.

If $\mathfrak{B}$ is an extension of $\mathfrak{A}$, and a is an element in $\mathfrak{A}$, then the constant symbol $\bar{a}$ in $L(\mathfrak{A})$ and $\bar{a}$ in $L(\mathfrak{B})$ are always assumed to be the same.

Let $k_0, \cdots, k_m$ be constant symbols not in $L(\mathfrak{A})$. The language $L(\mathfrak{A})(k_0, \cdots, k_m)$ is simply denoted by $L(\mathfrak{A};\ k_0, \cdots, k_m)$.

Let L' be an extended language of L, and let $\mathfrak{A}$ be a structure for L'. We denote by $\mathfrak{A}|L$ the reduct of $\mathfrak{A}$ to L.

Let Γ be a consistent set of sentences of L. Then Γ is called a theory of L, and the class of all models of Γ which are structures for L is denoted by $\mathcal{M}(\Gamma)$. If $\mathfrak{A}$ is a structure for L, then an extension of $\mathfrak{A}$ which is in $\mathcal{M}(\Gamma)$ is called an $\mathcal{M}(\Gamma)$-extension of $\mathfrak{A}$.

We regard cardinals as being identical with initial ordinals.

§1. **Algebraically closed structures and some kind of existentially closed structures.**

In this section, we shall introduce a new notion of algebraic extensions in some kind of elementary classes. By using this notion, we introduce the notion of algebraically closed structures. On the other hand, we define a certain kind of existentially closed structures. And we shall show their simple properties. Especially we show a relation between algebraically closed structures and existentially closed structures.

A sentence of the form:

$$\forall x_0 \cdots \forall x_n \exists y_0 \cdots \exists y_m \left(\Phi(x_0, \cdots, x_n) \to \Psi(x_0, \cdots, x_n, y_0, \cdots, y_m)\right)$$

is called a *special universal-existential sentence* (special $\forall$-$\exists$-sentence for short) of L, if $\Phi(x_0, \cdots, x_n)$ is a quantifier-free formula of L and $\Psi(x_0, \cdots, x_n, y_0, \cdots, y_m)$ is a positive quantifier-free formula of L, where either $\Phi(x_0, \cdots, x_n)$ or $\Psi(x_0, \cdots, x_n, y_0, \cdots, y_m)$ may be deleted.

Notice that any universal sentence ($\forall$-sentence for short) is logically equivalent to a special $\forall$-$\exists$-sentence, and any positive universal-existential sentence is identical with a special $\forall$-$\exists$-sentence.

Let T be a consistent set of sentences of L. If each member of T is a special $\forall$-$\exists$-sentence, we say that T is a *special universal-existential theory* (special $\forall$-$\exists$-theory for short) of L.

Let T be a special $\forall$-$\exists$-theory of L, and let S_T be the universal theory ($\forall$-theory for short) of L defined by

$$S_T = \{\Upsilon \mid \Upsilon \text{ is a } \forall\text{-sentence of } L \text{ such that } T \vdash \Upsilon\}.$$

Then clearly we have

$$\mathcal{M}(S_T) = \{\mathfrak{A} \mid \mathfrak{A} \subseteq \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathcal{M}(T)\}.$$

In this case, S_T is called the *supporting theory* of T, and $\mathcal{M}(S_T)$ is called the *supporting class* of $\mathcal{M}(T)$.

Notice that, as a special case, we can easily choose L and T such that T is the theory of commutative fields and S_T is the theory of commutative integral domains.

Let T be a special $\forall$-$\exists$-theory of L, and be defined by

$$T = \{\forall x_0 \cdots \forall x_{n_i} \exists y_0 \cdots \exists y_{m_i} \Upsilon_i \mid i \in I\},$$

where each Υ_i is a quantifier-free formula. The rank of T (denoted by rank T) is defined by

$$\text{rank } T = \sup\ (\{m_i + 1 \mid i \in I\} \cup \{1\})\,.$$

The rank of $\mathcal{M}(T)$ (denoted by rank $\mathcal{M}(T)$) is defined by

$$\text{rank } \mathcal{M}(T) = \min\ \{\text{rank } T' \mid T' \text{ is a special } \forall\text{-}\exists\text{-theory of } L \text{ such that } \mathcal{M}(T') = \mathcal{M}(T)\}\,.$$

We say that a special $\forall$-$\exists$-theory of T is in standard form, if rank T = rank $\mathcal{M}(T)$.

¿From now on, we assume that T is a special $\forall$-$\exists$-theory of L in standard form, and rank T = rank $\mathcal{M}(T) = \mathfrak{r}$. And S_T always denotes the supporting theory of T.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B}$. We say that $\mathfrak{B}$ is $(\mathfrak{A}, S_T)$-simple, if any $\mathfrak{A}$-homomorphism of $\mathfrak{B}$ onto an $\mathcal{M}(S_T)$-extension of $\mathfrak{A}$ is always an $\mathfrak{A}$-isomorphism, i.e., $\mathfrak{B}$ has no non-isomorphic $\mathfrak{A}$-homomorphism onto an $\mathcal{M}(S_T)$-extension of $\mathfrak{A}$.

Remark 1.1. The following two conditions are equivalent:

(1) $\mathfrak{B}$ is $(\mathfrak{A}, S_T)$-simple;

(2) For any atomic sentence Θ of $L(\mathfrak{B})$,
if $S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{B}) \cup \{\Theta\}$ is consistent then $\Theta \in D^+(\mathfrak{B})$.

Let $\mathfrak{A}, \mathfrak{B}$ be structures in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B}$, and let $b_0, \cdots, b_m \in \mathfrak{B}$. If $m < \mathfrak{r}$ and $\mathfrak{A}(b_0, \cdots, b_m)$ is $(\mathfrak{A}, S_T)$-simple, then we say that the sequence $\langle b_0, \cdots, b_m\rangle$ is algebraic (precisely speaking, S_T-algebraic) over $\mathfrak{A}$. We simply say that an element b is algebraic over $\mathfrak{A}$ if $\langle b\rangle$ is algebraic over $\mathfrak{A}$.

Let $\mathfrak{A}, \mathfrak{B}$ be structures in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B}$. We say that $\mathfrak{B}$ is a *primitive algebraic extension* (precisely speaking, primitive S_T-algebraic extension) of $\mathfrak{A}$, if there exist some elements $b_0, \cdots, b_m$ in $\mathfrak{B}$ such that $\mathfrak{B} = \mathfrak{A}(b_0, \cdots, b_m)$ and $\langle b_0, \cdots, b_m\rangle$ is algebraic over $\mathfrak{A}$.

Let $\mathfrak{A}, \mathfrak{B}$ be structures in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B}$. We say that $\mathfrak{B}$ is an *algebraic extension* (precisely speaking, S_T-algebraic extension) of $\mathfrak{A}$, if there exists an ascending chain

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots \subseteq \mathfrak{A}_\xi \subseteq \cdots$$

of substructures of $\mathfrak{B}$ which satisfies the following three conditions:

(1) $\mathfrak{B} = \mathfrak{A}_\mu$ for some ordinal μ;

(2) For each successor ordinal $\xi \leq \mu$, $\mathfrak{A}_\xi$ is a primitive algebraic extension of $\mathfrak{A}_{\xi-1}$;

(3) For each non-zero limit ordinal $\xi \leq \mu$, $\mathfrak{A}_\xi = \bigcup_{\eta<\xi} \mathfrak{A}_\eta$.

Notice that it follows immediately from the above definition that any algebraic extension of an algebraic extension of $\mathfrak{A}$ is an algebraic extension of $\mathfrak{A}$.

Let Ω be a structure in $\mathcal{M}(S_T)$. We say that Ω is *algebraically closed* (precisely speaking, S_T-algebraically closed), if there is no proper algebraic extension of Ω, i.e., $\Omega = \Omega'$ for every algebraic extension Ω' of Ω.

Let α be a finite or infinite cardinal (i.e., initial ordinal), and let $n \leq \omega$, where ω denotes the least infinite cardinal. A formula of $L_{\infty\omega}$ of the form:

$$\bigwedge_{\nu<\mu} \Theta_\nu(x_0, \cdots, x_m)$$

is called an (α, n)-formula over L, if $\mu < \alpha,\ m < n$, and each $\Theta_\nu(x_0, \cdots, x_m)$ is an atomic formula of L.

A sentence of $L_{\infty\omega}$ of the form:

$$\exists x_0 \cdots \exists x_m \Psi(x_0, \cdots, x_m)$$

is called an (α, n)-existential sentence over L, if $\Psi(x_0, \cdots, x_m)$ is an (α, n)-formula over L.

Deleting the restriction of $\mu < \alpha$ from the above definitions, we define the notions of an (∞, n)-formula over L and an (∞, n)-existential sentence over L.

Let Γ be a theory of L, and let $\mathfrak{A}$ be a structure in $\mathcal{M}(\Gamma)$. We say that $\mathfrak{A}$ is Γ-(α, n)-existentially closed, if for any (α, n)-existential sentence Φ over $L(\mathfrak{A})$ and for any $\mathcal{M}(\Gamma)$-extension $\mathfrak{B}$ of $\mathfrak{A}$, whenever $\bar{\mathfrak{B}} \models \Phi$ then $\bar{\mathfrak{A}} \models \Phi$.

The notion of Γ-(∞, n)-existential closedness is defined similarly.

We say that a set of sentences of $L_{\infty\omega}$ is consistent, if it has a model.

Remark 1.2. Let $\mathfrak{A} \in \mathcal{M}(\Gamma)$. Then the following two conditions are equivalent:

(1) $\mathfrak{A}$ is Γ-(α, n)-existentially (resp. Γ-(∞, n)-existentially) closed;

(2) For any (α, n)-existential (resp. (∞, n)-existential) sentence Φ over $L(\mathfrak{A})$,

if $\Gamma \cup D(\mathfrak{A}) \cup \{\Phi\}$ is consistent then $\bar{\mathfrak{A}} \models \Phi$.

Note that a structure $\mathfrak{A}$ in $\mathcal{M}(T)$ is T-(α, n)-existentially (resp. T-(∞, n)-existentially) closed if and only if $\mathfrak{A}$ is S_T-(α, n)-existentially (resp. S_T-(∞, n)-existentially) closed.

First we shall prove the following:

Theorem 1.3. *Let $\mathfrak{A} \in \mathcal{M}(S_T)$. If $\mathfrak{A}$ is S_T-$(\omega, \mathfrak{r})$-existentially closed, then $\mathfrak{A} \in \mathcal{M}(T)$. (Consequently, if $\mathfrak{A}$ is S_T-$(\infty, \mathfrak{r})$-existentially closed, then $\mathfrak{A} \in \mathcal{M}(T)$.)*

Proof. Let $\mathfrak{A}$ be an S_T-$(\omega, \mathfrak{r})$-existentially closed structure in $\mathcal{M}(S_T)$. And let Υ be any special $\forall$-$\exists$-sentence in T. We shall prove $\mathfrak{A} \models \Upsilon$ by separating Υ into the following three types:

(I) $\Upsilon = \forall x_0 \cdots \forall x_n \exists y_0 \cdots \exists y_m (\Phi(x_0, \cdots, x_n) \to \Psi(x_0, \cdots, x_n,\ y_0, \cdots, y_m))$;

(II) $\Upsilon = \forall x_0 \cdots \forall x_n \exists y_0 \cdots \exists y_m \Psi(x_0, \cdots, x_n,\ y_0, \cdots, y_m)$;

(III) $\Upsilon = \forall x_0 \cdots \forall x_n \neg \Phi(x_0, \cdots, x_n)$.

First we shall prove $\mathfrak{A} \models \Upsilon$ in the case of type (I). Let $a_0, \cdots, a_n$ be elements in $\mathfrak{A}$ such that $\bar{\mathfrak{A}} \models \Phi(\bar{a}_0, \cdots, \bar{a}_n)$.

It suffices to prove that

$$\bar{\mathfrak{A}} \models \exists y_0 \cdots \exists y_m \Psi(\bar{a}_0, \cdots, \bar{a}_n,\ y_0, \cdots, y_m).$$

Since $\mathfrak{A} \in \mathcal{M}(S_T)$, there exists a structure $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B} \in \mathcal{M}(T)$. Hence

$$\bar{\mathfrak{B}} \models \forall x_0 \cdots \forall x_n \exists y_0 \cdots \exists y_m (\Phi(x_0, \cdots, x_n) \to \Psi(x_0, \cdots, x_n,\ y_0, \cdots, y_m)),$$

and hence

$$\bar{\mathfrak{B}} \models \exists y_0 \cdots \exists y_m \Psi(\bar{a}_0, \cdots, \bar{a}_n,\ y_0, \cdots, y_m).$$

Therefore there exist elements $b_0, \cdots, b_m$ in $\mathfrak{B}$ such that

$$(*) \qquad \bar{\mathfrak{B}} \models \Psi(\bar{a}_0, \cdots, \bar{a}_n,\ \bar{b}_0, \cdots, \bar{b}_m).$$

Now Ψ can be logically equivalently rewritten in the following form:

$$\Psi \equiv \Psi_1 \vee \cdots \vee \Psi_p,$$

where each Ψ_i is a finitary conjunction of atomic formulas of L. And then by $(*)$, there is some Ψ_j such that

$$\bar{\mathfrak{B}} \models \Psi_j(\bar{a}_0, \cdots, \bar{a}_n,\ \bar{b}_0, \cdots, \bar{b}_m).$$

Hence we have

$$\bar{\mathfrak{B}} \models \exists y_0 \cdots \exists y_m \Psi_j(\bar{a}_0, \cdots, \bar{a}_n,\ y_0, \cdots, y_m).$$

Since $\mathfrak{A}$ is S_T-$(\omega, \mathfrak{r})$-existentially closed, we have

$$\bar{\mathfrak{A}} \models \exists y_0 \cdots \exists y_m \Psi_j(\bar{a}_0, \cdots, \bar{a}_n,\ y_0, \cdots, y_m).$$

Hence we have

$$\bar{\mathfrak{A}} \models \exists y_0 \cdots \exists y_m \Psi(\bar{a}_0, \cdots, \bar{a}_n,\ y_0, \cdots, y_m).$$

as desired.

In the case of type (II), we put $\Phi(x_0, \cdots, x_n) = (x_0 = x_0)$.

Then

$\forall x_0 \cdots \forall x_n \exists y_0 \cdots \exists y_m \Psi(x_0, \cdots, x_n,\ y_0, \cdots, y_m)$

$\equiv \forall x_0 \cdots \forall x_n \exists y_0 \cdots \exists y_m \left(\Phi(x_0, \cdots, x_n) \to \Psi(x_0, \cdots, x_n,\ y_0, \cdots, y_m)\right).$

Hence the proof in the case of type (I) implies the proof in the case of type (II).

Finally we have $\mathfrak{A} \models \Upsilon$ in the case of type (III), because $\Upsilon \in S_T$. This completes the proof of the theorem.

Let $\mathfrak{A}, \mathfrak{B}$ be structures in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B}$. If there exists an $\mathfrak{A}$-homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$, then we say that $\mathfrak{B}$ is a splitting extension of $\mathfrak{A}$ or that $\mathfrak{A}$ is a retract of $\mathfrak{B}$.

Theorem 1.4. *Let $\mathfrak{A}$ be a structure in $\mathcal{M}(S_T)$. Then the following three conditions are equivalent:*

(1) $\mathfrak{A}$ *is algebraically closed;*

(2) $\mathfrak{A}$ *is S_T-$(\infty, \mathfrak{r})$-existentially closed;*

(3) *For any $\mathcal{M}(S_T)$-extension $\mathfrak{B}$ of $\mathfrak{A}$, and for any elements $b_0, \cdots, b_m$ in $\mathfrak{B}$, if $m < \mathfrak{r}$ then $\mathfrak{A}(b_0, \cdots, b_m)$ is a splitting extension of $\mathfrak{A}$.*

Proof of $(1) \Rightarrow (2)$. We assume that $\mathfrak{A}$ is algebraically closed. Let Φ be any $(\infty, \mathfrak{r})$-existential sentence over $L(\mathfrak{A})$, and let

$$\Phi = \exists x_0 \cdots \exists x_m \bigwedge_{\nu<\mu} \Theta_\nu(x_0, \cdots, x_m),$$

where each $\Theta_\nu(x_0, \cdots, x_m)$ is an atomic formula of $L(\mathfrak{A})$. Suppose that

$$(**) \qquad S_T \cup D(\mathfrak{A}) \cup \{\Phi\} \text{ is consistent.}$$

It suffices to prove that $\bar{\mathfrak{A}} \models \Phi$.

¿From $(**)$, we have that

$$S_T \cup D(\mathfrak{A}) \cup \{\Theta_\nu(k_0, \cdots, k_m) \mid \nu < \mu\} \text{ is consistent,}$$

where $k_0, \cdots, k_m$ are new constant symbols. Therefore by using Zorn's Lemma and Compactness Theorem, it is easy to see that there exists a maximal set $\triangle$ of atomic sentences of $L(\mathfrak{A};\ k_0, \cdots, k_m)$ such that

$$S_T \cup D(\mathfrak{A}) \cup \{\Theta_\nu(k_0, \cdots, k_m) \mid \nu < \mu\} \cup \triangle \text{ is consistent.}$$

Let $\mathfrak{C}$ be a model of $S_T \cup D(\mathfrak{A}) \cup \{\Theta_\nu(k_0, \cdots, k_m) \mid \nu < \mu\} \cup \triangle$ which contains $\bar{\mathfrak{A}}$, and let $c_0, \cdots, c_m$ be the interpretations of $k_0, \cdots, k_m$ in $\mathfrak{C}$. Now let $\mathfrak{A}(c_0, \cdots, c_m)$ be the substructure of $\mathfrak{C}|L$ generated by $\mathfrak{A}$ and $c_0, \cdots, c_m$. Then obviously $\mathfrak{A}(c_0, \cdots, c_m)$ is a primitive algebraic extension of $\mathfrak{A}$. Since $\mathfrak{A}$ is algebraically closed, we have $c_i \in \mathfrak{A}$ $(i = 0, \cdots, m)$. Since

$$\mathfrak{C} \models \bigwedge_{\nu<\mu} \Theta_\nu(k_0, \cdots, k_m), \quad \text{i.e.,} \quad \bar{\mathfrak{C}} \models \bigwedge_{\nu<\mu} \Theta_\nu(\bar{c}_0, \cdots, \bar{c}_m),$$

we have

$$\bar{\mathfrak{A}} \models \bigwedge_{\nu<\mu} \Theta_\nu(\bar{c}_0, \cdots, \bar{c}_m).$$

Hence we have

$$\bar{\mathfrak{A}} \models \exists x_0 \cdots \exists x_m \bigwedge_{\nu<\mu} \Theta_\nu(x_0, \cdots, x_m).$$

This completes the proof of (1) $\Rightarrow$ (2).

Proof of (2) $\Rightarrow$ (3). We assume that $\mathfrak{A}$ is S_T-$(\infty, \mathfrak{r})$-existentially closed. Let $\mathfrak{B}$ be an $\mathcal{M}(S_T)$-extension of $\mathfrak{A}$, and let $b_0, \cdots, b_m \in \mathfrak{B}$ $(m < \mathfrak{r})$. We shall prove that $\mathfrak{A}(b_0, \cdots, b_m)$ is a splitting extension of $\mathfrak{A}$, i.e., there exists an $\mathfrak{A}$-homomorphism of $\mathfrak{A}(b_0, \cdots, b_m)$ onto $\mathfrak{A}$.

Let $\{\Theta_\nu(x_0, \cdots, x_m) \mid \nu < \mu\}$ be the set of all atomic formulas of $L(\mathfrak{A})$ such that

$$\overline{\mathfrak{A}(b_0, \cdots, b_m)} \models \Theta_\nu(\bar{b}_0, \cdots, \bar{b}_m).$$

Now let Φ be the $(\infty, \mathfrak{r})$-existential sentence over $L(\mathfrak{A})$ defined by

$$\Phi = \exists x_0 \cdots \exists x_m \bigwedge_{\nu<\mu} \Theta_\nu(x_0, \cdots, x_m).$$

Then clearly $\bar{\mathfrak{B}} \models \Phi$. Hence we have $\bar{\mathfrak{A}} \models \Phi$, because $\mathfrak{A}$ is S_T-$(\infty, \mathfrak{r})$-existentially closed. Therefore there exist elements $a_0, \cdots, a_m$ in $\mathfrak{A}$ such that

$$\bar{\mathfrak{A}} \models \bigwedge_{\nu<\mu} \Theta_\nu(\bar{a}_0, \cdots, \bar{a}_m).$$

Hence there exists an $\mathfrak{A}$-homomorphism of $\mathfrak{A}(b_0, \cdots, b_m)$ onto $\mathfrak{A}$ which maps b_i to a_i $(i = 0, \cdots, m)$. This completes the proof of (2) $\Rightarrow$ (3).

Proof of (3) $\Rightarrow$ (1). The condition (3) obviously implies that there is no proper primitive algebraic extension of $\mathfrak{A}$. Therefore it is obvious that the condition (3) implies the condition (1). This completes the proof of the theorem.

The following theorem is a direct consequence of Theorems 1.3 and 1.4.

Theorem 1.5. *Let Ω be a structure in $\mathcal{M}(S_T)$. If Ω is algebraically closed, then $\Omega \in \mathcal{M}(T)$.*

§2. **Existence of algebraically closed algebraic extensions.**

In this section, we shall give a certain condition on $\mathcal{M}(S_T)$ under which for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an algebraically closed algebraic extension of $\mathfrak{A}$.

Lemma 2.1. *Let $\mathfrak{A} \in \mathcal{M}(S_T)$, and let α be any finite or infinite cardinal (i.e., initial ordinal). Then there exists an algebraic extension $\mathfrak{B}$ of $\mathfrak{A}$ such that for any $(\alpha, \mathfrak{r})$-existential sentence Φ over $L(\mathfrak{A})$, if $S_T \cup D(\mathfrak{B}) \cup \{\Phi\}$ is consistent then $\bar{\mathfrak{B}} \models \Phi$.*

Proof. The proof of the lemma is trivial for $\alpha < 2$. Therefore, in the following, we shall prove the lemma in the case where $\alpha \geq 2$.

Let

$$\Psi_0,\ \Psi_1,\ \cdots,\ \Psi_\xi,\ \cdots \quad (\xi < \lambda)$$

be an enumeration of all $(\alpha, \mathfrak{r})$-existential sentences over $L(\mathfrak{A})$. We now inductively construct an ascending chain

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_\xi \subseteq \cdots \qquad (\xi \leq \lambda)$$

of $\mathcal{M}(S_T)$-extensions of $\mathfrak{A}$ as follows:

(1) The case where ξ is a non-zero limit ordinal. We put $\mathfrak{A}_\xi = \bigcup_{\eta<\xi} \mathfrak{A}_\eta$.

(2) The case where $\xi = \eta + 1$. We consider the following two cases (a) and (b).

(a) If $S_T \cup D(\mathfrak{A}_\eta) \cup \{\Psi_\eta\}$ is inconsistent, we put $\mathfrak{A}_\xi = \mathfrak{A}_\eta$.

(b) If $S_T \cup D(\mathfrak{A}_\eta) \cup \{\Psi_\eta\}$ is consistent, then $\mathfrak{A}_\xi$ is constructed as follows:

Let

$$\Psi_\eta = \exists x_0 \cdots \exists x_m \bigwedge_{\nu<\mu} \Theta_\nu(x_0, \cdots, x_m),$$

where $\Theta_\nu(x_0, \cdots, x_m)$ is an atomic formula of $L(\mathfrak{A})$. And let $k_0,\ \cdots,\ k_m$ be new constant symbols. Then $S_T \cup D(\mathfrak{A}_\eta) \cup \{\Theta_\nu(k_0, \cdots, k_m) \mid \nu < \mu\}$ is consistent. Hence there exists a maximal set $\triangle$ of atomic sentences of $L(\mathfrak{A}_\eta;\ k_0, \cdots, k_m)$ such that

$$S_T \cup D(\mathfrak{A}_\eta) \cup \{\Theta_\nu(k_0, \cdots, k_m) \mid \nu < \mu\} \cup \triangle \ \text{ is consistent.}$$

Let $\mathfrak{C}$ be a model of $S_T \cup D(\mathfrak{A}_\eta) \cup \{\Theta_\nu(k_0, \cdots, k_m) \mid \nu < \mu\} \cup \triangle$ which contains $\bar{\mathfrak{A}}_\eta$, and let $c_0, \cdots, c_m$ be the interpretations of $k_0, \cdots, k_m$ in $\mathfrak{C}$. Now we put $\mathfrak{A}_\xi = \mathfrak{A}_\eta(c_0, \cdots, c_m)$, where $\mathfrak{A}_\eta(c_0, \cdots, c_m)$ denotes the substructure of $\mathfrak{C}|L$ generated by $\mathfrak{A}_\eta$ and $c_0, \cdots, c_m$. Then obviously $\mathfrak{A}_\xi$ is a primitive algebraic extension of $\mathfrak{A}_\eta$, and $\bar{\mathfrak{A}}_\xi \models \Psi_\eta$.

Now we put

$$\mathfrak{B} = \mathfrak{A}_\lambda.$$

Then it is clear that $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$.

Let Φ be any $(\alpha, \mathfrak{r})$-existential sentence over $L(\mathfrak{A})$ such that $S_T \cup D(\mathfrak{B}) \cup \{\Phi\}$ is consistent. Then $\Phi = \Psi_\eta$ for some $\eta < \lambda$. Since $S_T \cup D(\mathfrak{B}) \cup \{\Phi\}$ is consistent, we have that

$$S_T \cup D(\mathfrak{A}_\eta) \cup \{\Phi\} \quad \text{is consistent.}$$

Hence $\bar{\mathfrak{A}}_{\eta+1} \models \Phi$. Since Φ is existential, we have $\bar{\mathfrak{B}} \models \Phi$. This completes the proof.

Theorem 2.2. *Let $\mathfrak{A} \in \mathcal{M}(S_T)$, and let α be any cardinal. Then there exists an algebraic extension of $\mathfrak{A}$ which is S_T-$(\alpha, \mathfrak{r})$-existentially closed.*

Proof. In order to prove the theorem, it suffices to prove it in the case where $\alpha \geq 2$. Hence, in the following, we shall prove the theorem in the case where $\alpha \geq 2$.

The definition of "regular cardinal" is quoted from [1 ; G. Grätzer : Universal Algebra, P.14].

Now assume that $\alpha \geq 2$, and let β be the least regular cardinal greater than α. We construct an ascending chain

$$\mathfrak{A} = \mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots \subseteq \mathfrak{B}_\xi \subseteq \cdots \quad (\xi < \beta)$$

of $\mathcal{M}(S_T)$-extensions of $\mathfrak{A}$ inductively as follows:

(1) If ξ is a non-zero limit ordinal, we put $\mathfrak{B}_\xi = \bigcup_{\eta<\xi} \mathfrak{B}_\eta$.

(2) If $\xi = \eta + 1$, $\mathfrak{B}_\xi$ is an algebraic extension of $\mathfrak{B}_\eta$ such that for any $(\alpha, \mathfrak{r})$-existential sentence Ψ over $L(\mathfrak{B}_\eta)$, whenever $S_T \cup D(\mathfrak{B}_\xi) \cup \{\Psi\}$ is consistent then $\bar{\mathfrak{B}}_\xi \models \Psi$. The existence of such a $\mathfrak{B}_\xi$ is obvious by the above lemma.

Now we put

$$\mathfrak{C} = \bigcup_{\xi<\beta} \mathfrak{B}_\xi.$$

Then it is easy to see that $\mathfrak{C}$ is an algebraic extension of $\mathfrak{A}$.

In the following, we shall prove that $\mathfrak{C}$ is S_T-$(\alpha, \mathfrak{r})$-existentially closed.

Let Φ be any $(\alpha, \mathfrak{r})$-existential sentence over $L(\mathfrak{C})$ such that $S_T \cup D(\mathfrak{C}) \cup \{\Phi\}$ is consistent. It suffices to prove that $\bar{\mathfrak{C}} \models \Phi$. Now let

$$(\#): \qquad \bar{c}_0,\ \bar{c}_1,\ \cdots,\ \bar{c}_\eta,\ \cdots$$

be an enumeration of all constant symbols in $\{\bar{c} \mid c \in \mathfrak{C}\}$ which occur in Φ. Since Φ is an $(\alpha, \mathfrak{r})$-existential sentence, the length of (#) is less than β. Since $\mathfrak{C} = \bigcup_{\xi<\beta} \mathfrak{B}_\xi$, each c_η is in some $\mathfrak{B}_\xi$. Hence there exists an ordinal λ less than β such that (#) can be embedded into $\{\bar{c} \mid c \in \mathfrak{B}_\lambda\}$ in some order, because β is regular. Hence Φ can be considered as an $(\alpha, \mathfrak{r})$-existential sentence over $L(\mathfrak{B}_\lambda)$, and obviously $S_T \cup D(\mathfrak{B}_{\lambda+1}) \cup \{\Phi\}$ is consistent. Therefore we have $\bar{\mathfrak{B}}_{\lambda+1} \models \Phi$, and therefore $\bar{\mathfrak{C}} \models \Phi$. This completes the proof.

Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B}$, and let $b_0, \cdots, b_m \in \mathfrak{B}$. We say that $\langle b_0, \cdots, b_m \rangle$ is α-definably algebraic (precisely speaking, α-definably S_T-algebraic) over $\mathfrak{A}$, if there exists an $(\alpha, \mathfrak{r})$-formula $\Psi(x_0, \cdots, x_m)$ over $L(\mathfrak{A})$ which satisfies the following two conditions:

(1) $\overline{\mathfrak{A}(b_0, \cdots, b_m)} \models \Psi(\bar{b}_0, \cdots, \bar{b}_m)$;

(2) For any $\mathcal{M}(S_T)$-extension $\mathfrak{C}$ of $\mathfrak{A}$, and for arbitrary elements $c_0, \cdots, c_m$ in $\mathfrak{C}$, if $\overline{\mathfrak{A}(c_0, \cdots, c_m)} \models \Psi(\bar{c}_0, \cdots, \bar{c}_m)$ then $\mathfrak{A}(b_0, \cdots, b_m)$ and $\mathfrak{A}(c_0, \cdots, c_m)$ are $\mathfrak{A}$-isomorphic by the correspondence $b_i \leftrightarrow c_i$ $(i = 0, \cdots, m)$.

The $(\alpha, \mathfrak{r})$-formula $\Psi(x_0, \cdots, x_m)$ over $L(\mathfrak{A})$ which satisfies the above two conditions (1) and (2) is called a characteristic formula of $\langle b_0, \cdots, b_m \rangle$ over $(\mathfrak{A}, S_T)$.

Remark 2.3. The above condition (2) can be changed by the following condition:

(2)′ For any atomic sentence Θ of $L(\mathfrak{A};\ k_0, \cdots, k_m)$,
if $S_T \cup D(\mathfrak{A}) \cup \{\Psi(k_0, \cdots, k_m)\} \cup \{\Theta\}$ is consistent then
$S_T \cup D(\mathfrak{A}) \cup \{\Psi(k_0, \cdots, k_m)\} \models \Theta$.

Remark 2.4. If $\langle b_0, \cdots, b_m\rangle$ is α-definably algebraic over $\mathfrak{A}$, then $\langle b_0, \cdots, b_m\rangle$ is algebraic over $\mathfrak{A}$. Conversely, if $\langle b_0, \cdots, b_m\rangle$ is algebraic over $\mathfrak{A}$, then for a sufficiently large cardinal α, $\langle b_0, \cdots, b_m\rangle$ is α-definably algebraic over $\mathfrak{A}$.

We say that $\mathcal{M}(S_T)$ has the *algebraic-extension-definable property* (AEDP for short), if for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists a cardinal α such that for any algebraic extension $\mathfrak{B}$ of $\mathfrak{A}$, and for any elements $c_0, \cdots, c_m$ in any $\mathcal{M}(S_T)$-extension of $\mathfrak{B}$, whenever $\langle c_0, \cdots, c_m\rangle$ is algebraic over $\mathfrak{B}$ then $\langle c_0, \cdots, c_m\rangle$ is α-definably algebraic over $\mathfrak{B}$.

The least cardinal α for $\mathfrak{A}$ which satisfies the above condition is called the algebraic-extension-definable degree (AED-degree for short) of $\mathfrak{A}$.

Theorem 2.5. *Assume that $\mathcal{M}(S_T)$ has the AEDP. Then for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an algebraically closed algebraic extension of $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be any structure in $\mathcal{M}(S_T)$, and let α be the AED-degree of $\mathfrak{A}$. Then by Theorem 2.2, there exists an algebraic extension Ω of $\mathfrak{A}$ which is S_T-$(\alpha, \mathfrak{r})$-existentially closed.

In the following, we shall prove that Ω is algebraically closed.

Let Ω' be any $\mathcal{M}(S_T)$-extension of Ω, and let $b_0, \cdots, b_m \in \Omega'$ such that $\langle b_0, \cdots, b_m\rangle$ is algebraic over Ω. It suffices to prove that $\Omega(b_0, \cdots, b_m) = \Omega$.

Since α is the AED-degree of $\mathfrak{A}$, $\langle b_0, \cdots, b_m\rangle$ is α-definably algebraic over Ω. Hence there exists an $(\alpha, \mathfrak{r})$-formula $\Psi(x_0, \cdots, x_m)$ over $L(\Omega)$ which is a characteristic formula of $\langle b_0, \cdots, b_m\rangle$ over (Ω, S_T). Hence we have

$$\overline{\Omega(b_0, \cdots, b_m)} \models \Psi(\bar{b}_0, \cdots, \bar{b}_m).$$

Since Ω is S_T-$(\alpha, \mathfrak{r})$-existentially closed, we have

$$\bar{\Omega} \models \exists x_0 \cdots \exists x_m \Psi(x_0, \cdots, x_m).$$

Hence there exist elements $c_0, \cdots, c_m$ in Ω such that

$$\bar{\Omega} \models \Psi(\bar{c}_0, \cdots, \bar{c}_m).$$

Therefore we have

$$\overline{\Omega(c_0, \cdots, c_m)} \models \Psi(\bar{c}_0, \cdots, \bar{c}_m).$$

Since $\Psi(x_0, \cdots, x_m)$ is a characteristic formula of $\langle b_0, \cdots, b_m\rangle$ over (Ω, S_T), $\Omega(b_0, \cdots, b_m)$ and $\Omega(c_0, \cdots, c_m)$ are Ω-isomorphic by the correspondence $b_i \leftrightarrow c_i$ $(i = 0, \cdots, m)$. Since $c_0, \cdots, c_m \in \Omega$, we can immediately obtain

$$\Omega(b_0, \cdots, b_m) = \Omega.$$

This completes the proof.

§3. **Certain properties of algebraic extensions.**

In this section, we shall discuss certain properties of algebraic extensions. Especially, we shall give a certain condition on $\mathcal{M}(S_T)$ under which for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, all algebraically closed algebraic extensions of $\mathfrak{A}$ are $\mathfrak{A}$-isomorphic one another.

The following lemma can be obtained obviously.

Lemma 3.1. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be structures in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$. Assume that $\mathfrak{C}$ is $(\mathfrak{A}, S_T)$-simple. Then $\mathfrak{C}$ is $(\mathfrak{B}, S_T)$-simple.*

Next we shall prove the following two lemmas:

Lemma 3.2. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be structures in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$. Assume that $\mathfrak{C}$ is $(\mathfrak{B}, S_T)$-simple and that $\mathfrak{B}$ is $(\mathfrak{A}, S_T)$-simple. Then $\mathfrak{C}$ is $(\mathfrak{A}, S_T)$-simple.*

Proof. Suppose that $\mathfrak{C}$ is not $(\mathfrak{A}, S_T)$-simple. Then there exists a non-isomorphic $\mathfrak{A}$-homomorphism ϕ of $\mathfrak{C}$ onto some $\mathcal{M}(S_T)$-extension $\mathfrak{D}$ of $\mathfrak{A}$. Since $\mathfrak{B}$ is $(\mathfrak{A}, S_T)$-simple, $\phi|\mathfrak{B}$ is an $\mathfrak{A}$-isomorphism, where $\phi|\mathfrak{B}$ denotes the $\mathfrak{A}$-homomorphism of $\mathfrak{B}$ which is the restriction of ϕ. Hence there exists an $\mathcal{M}(S_T)$-extension $\mathfrak{D}'$ of $\mathfrak{B}$ such that there exists an isomorphism ψ of $\mathfrak{D}$ onto $\mathfrak{D}'$ which satisfies $\psi\phi(b) = b$ (i.e., $\psi(\phi(b)) = b$) for all elements b in $\mathfrak{B}$. Obviously $\psi\phi$ is a non-isomorphic $\mathfrak{B}$-homomorphism of $\mathfrak{C}$ onto $\mathfrak{D}'$. Hence $\mathfrak{C}$ is not $(\mathfrak{B}, S_T)$-simple. This contradicts the assumption.

Lemma 3.3. *Let λ be any ordinal, and let*

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_\xi \subseteq \cdots \qquad (\xi \leq \lambda)$$

be an ascending chain of structures in $\mathcal{M}(S_T)$ such that the following two conditions hold for each non-zero ordinal $\xi \leq \lambda$:

(1) *If $\xi = \eta + 1$, then $\mathfrak{A}_\xi$ is $(\mathfrak{A}_\eta, S_T)$-simple;*

(2) *If ξ is a non-zero limit ordinal, then $\mathfrak{A}_\xi = \bigcup_{\eta<\xi} \mathfrak{A}_\eta$.*

Then $\mathfrak{A}_\lambda$ is $(\mathfrak{A}, S_T)$-simple.

Proof. In order to prove the lemma, it suffices to prove that $\mathfrak{A}_\xi$ is $(\mathfrak{A}, S_T)$-simple, on the assumption that

$$\mathfrak{A}_0,\ \mathfrak{A}_1,\ \cdots,\ \mathfrak{A}_\eta,\ \cdots \qquad (\eta < \xi)$$

are all $(\mathfrak{A}, S_T)$-simple.

(1) If ξ is a successor ordinal, it follows by Lemma 3.2 that $\mathfrak{A}_\xi$ is $(\mathfrak{A}, S_T)$-simple.

(2) If ξ is a limit ordinal, we shall prove it as follows:
If $\xi = 0$, then obviously $\mathfrak{A}_\xi$ is $(\mathfrak{A}, S_T)$-simple. Hence, in the following, we assume that ξ is a non-zero limit ordinal.

Let Θ be an atomic sentence of $L(\mathfrak{A}_\xi)$ such that

$$(+) \qquad S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\xi) \cup \{\Theta\} \ \text{ is consistent.}$$

Since the number of constant symbols occurring in Θ is finite, there exists an ordinal η less than ξ such that Θ is an atomic sentence of $L(\mathfrak{A}_\eta)$. From (+), we have that

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\eta) \cup \{\Theta\} \text{ is consistent.}$$

Hence we have $\Theta \in D^+(\mathfrak{A}_\eta)$, because $\mathfrak{A}_\eta$ is $(\mathfrak{A}, S_T)$-simple. Therefore $\Theta \in D^+(\mathfrak{A}_\xi)$. This means that $\mathfrak{A}_\xi$ is $(\mathfrak{A}, S_T)$-simple.

The following theorem is a direct consequence of Lemma 3.3.

Theorem 3.4. *For any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, every algebraic extension of $\mathfrak{A}$ is $(\mathfrak{A}, S_T)$-simple.*

We say that $\mathcal{M}(S_T)$ has the *conditional amalgamation property* (CAP for short), if the following condition holds: For any three structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ in $\mathcal{M}(S_T)$, if $\mathfrak{B}$ and $\mathfrak{C}$ are primitive algebraic extensions of $\mathfrak{A}$ then there exists an $\mathcal{M}(S_T)$-extension $\mathfrak{D}$ of $\mathfrak{B}$ such that $\mathfrak{C}$ can be $\mathfrak{A}$-embedded into $\mathfrak{D}$.

For any structure $\mathfrak{A}$, the domain of $\mathfrak{A}$ is denoted by $\mathrm{dom}(\mathfrak{A})$.

The following lemma can be easily obtained from the definition of the CAP:

Lemma 3.5. *$\mathcal{M}(S_T)$ has the CAP if and only if for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$ and any two primitive algebraic extensions $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathrm{dom}(\mathfrak{B}) \cap \mathrm{dom}(\mathfrak{C}) = \mathrm{dom}(\mathfrak{A})$,*

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{B}) \cup D^+(\mathfrak{C}) \text{ \textit{is consistent}.}$$

Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be structures for L such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$. If $\mathrm{dom}(\mathfrak{B}) \cap \mathrm{dom}(\mathfrak{C}) = \mathrm{dom}(\mathfrak{A})$, then we denote by $L(\mathfrak{A};\ \mathfrak{B},\ \mathfrak{C})$ the language $L(\mathfrak{B}) \cup L(\mathfrak{C})$.

Theorem 3.6. *Suppose that $\mathcal{M}(S_T)$ has the CAP. Let $\mathfrak{A} \in \mathcal{M}(S_T)$, and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be algebraic extensions of $\mathfrak{A}$. Then there exist an algebraic extension $\mathfrak{U}$ of $\mathfrak{A}$ and $\mathfrak{A}$-embeddings ϕ_i of $\mathfrak{M}_i$ into $\mathfrak{U}$ $(i = 1,\ 2)$ such that*

(1) *$\mathfrak{U}$ is an algebraic extension of $\phi_i(\mathfrak{M}_i)$ $(i = 1,\ 2)$;*

(2) *$\mathfrak{U}$ is generated by $\phi_1(\mathfrak{M}_1)$ and $\phi_2(\mathfrak{M}_2)$.*

Proof. To prove the theorem, it suffices to prove it in the case where $\mathrm{dom}(\mathfrak{M}_1) \cap \mathrm{dom}(\mathfrak{M}_2) = \mathrm{dom}(\mathfrak{A})$. Hence we shall prove the theorem on the assumption that $\mathrm{dom}(\mathfrak{M}_1) \cap \mathrm{dom}(\mathfrak{M}_2) = \mathrm{dom}(\mathfrak{A})$.

Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are algebraic extensions of $\mathfrak{A}$, there exist two chains of suitable lengths

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_\xi \subseteq \cdots \subseteq \mathfrak{A}_\lambda = \mathfrak{M}_1,$$

$$\mathfrak{A} = \mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots \subseteq \mathfrak{B}_\eta \subseteq \cdots \subseteq \mathfrak{B}_\mu = \mathfrak{M}_2$$

which satisfy the following four conditions:

(1) For each successor ordinal $\xi \leq \lambda$, $\mathfrak{A}_\xi$ is a primitive algebraic extension of $\mathfrak{A}_{\xi-1}$;

(2) For each non-zero limit ordinal $\xi \leq \lambda$, $\mathfrak{A}_\xi = \bigcup_{\zeta<\xi} \mathfrak{A}_\zeta$;

(3) For each successor ordinal $\eta \leq \mu$, $\mathfrak{B}_\eta$ is a primitive algebraic extension of $\mathfrak{B}_{\eta-1}$;

(4) For each non-zero limit ordinal $\eta \leq \mu$, $\mathfrak{B}_\eta = \bigcup_{\chi<\eta} \mathfrak{B}_\chi$;

First we shall prove the following

Assertion: There exists a set

$$\{\triangle_{(\xi,\eta)} \mid \xi \leq \lambda,\ \eta \leq \mu\}$$

which satisfies the following five conditions:

(C_1) Each $\triangle_{(\xi,\eta)}$ is a maximal set of atomic sentences of $L(\mathfrak{A};\ \mathfrak{A}_\xi,\ \mathfrak{B}_\eta)$ such that

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\xi) \cup D^+(\mathfrak{B}_\eta) \cup \triangle_{(\xi,\eta)} \quad \text{is consistent;}$$

(C_2) If $\xi = \zeta + 1$, then $\triangle_{(\xi,\eta)} \supseteq \triangle_{(\zeta,\eta)}$;

(C_3) If $\eta = \chi + 1$, then $\triangle_{(\xi,\eta)} \supseteq \triangle_{(\xi,\chi)}$;

(C_4) If ξ is a non-zero limit ordinal, then $\triangle_{(\xi,\eta)} = \bigcup_{\zeta<\xi} \triangle_{(\zeta,\eta)}$;

(C_5) If η is a non-zero limit ordinal, then $\triangle_{(\xi,\eta)} = \bigcup_{\chi<\eta} \triangle_{(\xi,\chi)}$.

In order to prove the existence of such a set $\{\triangle_{(\xi,\eta)} \mid \xi \leq \lambda,\ \eta \leq \mu\}$, we construct $\triangle_{(\xi,\eta)}$ by separating into the following three cases:

(I) The case where $\eta = 0$.

In this case, we put $\triangle_{(\xi,0)} = D^+(\mathfrak{A}_\xi)$. Then $\triangle_{(\xi,0)}$ satisfies (C_1) because $\mathfrak{A}_\xi$ is $(\mathfrak{A}, S_T)$-simple. Obviously $\triangle_{(\xi,0)}$ satisfies (C_2), and $\triangle_{(\xi,0)}$ satisfies (C_4) because $D^+(\mathfrak{A}_\xi) = D^+(\bigcup_{\zeta<\xi} \mathfrak{A}_\zeta) = \bigcup_{\zeta<\xi} D^+(\mathfrak{A}_\zeta)$ for any non-zero limit ordinal ξ not greater than λ. And (C_3), (C_5) are vacuous.

(II) The case where $\xi = 0$.

We put $\triangle_{(0,\eta)} = D^+(\mathfrak{B}_\eta)$ similarly to the above case.

(III) The case where $\xi > 0$ and $\eta > 0$.

We construct $\triangle_{(\xi,\eta)}$ inductively on the lexicographic order of the index set $\{(\xi,\eta) \mid 0 < \xi \leq \lambda,\ 0 < \eta \leq \mu\}$ as follows:

(1) First we construct $\triangle_{(1,1)}$.

Since $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are both primitive algebraic extensions of $\mathfrak{A}$, it follows from the CAP and Lemma 3.5 that

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_1) \cup D^+(\mathfrak{B}_1) \quad \text{is consistent.}$$

Therefore there exists a maximal set $\triangle_{(1,1)}$ of atomic sentences of $L(\mathfrak{A};\ \mathfrak{A}_1,\ \mathfrak{B}_1)$ such that

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_1) \cup D^+(\mathfrak{B}_1) \cup \triangle_{(1,1)} \quad \text{is consistent.}$$

Obviously we have $\triangle_{(1,0)} = D^+(\mathfrak{A}_1) \subseteq \triangle_{(1,1)}$ and $\triangle_{(0,1)} = D^+(\mathfrak{B}_1) \subseteq \triangle_{(1,1)}$.

(2) We assume that $\{\triangle_{(\xi,\eta)} \mid (1,1) \leq (\xi,\eta) < (\sigma,\tau)\}$ has been already constructed so that it satisfies the conditions $(C_1)\sim(C_5)$, where $(1,1) < (\sigma,\tau) \leq (\lambda,\mu)$. In the following, we shall show that $\triangle_{(\sigma,\tau)}$ can be constructed so that $\{\triangle_{(\xi,\eta)} \mid (1,1) \leq (\xi,\eta) \leq (\sigma,\tau)\}$ satisfies the conditions $(C_1)\sim(C_5)$.

(a) The case where $\sigma = \zeta + 1$ and $\tau = \chi + 1$.

By the assumption of induction or by virtue of (I),

$$\Gamma_{(\sigma,\chi)} \overset{\mathrm{d}}{=} S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\sigma) \cup D^+(\mathfrak{B}_\chi) \cup \triangle_{(\sigma,\chi)}$$

is consistent, where "$\overset{\mathrm{d}}{=}$" is read "defined as". Let $\mathfrak{C}^*$ be a model of $\Gamma_{(\sigma,\chi)}$ which contains $\bar{\mathfrak{A}}$. And let $\mathfrak{A}^*_\sigma$ (resp. $\mathfrak{A}^*_\zeta$) be the substructure of $\mathfrak{C}^*|L(\mathfrak{A}_\sigma)$ (resp. $\mathfrak{C}^*|L(\mathfrak{A}_\zeta)$) generated by the interpretations of all constant symbols of $L(\mathfrak{A}_\sigma)$ (resp. $L(\mathfrak{A}_\zeta)$). Then it is easy to see that the domain of $\mathfrak{A}^*_\sigma$ exactly consists of the interpretations of all constant symbols of $L(\mathfrak{A}_\sigma)$, because $\mathfrak{A}^*_\sigma$ is a model of $D^+(\mathfrak{A}_\sigma)$. Similarly the domain of $\mathfrak{A}^*_\zeta$ consists of the interpretations of all constant symbols of $L(\mathfrak{A}_\zeta)$. Now we put $\mathfrak{A}'_\sigma = \mathfrak{A}^*_\sigma|L$ and $\mathfrak{A}'_\zeta = \mathfrak{A}^*_\zeta|L$. Let ϕ be the mapping of $\mathrm{dom}(\mathfrak{A}_\sigma)$ onto $\mathrm{dom}(\mathfrak{A}'_\sigma)$ which maps each element a in $\mathfrak{A}_\sigma$ to a^* in $\mathfrak{A}'_\sigma$, where a^* denotes the interpretation of $\bar{a}$ in $\mathfrak{A}^*_\sigma$. Then it can be easily seen that ϕ is an $\mathfrak{A}$-isomorphism of $\mathfrak{A}_\sigma$ onto $\mathfrak{A}'_\sigma$, because $\mathfrak{A}^*_\sigma$ is a model of $S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\sigma)$ and $\mathfrak{A}_\sigma$ is $(\mathfrak{A}, S_T)$-simple.

Next we let $\mathfrak{B}^*_\chi$ be the substructure of $\mathfrak{C}^*|L(\mathfrak{B}_\chi)$ generated by the interpretations of all constant symbols of $L(\mathfrak{B}_\chi)$. And we put $\mathfrak{B}'_\chi = \mathfrak{B}^*_\chi|L$.

Moreover let $\mathfrak{C}^*_{(\sigma,\chi)}$ (resp. $\mathfrak{C}^*_{(\zeta,\chi)}$) be the substructure of $\mathfrak{C}^*|L(\mathfrak{A};\mathfrak{A}_\sigma,\mathfrak{B}_\chi)$ (resp. $\mathfrak{C}^*|L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\chi)$) generated by the interpretations of all constant symbols of $L(\mathfrak{A};\mathfrak{A}_\sigma,\mathfrak{B}_\chi)$ (resp. $L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\chi)$). And put $\mathfrak{C}_{(\sigma,\chi)} = \mathfrak{C}^*_{(\sigma,\chi)}|L$ and $\mathfrak{C}_{(\zeta,\chi)} = \mathfrak{C}^*_{(\zeta,\chi)}|L$. Then obviously we have

(∗) $\quad \mathfrak{C}_{(\sigma,\chi)}$ (resp. $\mathfrak{C}_{(\zeta,\chi)}$) is generated by $\mathfrak{A}'_\sigma$ and $\mathfrak{B}'_\chi$ (resp. $\mathfrak{A}'_\zeta$ and $\mathfrak{B}'_\chi$).

Since $\mathfrak{A}_\sigma$ is a primitive algebraic extension of $\mathfrak{A}_\zeta$, there exist some elements $a_0, \cdots, a_m$ in $\mathfrak{A}_\sigma$ such that

$$\mathfrak{A}_\sigma = \mathfrak{A}_\zeta(a_0, \cdots, a_m) \quad \text{and} \quad m < \mathfrak{r}.$$

Hence by using the $\mathfrak{A}$-isomorphism ϕ, we get

$$\mathfrak{A}'_\sigma = \mathfrak{A}'_\zeta(a^*_0, \cdots, a^*_m),$$

where $a^*_0, \cdots, a^*_m$ are the interpretations of $\bar{a}_0, \cdots, \bar{a}_m$ in $\mathfrak{A}^*_\sigma$. And hence by using (∗), we get

$$(**)\qquad \mathfrak{C}_{(\sigma,\chi)} = \mathfrak{C}_{(\zeta,\chi)}(a_0^*, \cdots, a_m^*).$$

Since $\triangle_{(\sigma,\chi)}$ is a maximal set of atomic sentences of $L(\mathfrak{A};\ \mathfrak{A}_\sigma,\ \mathfrak{B}_\chi)$ such that

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\sigma) \cup D^+(\mathfrak{B}_\chi) \cup \triangle_{(\sigma,\chi)} \text{ is consistent,}$$

it is easy to see that $\mathfrak{C}_{(\sigma,\chi)}$ is $(\mathfrak{A}, S_T)$-simple. Hence by Lemma 3.1, $\mathfrak{C}_{(\sigma,\chi)}$ is $(\mathfrak{C}_{(\zeta,\chi)}, S_T)$-simple. Therefore by combining with $(**)$, we obtain the following:

$$(***)\qquad \mathfrak{C}_{(\sigma,\chi)} \text{ is a primitive algebraic extension of } \mathfrak{C}_{(\zeta,\chi)}.$$

Once more, by the assumption of induction or by virtue of (II),

$$\Gamma_{(\zeta,\tau)} \overset{\mathrm{d}}{=} S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\zeta) \cup D^+(\mathfrak{B}_\tau) \cup \triangle_{(\zeta,\tau)}$$

is consistent. Let $\mathfrak{D}^*$ be a model of $\Gamma_{(\zeta,\tau)}$ which contains $\bar{\mathfrak{A}}$. And let $\mathfrak{D}^*_{(\zeta,\tau)}$ (resp. $\mathfrak{D}^*_{(\zeta,\chi)}$) be the substructure of $\mathfrak{D}^*|L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\tau)$ (resp. $\mathfrak{D}^*|L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\chi)$) generated by the interpretations of all constant symbols of $L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\tau)$ (resp. $L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\chi)$). And we put $\mathfrak{D}_{(\zeta,\tau)} = \mathfrak{D}^*_{(\zeta,\tau)}|L$ and $\mathfrak{D}_{(\zeta,\chi)} = \mathfrak{D}^*_{(\zeta,\chi)}|L$. Then in a way similar to that of the proof of $(***)$, we can prove that $\mathfrak{D}_{(\zeta,\tau)}$ is a primitive algebraic extension of $\mathfrak{D}_{(\zeta,\chi)}$.

Now we put

$$\Gamma_{(\zeta,\chi)} = S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\zeta) \cup D^+(\mathfrak{B}_\chi) \cup \triangle_{(\zeta,\chi)}.$$

Then it is obvious that both $\mathfrak{C}^*_{(\zeta,\chi)}$ and $\mathfrak{D}^*_{(\zeta,\chi)}$ are models of $\Gamma_{(\zeta,\chi)}$, and each of them is generated by the interpretations of all constant symbols of $L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\chi)$. Since $\triangle_{(\zeta,\chi)}$ is a maximal set of atomic sentences of $L(\mathfrak{A};\mathfrak{A}_\zeta,\mathfrak{B}_\chi)$ such that

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\zeta) \cup D^+(\mathfrak{B}_\chi) \cup \triangle_{(\zeta,\chi)} \text{ is consistent,}$$

it is easy to see that there exists an isomorphism ψ of $\mathfrak{C}^*_{(\zeta,\chi)}$ onto $\mathfrak{D}^*_{(\zeta,\chi)}$. Hence there exists an $\mathcal{M}(S_T)$-extension $\mathfrak{C}^{\#}_{(\sigma,\chi)}$ of $\mathfrak{D}_{(\zeta,\chi)}$ such that there exists an $\mathfrak{A}$-isomorphism of $\mathfrak{C}_{(\sigma,\chi)}$ onto $\mathfrak{C}^{\#}_{(\sigma,\chi)}$ which is an extension of ψ. Therefore by the CAP, there exists an $\mathcal{M}(S_T)$-extension $\mathfrak{C}^{\#\#}$ of $\mathfrak{C}^{\#}_{(\sigma,\chi)}$ such that $\mathfrak{D}_{(\zeta,\tau)}$ can be $\mathfrak{D}_{(\zeta,\chi)}$-embedded into $\mathfrak{C}^{\#\#}$. Hence it is easy to see that

$$\Gamma_{(\sigma,\chi)} \cup \Gamma_{(\zeta,\tau)}$$
$$(\text{i.e., } S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\sigma) \cup D^+(\mathfrak{B}_\tau) \cup \triangle_{(\sigma,\chi)} \cup \triangle_{(\zeta,\tau)})$$

is consistent. Therefore there exists a maximal set $\triangle_{(\sigma,\tau)}$ of atomic sentences of $L(\mathfrak{A};\mathfrak{A}_\sigma,\mathfrak{B}_\tau)$ such that

$$S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{A}_\sigma) \cup D^+(\mathfrak{B}_\tau) \cup \triangle_{(\sigma,\chi)} \cup \triangle_{(\zeta,\tau)} \cup \triangle_{(\sigma,\tau)}$$

is consistent. Thus $\{\triangle_{(\xi,\eta)} \mid (1,1) \leq (\xi,\eta) \leq (\sigma,\tau)\}$ can be constructed, and obviously it satisfies the conditions $(\mathrm{C}_1) \sim (\mathrm{C}_5)$.

(b) The case where $\sigma = \zeta + 1$ and τ is a limit ordinal.

Put $\triangle_{(\sigma,\tau)} = \bigcup_{\chi<\tau} \triangle_{(\sigma,\chi)}$. Then we can easily prove that $\{\triangle_{(\xi,\eta)} \mid (1,1) \leq (\xi,\eta) \leq (\sigma,\tau)\}$ satisfies the conditions $(\mathrm{C}_1) \sim (\mathrm{C}_5)$.

(c) The case where σ is a limit ordinal and $\tau = \chi + 1$.

Similar to the case above.

(d) The case where both σ and τ are limit ordinals.

Since

$$\bigcup_{\zeta<\sigma} \triangle_{(\zeta,\tau)} = \bigcup_{\zeta<\sigma} \bigcup_{\chi<\tau} \triangle_{(\zeta,\chi)} = \bigcup_{\chi<\tau} \bigcup_{\zeta<\sigma} \triangle_{(\zeta,\chi)} = \bigcup_{\chi<\tau} \triangle_{(\sigma,\chi)},$$

if we put $\triangle_{(\sigma,\tau)} = \bigcup_{\zeta<\sigma} \triangle_{(\zeta,\tau)}$, then we can easily prove that $\{\triangle_{(\xi,\eta)} \mid (1,1) \leq (\xi,\eta) \leq (\sigma,\tau)\}$ satisfies the conditions $(\mathrm{C}_1) \sim (\mathrm{C}_5)$. This completes the proof of the Assertion.

It follows from the Assertion that

$$\Gamma_{(\lambda,\mu)} \stackrel{\mathrm{d}}{=} S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{M}_1) \cup D^+(\mathfrak{M}_2) \cup \triangle_{(\lambda,\mu)}$$

is consistent. Hence there exists a model $\mathfrak{U}^\#$ of $\Gamma_{(\lambda,\mu)}$ which contains $\bar{\mathfrak{A}}$. Let $\mathfrak{U}^*$ be the substructure of $\mathfrak{U}^\#|L(\mathfrak{A};\mathfrak{M}_1,\mathfrak{M}_2)$ generated by the interpretations of all constant symbols of $L(\mathfrak{A};\mathfrak{M}_1,\mathfrak{M}_2)$, and let $\mathfrak{U} = \mathfrak{U}^*|L$.

In the following, we shall prove that $\mathfrak{U}$ is a structure desired in the theorem.

Let $\mathfrak{M}_1^*$ (resp. $\mathfrak{M}_2^*$) be the substructure of $\mathfrak{U}^*|L(\mathfrak{M}_1)$ (resp. $\mathfrak{U}^*|L(\mathfrak{M}_2)$) generated by the interpretations of all constant symbols of $L(\mathfrak{M}_1)$ (resp. $L(\mathfrak{M}_2)$), and let $\mathfrak{M}_1' = \mathfrak{M}_1^*|L$ and $\mathfrak{M}_2' = \mathfrak{M}_2^*|L$. Then $\mathfrak{M}_i'$ and $\mathfrak{M}_i$ are $\mathfrak{A}$-isomorphic ($i = 1,\ 2$), because $\mathfrak{M}_i^*$ is a model of $S_T \cup D(\mathfrak{A}) \cup D^+(\mathfrak{M}_i)$ and $\mathfrak{M}_i$ is $(\mathfrak{A}, S_T)$-simple.

For each ordinal $\xi \leq \lambda$, let $\mathfrak{C}_\xi^*$ be the substructure of $\mathfrak{U}^*|L(\mathfrak{A};\mathfrak{A}_\xi,\mathfrak{M}_2)$ generated by the interpretations of all constant symbols of $L(\mathfrak{A};\mathfrak{A}_\xi,\mathfrak{M}_2)$, and let $\mathfrak{C}_\xi = \mathfrak{C}_\xi^*|L$. Then obviously we have

$$\mathfrak{M}_2' = \mathfrak{C}_0 \subseteq \mathfrak{C}_1 \subseteq \cdots \subseteq \mathfrak{C}_\xi \subseteq \cdots \subseteq \mathfrak{C}_\lambda = \mathfrak{U}.$$

Now we shall show that this chain has the following two properties:

(P_1) If ξ is a non-zero limit ordinal, then $\mathfrak{C}_\xi = \bigcup_{\zeta<\xi} \mathfrak{C}_\zeta$;

(P_2) If $\xi = \zeta + 1$, then $\mathfrak{C}_\xi$ is a primitive algebraic extension of $\mathfrak{C}_\zeta$.

Since it is obvious that the chain has (P_1), we shall show that the chain has (P_2).

Now we assume that $\xi = \zeta + 1$. Then in almost the same way as we proved $(***)$ in the proof of the Assertion, we can prove that $\mathfrak{C}_\xi$ is a primitive algebraic extension of $\mathfrak{C}_\zeta$.

Thus we can obtain that $\mathfrak{U}$ is an algebraic extension of $\mathfrak{M}'_2$. Similarly, we can obtain that $\mathfrak{U}$ is an algebraic extension of $\mathfrak{M}'_1$. Now obviously $\mathfrak{U}$ is an algebraic extension of $\mathfrak{A}$.

Moreover $\mathfrak{U}$ is generated by $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$, because $\mathfrak{U} = \mathfrak{U}^*|L$ and $\mathfrak{U}^*$ is the substructure of $\mathfrak{U}^\#|L(\mathfrak{A}; \mathfrak{M}_1, \mathfrak{M}_2)$ generated by the interpretations of all constant symbols of $L(\mathfrak{A}; \mathfrak{M}_1, \mathfrak{M}_2)$, and because $\mathfrak{M}'_i = \mathfrak{M}^*_i|L$ and $\mathfrak{M}^*_i$ is the substructure of $\mathfrak{U}^*|L(\mathfrak{M}_i)$ generated by the interpretations of all constant symbols of $L(\mathfrak{M}_i)$ $(i = 1,\ 2)$. This completes the proof of the theorem.

As important application of the above theorem, we shall show the following two theorems:

Theorem 3.7. *Suppose that $\mathcal{M}(S_T)$ has the CAP. Let $\mathfrak{A} \in \mathcal{M}(S_T)$. Let Ω be an algebraically closed algebraic extension of $\mathfrak{A}$, and let $\mathfrak{M}$ be any algebraic extension of $\mathfrak{A}$. Then there exists an $\mathfrak{A}$-embedding ϕ of $\mathfrak{M}$ into Ω such that Ω is an algebraic extension of $\phi(\mathfrak{M})$.*

Proof. By using Theorem 3.6, it is easy to see that there exist an algebraic extension $\mathfrak{U}$ of Ω and an $\mathfrak{A}$-embedding ϕ of $\mathfrak{M}$ into $\mathfrak{U}$ such that $\mathfrak{U}$ is an algebraic extension of $\phi(\mathfrak{M})$. Since Ω is algebraically closed, we have $\mathfrak{U} = \Omega$. This completes the proof.

Theorem 3.8. *Suppose that $\mathcal{M}(S_T)$ has the CAP. Let $\mathfrak{A} \in \mathcal{M}(S_T)$, and let Ω_1 and Ω_2 be any two algebraically closed algebraic extensions of $\mathfrak{A}$. Then Ω_1 and Ω_2 are $\mathfrak{A}$-isomorphic.*

Proof. By Theorem 3.7, there exists an $\mathfrak{A}$-embedding ϕ of Ω_2 into Ω_1 such that Ω_1 is an algebraic extension of $\phi(\Omega_2)$. Since $\phi(\Omega_2)$ is algebraically closed, we have $\Omega_1 = \phi(\Omega_2)$. Hence Ω_1 and Ω_2 are $\mathfrak{A}$-isomorphic.

§4. **Algebraic closures.**

First we shall introduce a new notion of algebraic closures, which is a generalization of the usual one in the commutative field theory. Our main purpose of this section is to give some conditions on $\mathcal{M}(S_T)$ each of which is necessary and sufficient for the existence of algebraic closures.

Let $\mathfrak{A}$ and Ω be structures in $\mathcal{M}(S_T)$. We say that Ω is an *algebraic closure* of $\mathfrak{A}$, if the following two conditions hold:

(1) Ω is an algebraically closed algebraic extension of $\mathfrak{A}$;

(2) For any algebraic extension $\mathfrak{B}$ of $\mathfrak{A}$, there exists an $\mathfrak{A}$-embedding ϕ of $\mathfrak{B}$ into Ω such that Ω is an algebraic extension of $\phi(\mathfrak{B})$.

The above definition of an algebraic closure is somewhat different from the usual one. But it seems to the author that the above definition is a better one in general classes.

Theorem 4.1. *Let $\mathfrak{A}$ be a structure in $\mathcal{M}(S_T)$. If there exist algebraic closures of $\mathfrak{A}$, then they are $\mathfrak{A}$-isomorphic one another.*

Proof. Similar to the proof of Theorem 3.8.

Let $\mathfrak{A}$ be any structure for L. We denote by $|\mathfrak{A}|$ the cardinality of the domain of $\mathfrak{A}$.

Lemma 4.2. *Assume that for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists a cardinal α such that $|\mathfrak{B}| \le \alpha$ for all algebraic extensions $\mathfrak{B}$ of $\mathfrak{A}$. Then $\mathcal{M}(S_T)$ has the AEDP.*

Proof. Let $\mathfrak{A}$ be any structure in $\mathcal{M}(S_T)$. Then by the assumption in the lemma, there exists a cardinal α such that $|\mathfrak{B}| \le \alpha$ for all algebraic extensions $\mathfrak{B}$ of $\mathfrak{A}$. Let L^* be a first order language formed from L by adding new constant symbols indexed by all ordinals less than α. And let β be the successor cardinal of the cardinality of the set of all atomic formulas of L^*. In the following, we shall prove that the AED-degree of $\mathfrak{A}$ is less than or equal to β.

Now let $\mathfrak{B}$ be any algebraic extension of $\mathfrak{A}$, and let $c_0, \cdots, c_m$ be any elements in an $\mathcal{M}(S_T)$-extension of $\mathfrak{B}$ such that $\langle c_0, \cdots, c_m\rangle$ is algebraic over $\mathfrak{B}$. It suffices to show that $\langle c_0, \cdots, c_m\rangle$ is β-definably algebraic over $\mathfrak{B}$.

Let Σ be the set of all atomic formulas $\Theta(x_0, \cdots, x_m)$ of $L(\mathfrak{B})$ such that

$$\overline{\mathfrak{B}(c_0, \cdots, c_m)} \models \Theta(\bar{c}_0, \cdots, \bar{c}_m),$$

and let

$$\Theta_0(x_0, \cdots, x_m),\ \Theta_1(x_0, \cdots, x_m),\ \cdots,\ \Theta_\xi(x_0, \cdots, x_m),\ \cdots \quad (\xi < \mu)$$

be an enumeration of all formulas of Σ. Then $\mu < \beta$, because $|\mathfrak{B}| \le \alpha$. Let

$$\Psi(x_0, \cdots, x_m) = \bigwedge_{\xi<\mu} \Theta_\xi(x_0, \cdots, x_m).$$

Then $\Psi(x_0, \cdots, x_m)$ is a $(\beta, \mathfrak{r})$-formula over $L(\mathfrak{B})$. And obviously

$$\overline{\mathfrak{B}(c_0, \cdots, c_m)} \models \Psi(\bar{c}_0, \cdots, \bar{c}_m).$$

Let $d_0, \cdots, d_m$ be arbitrary elements in any $\mathcal{M}(S_T)$-extension of $\mathfrak{B}$ such that

$$\overline{\mathfrak{B}(d_0, \cdots, d_m)} \models \Psi(\bar{d}_0, \cdots, \bar{d}_m).$$

Then it is easy to see that there exists a $\mathfrak{B}$-homomorphism ϕ of $\mathfrak{B}(c_0, \cdots, c_m)$ onto $\mathfrak{B}(d_0, \cdots, d_m)$ which maps c_i to d_i $(i = 0, \cdots, m)$. Since $\mathfrak{B}(c_0, \cdots, c_m)$ is $(\mathfrak{B}, S_T)$-simple, ϕ is a $\mathfrak{B}$-isomorphism. Therefore $\Psi(x_0, \cdots, x_m)$ is a characteristic formula of $\langle c_0, \cdots, c_m\rangle$ over $(\mathfrak{B}, S_T)$. Thus $\langle c_0, \cdots, c_m\rangle$ is β-definably algebraic over $\mathfrak{B}$. This completes the proof.

We are now in the position to state the following main theorem:

Theorem 4.3. *The following five conditions are equivalent:*

(1) *For any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an algebraic closure of $\mathfrak{A}$;*

(2) *For any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an $\mathcal{M}(S_T)$-extension $\mathfrak{B}$ of $\mathfrak{A}$ such that every algebraic extension of $\mathfrak{A}$ can be $\mathfrak{A}$-embedded into $\mathfrak{B}$;*

(3) *$\mathcal{M}(S_T)$ has the CAP, and for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists a structure $\mathfrak{M}$ for L such that every algebraic extension of $\mathfrak{A}$ can be embedded into $\mathfrak{M}$;*

(4) *$\mathcal{M}(S_T)$ has the CAP, and for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists a cardinal α such that $|\mathfrak{B}| \leq \alpha$ for all algebraic extensions $\mathfrak{B}$ of $\mathfrak{A}$;*

(5) *$\mathcal{M}(S_T)$ has the AEDP and the CAP.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) follows immediately from Lemma 4.2.

(5) $\Rightarrow$ (1) is obvious from Theorem 2.5 and 3.7.

In the rest of this section, we shall give some theorems, which may be convenient to examine the existence of algebraic closures.

We say that $\mathcal{M}(S_T)$ has the *half-conditional amalgamation property* (HCAP for short), if the following condition holds: For any three structures $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ in $\mathcal{M}(S_T)$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{C}$ is a primitive algebraic extension of $\mathfrak{A}$ then there exists an $\mathcal{M}(S_T)$-extension $\mathfrak{D}$ of $\mathfrak{B}$ such that $\mathfrak{C}$ can be $\mathfrak{A}$-embedded into $\mathfrak{D}$.

Lemma 4.4. *Assume that $\mathcal{M}(S_T)$ has the HCAP. Let Ω be a structure in $\mathcal{M}(S_T)$ which is algebraically closed, and let $\mathfrak{A}$ be a substructure of Ω. Then any primitive algebraic extension of $\mathfrak{A}$ can be $\mathfrak{A}$-embedded into Ω.*

Proof. Let $\mathfrak{B}$ be any primitive algebraic extension of $\mathfrak{A}$. Then we can put

$$\mathfrak{B} = \mathfrak{A}(b_0, \cdots, b_m),$$

where $\langle b_0, \cdots, b_m \rangle$ is algebraic over $\mathfrak{A}$. Since $\mathcal{M}(S_T)$ has the HCAP, there exists an $\mathcal{M}(S_T)$-extension Ω^* of Ω such that an $\mathfrak{A}$-embedding ϕ of $\mathfrak{A}(b_0, \cdots, b_m)$ into Ω^* exists. We put $c_i = \phi(b_i)$ $(i = 0, \cdots, m)$. Then by Theorem 1.4, $\Omega(c_0, \cdots, c_m)$ is a splitting extension of Ω. Hence there exists an Ω-homomorphism ψ of $\Omega(c_0, \cdots, c_m)$ onto Ω. Put $\psi' = \psi | \mathfrak{A}(c_0, \cdots, c_m)$. Then ψ' is an $\mathfrak{A}$-homomorphism of $\mathfrak{A}(c_0, \cdots, c_m)$ into Ω. Hence $\psi'\phi$ is an $\mathfrak{A}$-homomorphism of $\mathfrak{A}(b_0, \cdots, b_m)$ into Ω. Since $\mathfrak{A}(b_0, \cdots, b_m)$ is a primitive algebraic extension of $\mathfrak{A}$, $\mathfrak{A}(b_0, \cdots, b_m)$ is $(\mathfrak{A}, S_T)$-simple. Hence $\psi'\phi$ is an $\mathfrak{A}$-embedding of $\mathfrak{A}(b_0, \cdots, b_m)$ into Ω. This completes the proof.

Theorem 4.5. *Assume that $\mathcal{M}(S_T)$ has the HCAP. Let Ω be a structure in $\mathcal{M}(S_T)$ which is algebraically closed, and let $\mathfrak{A}$ be a substructure of Ω. Then every algebraic extension of $\mathfrak{A}$ can be $\mathfrak{A}$-embedded into Ω.*

Proof. Let $\mathfrak{B}$ be any algebraic extension of $\mathfrak{A}$. Then there exists an ascending chain

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_\xi \subseteq \cdots$$

of substructures of $\mathfrak{B}$ which satisfies the following three conditions:

(1) $\mathfrak{B} = \mathfrak{A}_\lambda$ for some ordinal λ;

(2) For each successor ordinal $\xi \leq \lambda$, $\mathfrak{A}_\xi$ is a primitive algebraic extension of $\mathfrak{A}_{\xi-1}$;

(3) For each non-zero limit ordinal $\xi \leq \lambda$, $\mathfrak{A}_\xi = \bigcup_{\eta<\xi} \mathfrak{A}_\eta$.

Now we inductively construct a chain

$$\mathfrak{A} = \mathfrak{A}_0^* \subseteq \mathfrak{A}_1^* \subseteq \cdots \subseteq \mathfrak{A}_\xi^* \subseteq \cdots \quad (\xi \leq \lambda)$$

of substructures of Ω, and a chain

$$\theta_0 \subseteq \theta_1 \subseteq \cdots \subseteq \theta_\xi \subseteq \cdots \quad (\xi \leq \lambda)$$

of isomorphisms so that they satisfy the following three conditions:

(1) θ_0 is the identity isomorphism of $\mathfrak{A}_0$ onto $\mathfrak{A}_0^*$;

(2) If $\xi = \eta + 1$, then θ_ξ is an isomorphism of $\mathfrak{A}_\xi$ onto $\mathfrak{A}_\xi^*$ which is an extension of θ_η;

(3) If ξ is a non-zero limit ordinal, then

$$\mathfrak{A}_\xi^* = \bigcup_{\eta<\xi} \mathfrak{A}_\eta^* \text{ and } \theta_\xi = \bigcup_{\eta<\xi} \theta_\eta.$$

It is obvious from Lemma 4.4 that such two chains can be constructed.

Now we put $\mathfrak{B}^* = \mathfrak{A}_\lambda^*$ and $\theta = \theta_\lambda$. Then clearly $\mathfrak{B}^* \subseteq \Omega$ and θ is an $\mathfrak{A}$-isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}^*$. Hence $\mathfrak{B}$ can be $\mathfrak{A}$-embedded into Ω. This completes the proof.

Theorem 4.6. *Assume that $\mathcal{M}(S_T)$ has the HCAP. If for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an $\mathcal{M}(S_T)$-extension of $\mathfrak{A}$ which is algebraically closed, then for any structure $\mathfrak{B}$ in $\mathcal{M}(S_T)$, there exists an algebraic closure of $\mathfrak{B}$.*

Proof. This theorem can be immediately obtained from Theorem 4.5 and (2) $\Rightarrow$ (1) of Theorem 4.3.

If for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an $\mathcal{M}(S_T)$-extension $\mathfrak{B}$ of $\mathfrak{A}$ such that every $\mathcal{M}(S_T)$-extension of $\mathfrak{B}$ is a splitting extension of $\mathfrak{B}$, we say that $\mathcal{M}(S_T)$ is absolutely retractively complete.

As a direct consequence of Theorem 4.6 and (3) $\Rightarrow$ (1) of Theorem 1.4, we can obtain the following:

Corollary 4.7. *Assume that $\mathcal{M}(S_T)$ has the HCAP. If $\mathcal{M}(S_T)$ is absolutely retractively complete, then for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an algebraic closure of $\mathfrak{A}$.*

Remark 4.8. Let V be a variety. Then V is absolutely retractively complete if and only if it is residually small (Cf. [5]).

§5. **Additional notes.**

In our discussion above, we have not discussed whether each element of an algebraic extension of a structure $\mathfrak{A}$ is algebraic over $\mathfrak{A}$ or not. In this section, we shall give some equivalent conditions each of which implies that every element of an algebraic extension of a structure $\mathfrak{A}$ is algebraic over $\mathfrak{A}$.

Let $\mathfrak{A}$ and $\mathfrak{C}$ be structure in $\mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{C}$. We say that $\mathfrak{C}$ is $(\mathfrak{A}, S_T)$-strongly simple, if every structure $\mathfrak{B}$ satisfying $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$ is $(\mathfrak{A}, S_T)$-simple.

The following theorem can be easily obtained from the definition of an algebraic extension and Lemma 3.1.

Theorem 5.1. *Let* $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}(S_T)$ *such that* $\mathfrak{A} \subseteq \mathfrak{B}$. *If* $\mathfrak{B}$ *is* $(\mathfrak{A}, S_T)$*-strongly simple, then* $\mathfrak{B}$ *is an algebraic extension of* $\mathfrak{A}$.

We say that $\mathcal{M}(S_T)$ has the *strong-simpleness property* (SSP for short), if for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, every algebraic extension of $\mathfrak{A}$ is $(\mathfrak{A}, S_T)$-strongly simple.

Now we can easily obtain the following theorem:

Theorem 5.2. *The following four conditions are equivalent:*

(1) $\mathcal{M}(S_T)$ *has the SSP;*

(2) *For any three structures* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *in* $\mathcal{M}(S_T)$ *such that* $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$, *if* $\mathfrak{C}$ *is an algebraic extension of* $\mathfrak{A}$ *then every element of* $\mathfrak{C}$ *is algebraic over* $\mathfrak{B}$*;*

(3) *For any four structures* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ *in* $\mathcal{M}(S_T)$ *such that* $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}$, *if* $\mathfrak{D}$ *is an algebraic extension of* $\mathfrak{A}$ *then* $\mathfrak{C}$ *is an algebraic extension of* $\mathfrak{B}$*;*

(4) *For any three structures* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *in* $\mathcal{M}(S_T)$ *such that* $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$, *if* $\mathfrak{C}$ *is an algebraic extension of* $\mathfrak{A}$ *then* $\mathfrak{B}$ *is an algebraic extension of* $\mathfrak{A}$.

Next we shall prove the following:

Theorem 5.3. *Assume that* $\mathcal{M}(S_T)$ *has the SSP. Let* $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}(S_T)$. *Then the following two conditions are equivalent:*

(1) $\mathfrak{B}$ *is an algebraically closed algebraic extension of* $\mathfrak{A}$*;*

(2) $\mathfrak{B}$ *is a maximal* $(\mathfrak{A}, S_T)$*-strongly simple extension of* $\mathfrak{A}$, *— i.e.,* $\mathfrak{B}$ *is an* $(\mathfrak{A}, S_T)$*-strongly simple extension of* $\mathfrak{A}$ *and no proper* $\mathcal{M}(S_T)$*-extension of* $\mathfrak{B}$ *is* $(\mathfrak{A}, S_T)$*-strongly simple.*

Proof of (1) $\Rightarrow$ (2). Suppose the condition (1) holds. Then by the assumption, $\mathfrak{B}$ is $(\mathfrak{A}, S_T)$-strongly simple. Now suppose that $\mathfrak{B}$ is not a maximal $(\mathfrak{A}, S_T)$-strongly simple extension of $\mathfrak{A}$. Then there exists a proper $\mathcal{M}(S_T)$-extension $\mathfrak{C}$ of $\mathfrak{B}$ which is $(\mathfrak{A}, S_T)$-strongly simple. Take an element c such that $c \in \mathfrak{C}$ and $c \notin \mathfrak{B}$. Then $\mathfrak{B}(c)$ is $(\mathfrak{A}, S_T)$-simple. Hence $\mathfrak{B}(c)$ is $(\mathfrak{B}, S_T)$-simple. Therefore $\mathfrak{B}(c)$ is a proper primitive algebraic extension of $\mathfrak{B}$. This contradicts the fact that $\mathfrak{B}$ is algebraically closed. Therefore $\mathfrak{B}$ is a maximal $(\mathfrak{A}, S_T)$-strongly simple extension of $\mathfrak{A}$.

Proof of (2) $\Rightarrow$ (1). Suppose the condition (2) holds. Then by Theorem 5.1, $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$. We must prove that $\mathfrak{B}$ is algebraically closed. Now suppose that $\mathfrak{B}$ is not algebraically closed. Then there exists a proper algebraic extension $\mathfrak{D}$ of $\mathfrak{B}$. Obviously $\mathfrak{D}$ is an algebraic extension of $\mathfrak{A}$. Therefore by the assumption, $\mathfrak{D}$ is $(\mathfrak{A}, S_T)$-strongly simple extension of $\mathfrak{A}$. This contradicts the condition (2). Therefore $\mathfrak{B}$ is algebraically clossed.

As a direct consequence of (1) $\Rightarrow$ (3) in Theorem 5.2, we have the following:

Theorem 5.4. *Assume that* $\mathcal{M}(S_T)$ *has the SSP, and let* $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}(S_T)$. *If* $\mathfrak{B}$ *is an algebraically closed algebraic extension of* $\mathfrak{A}$, *then* $\mathfrak{B}$ *is a minimal algebraically closed extension of* $\mathfrak{A}$,*— i.e.,* $\mathfrak{B}$ *is an algebraically closed extension of* $\mathfrak{A}$ *and no proper substructure of* $\mathfrak{B}$ *containing* $\mathfrak{A}$ *is algebraically closed.*

In order to consider the counterpart of this theorem, we shall first give the following:

Lemma 5.5. *Assume that $\mathcal{M}(S_T)$ has the HCAP. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathcal{M}(S_T)$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is a proper substructure of $\mathfrak{C}$. If $\mathfrak{C}$ is a minimal algebraically closed extension of $\mathfrak{A}$, then there exists a proper primitive algebraic extension of $\mathfrak{B}$ which is contained in $\mathfrak{C}$.*

Proof. Since $\mathfrak{B}$ is not algebraically closed, there exists a proper primitive algebraic extension $\mathfrak{D}$ of $\mathfrak{B}$. By Lemma 4.4, there exists a $\mathfrak{B}$-embedding ϕ of $\mathfrak{D}$ into $\mathfrak{C}$. Obviouly, $\phi(\mathfrak{D})$ is a proper primitive algebraic extension of $\mathfrak{B}$, and $\phi(\mathfrak{D}) \subseteq \mathfrak{C}$.

Now we have the following:

Theorem 5.6. *Assume that $\mathcal{M}(S_T)$ has the HCAP, and let $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}(S_T)$. If $\mathfrak{B}$ is a minimal algebraically closed extension of $\mathfrak{A}$, then $\mathfrak{B}$ is an algebraically closed algebraic extension of $\mathfrak{A}$, moreover $\mathfrak{B}$ is an algebraic closure of $\mathfrak{A}$.*

Proof. It suffices to show, on the supposition in the theorem, that $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$. If $\mathfrak{B} = \mathfrak{A}$, it is clear that $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$. If $\mathfrak{B} \neq \mathfrak{A}$, it can be easily shown from the above lemma and the definition of an algebraic extension that $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$.

Let $\mathfrak{A}$ be a structure in $\mathcal{M}(S_T)$. We say that $\mathfrak{A}$ is injective in $\mathcal{M}(S_T)$, if the following condition holds: For any two structures $\mathfrak{B}$ and $\mathfrak{C}$ in $\mathcal{M}(S_T)$, if there exists a homomorphism θ of $\mathfrak{B}$ into $\mathfrak{A}$ and an embedding ϕ of $\mathfrak{B}$ into $\mathfrak{C}$, then there exists a homomorphism ψ of $\mathfrak{C}$ into $\mathfrak{A}$ such that $\psi(\phi(b)) = \theta(b)$ for all elements b in $\mathfrak{B}$.

If for any structure $\mathfrak{A}$ in $\mathcal{M}(S_T)$, there exists an $\mathcal{M}(S_T)$-extension of $\mathfrak{A}$ which is injective in $\mathcal{M}(S_T)$, then we say that $\mathcal{M}(S_T)$ is injectively complete (Cf. [1; P.107]).

Suppose that $\mathcal{M}(S_T)$ is injectively complete. Then it is easy to see that $\mathcal{M}(S_T)$ has the HCAP and the SSP, and satisfies the condition (2) of Theorem 4.3. Hence in this case, Theorems 4.3, 5.2, 5.3, 5.4, and 5.6 can be effectively applied to $\mathcal{M}(S_T)$.

As is well-known, the varieties of Abelian groups, Boolean algebras, and semilattices are injectively complete universal classes. Moreover it is easy to see that the class of partially ordered sets and the class of totally ordered sets (defined by the ordering relation $\leq$) are universal and injectively complete. Hence Theorems 4.3, 5.2, 5.3, 5.4, and 5.6 can be effectively applied to these varieties and classes.

References

[1] G. Grätzer: Universal Algebra, D. Van Nostrand Company, Princeton.

[2] R. S. Pierce: Introduction to the Theory of Abstract Algebras, Holt, Rinehart and Winston, New York, 1968.

[3] K. Shoda: Zur Theorie der algebraischen Erweiterungen, Osaka Mathematical Journal Vol.4, No.2 (1952), 133–143.

[4] K. Shoda: Berichtigungen zu den Arbeiten über die Erweiterungen algebraischer Systeme, Osaka Mathematical Journal Vol.9 (1957), 239–240.

[5] W. Taylor: Residually small varieties, Algebra Universalis Vol.2 (1972), 33–53.

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