# ANTI-HOMOGENEOUS PARTITIONS OF A TOPOLOGICAL SPACE

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Abstract. We prove the consistency (modulo supercompact) of a negative answer to the Cantor discontinuum partition problem (i.e., some Hausdorff compact space cannot be partitioned to two sets not containing a closed copy of Cantor discontinuum). In this model we have CH. Without CH we get consistency results using a pcf assumption, close relatives of which are necessary for such results; so we try to deal with equiconsistency.

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## Annotated Content

§1 General spaces: consistency from strong assumptions

[We define $X^* \rightarrow (Y^*)^1_\theta$ for topological spaces $X^*, Y^*$. Then starting with a Hausdorff space $Y^*$ with $\theta$ points such that any set of $< \sigma < \theta$ members is discrete and $\kappa = \kappa^{<\kappa} \in (\theta, \lambda)$ and appropriate $\mathcal{A} \subseteq [\lambda]^\theta$ such that any two members has intersection $< \sigma$, we force appropriate $X^*$. We then show that the assumption holds under appropriate pcf assumption and finish with some improvements, varying the topological and set theoretical assumptions.]

§2 Consistency from supercompact, with clopen basis

[We deal here with the set theoretic assumption. We show that the assumptions can be gotten from supercompact for the case we agree to have CH, relying on earlier consistency results. We also investigate the order of consistency between relatives of "$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary with no stationary subset in $I[\lambda]$", and existence of non trivial (in an appropriate sense) of $\mathcal{A} \subseteq [\lambda]^\kappa, [A_1 \neq A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma]$.]

§3 Equi-consistency

[We show that some versions of the topological question and suitable combinatorial questions are equi-consistents. See [Sh:108], [HJSh:249], [Sh:460], [HJSh:697]. We then indicate the changes needed for the not necessarily closed subspace case colouring by more colours and other spaces. For discussion see [Sh:666],§1.]

§4 Decomposing families of almost disjoint functions

## §1 General spaces: consistency from strong assumptions

In our main theorem, 1.2, we give set theoretic sufficient conditions for being able to force counterexamples to the Cantor discontinuum partition problem, possibly replacing the Cantor discontinuum by any other space. It has a version for spaces with clopen basis. Then (in claim 1.4) we connect this to pcf theory: after easy forcing the assumptions of Theorem 1.2 can be proved, if we start with a suitable (strong) pcf assumption (whose status is not known). Then in claim 1.7 we deal with variants of the theorem, weakening the topological and/or set theoretic assumptions. Further variants are discussed in the end of the section ($T_3$ spaces without clopen basis and variants of 1.4) This continues Juhasz Hajnal Shelah [HJSH:249]. [Sh 460] if $2^{\aleph_0} > \aleph_\omega$ then it is doubtful if $(\exists X)(X \rightarrow (\text{Cantor discontinuum})^1_{\aleph_0})$ is consistent; e.g. if $|\mathfrak{a}| \leq \aleph_\alpha \Rightarrow |\text{pcf}(\mathfrak{a})| \leq \aleph_{\alpha+732}$ or if $\mathbf{V} = \mathbf{V}_1^{\mathbb{Q}}$ where $\mathbb{Q}$ is a c.c.c. forcing making the continuum $\geq \beth_\omega^{\mathbf{V}_1}$, then there is no such space. On the case with $\geq$ cf$(\theta)$ colours see 4.17. Bill Weiss has proved the existence of such partitions under $\mathbf{V} = \mathbf{L}$.

Recall

**1.1 Definition.** Let $n \in [1, \omega)$ (though we concentrate on $n = 1$) .
1) We say $X^* \rightarrow (Y^*)^n_\theta$, <u>if</u> $X^*, Y^*$ are topological spaces and for every $h : [X^*]^n \rightarrow \theta$ there is a closed subspace $Y$ of $X^*$, homeomorphic to $Y^*$ such that $h \restriction [Y]^n$ is constant (if $n = 1$ we may write $h : X^* \rightarrow \theta$ and $h \restriction Y$).
2) If we omit the "closed", we shall write $\rightarrow_w$ instead of $\rightarrow$. We write $(Y^*)^n_{<\theta}$ meaning: for every $h : [X^*]^n \rightarrow \gamma < \theta$. We use $\nrightarrow, \nrightarrow_w$ for the negations.

**1.2 Theorem.** *Assume*

$(A)$ $(i)$ $\lambda > \kappa > \theta > \sigma \geq \aleph_0$ *and* $\kappa = \kappa^{<\kappa}$
$(ii)$ $(\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)$ *and* $\kappa > \theta^* \geq \theta$

$(B)_1$ $\mathcal{A} \subseteq [\lambda]^\theta$ *and*
$A_1 \neq A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma$

$(B)_2$ $\mathcal{A}$ *is* $(< \kappa)$*-free (or* $\kappa$*-free) which means: if* $\mathcal{A}' \subseteq \mathcal{A}, |\mathcal{A}'| < \kappa$ <u>*then*</u> *for some list* $\{A_\varepsilon : \varepsilon < \zeta\}$ *of* $\mathcal{A}'$*, for every* $\varepsilon < \zeta$ *we have*
$$|A_\varepsilon \cap \bigcup_{\xi<\varepsilon} A_\xi| < \sigma$$

$(C)$ *if* $F : \lambda \rightarrow [\lambda]^{\leq\kappa}$, <u>*then*</u> *some* $A \in \mathcal{A}$ *(or just some* $A$ *such that* $(\exists A')(A \subseteq A'$ & $|A| = \theta$ & $A' \in \mathcal{A})$ *is* $F$*-free which means*

$(*)$ *for* $\alpha \neq \beta$ *from* $A$ *we have* $\alpha \notin F(\beta)$

$(D)$ $Y^*$ *is a Hausdorff space with set of points* $\theta$ *and a basis* $\mathcal{B} = \{b_i : i < \theta^*\}$

$(E)$ *if* $Y$ *is a subset of* $Y^*$ *with* $< \sigma$ *points,* <u>*then*</u> $Y$ *is a discrete subset, i.e. there is a sequence of open (for* $Y^*$*) pairwise disjoint sets* $\langle \mathcal{U}_y : y \in Y \rangle$*, such that* $y \in \mathcal{U}_y$
*(if* $\sigma = \aleph_0$ *this follows from being Hausdorff).*

<u>*Then*</u>
*1) for some* $\kappa$*-complete* $\kappa^+$*-c.c. forcing notion* $\mathbb{P}$*, in* $\mathbf{V}^{\mathbb{P}}$ *there is* $X^*$ *such that:*

$(a)$ $X^*$ *is a Hausdorff topological space with* $\lambda$ *points and basis of size* $|\mathcal{A}| + \theta^*$

(b) $X^* \to (Y^*)^1_{<\,\mathrm{cf}(\theta)}$ *(that is, if* $X^* = \bigcup\limits_{i<i(*)} X_i$ *where* $i(*) < \mathrm{cf}(\theta)$*)* *then* *some closed subspace* $Y$ *of* $X^*$ *homeomorphic to* $Y^*$ *is included in some single* $X_i$ *(i.e.* $(\exists i)(Y \subseteq X_i)$*)).*

*2) If in addition* $Y^*$ *has a clopen basis* $\mathcal{B}$ *of cardinality* $\le \theta^*$ *such that the union of any* $< \sigma$ *members of* $\mathcal{B}$ *is clopen,* *then* *we can require that* $X^*$ *has a clopen basis.*

*1.3 Remark.* We may define the conditions historically (see [ShSt 258], [RoSh 733], so put only the required conditions). Then we can allow $\theta^* = \kappa$, but see 1.7.

*Proof.* We write the proof for part (1) and indicate the changes for part (2). Without loss of generality

$\bigotimes_1$ $(\forall \alpha < \beta < \lambda)(\forall B \in [\lambda]^{<\lambda})(\exists^{\ge \kappa^+} A \in \mathcal{A})[\{\alpha,\beta\} \subseteq A \ \& \ A \cap B \subseteq \{\alpha,\beta\}]$.

[Why? As we can use $\big\{\{2\alpha : \alpha \in A\} : A \in \mathcal{A}\big\}$, without loss of generality $\bigcup\{A : A \in \mathcal{A}\} = \{2\alpha : \alpha < \lambda\}$ and choose $A_{\alpha,\beta,\gamma} \in [\lambda]^\theta$ for $\alpha < \beta < \gamma < \lambda$ such that $\{\alpha,\beta\} \subseteq A_{\alpha,\beta,\gamma}$ and $\langle A_{\alpha,\beta,\gamma} \backslash \{\alpha,\beta\} : \alpha < \beta < \gamma < \lambda\rangle$ are pairwise disjoint subsets of $\{2\alpha+1 : \alpha < \lambda\}$, each of cardinality $\theta$ and we replace $\mathcal{A}$ by $\mathcal{A}^* =: \mathcal{A} \cup \{A_{\alpha,\beta,\gamma} : \alpha < \beta < \gamma < \lambda\}$. Now clauses (A), (D), (E) are not affected. Clearly clause $(B)_1$ holds (i.e. $\mathcal{A}^* \subseteq [\lambda]^\theta$ and $A \ne B \in \mathcal{A}^* \Rightarrow |A \cap B| < \sigma$). Also clause (C) is inherited by any extension of the original $\mathcal{A}$. Lastly for clause $(B)_2$, if $\mathcal{A}' \subseteq \mathcal{A}^*, |\mathcal{A}'| < \kappa$, let $\langle A_\zeta : \zeta < \zeta^*\rangle$ be a list of $\mathcal{A}' \cap \mathcal{A}$ as guaranteed by $(B)_2$ and let $\langle A_\zeta : \zeta \in [\zeta^*, \zeta^* + |\mathcal{A}' \backslash \mathcal{A}|)\rangle$ list with no repetitions $\mathcal{A}' \backslash \mathcal{A}$, now check.]

$\bigotimes_2$ $\mathcal{B}$ is a basis of $Y^*$ of cardinality $\theta^*$, and for part (2), $\mathcal{B}$ is as there.

[Why? Straight.]

Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$ and $\mathcal{B} = \{b_i : i < \theta^*\}$.
We define a forcing notion $\mathbb{P}$:

$p \in \mathbb{P}$ has the form $p = (u, u_*, v, v_*, \bar{w}) = (u^p, u^p_*, v^p, v^p_*, \bar{w}^p)$ such that:

($\alpha$) $u_* \subseteq u \in [\lambda]^{<\kappa}$

($\beta$) $v_* \subseteq v \in [\lambda^*]^{<\kappa}$

($\gamma$) $\bar{w} = \bar{w}^p = \langle w_{\zeta,i} : \zeta \in v_* \text{ and } i < \theta^*\rangle = \langle w^p_{\zeta,i} : \zeta \in v_*, i < \theta^*\rangle$

($\delta$) $w_{\zeta,i} \subseteq u_*$ and
$b_i \cap b_j = \emptyset \Rightarrow w_{\zeta,i} \cap w_{\zeta,j} = \emptyset$; this is toward being Hausdorff

($\varepsilon$) $\zeta \in v_* \Rightarrow A_\zeta \subseteq u$

($\zeta$) letting $A^p_\zeta =: \cup\{w_{\zeta,i} : i < \theta^*\} \cap A_\zeta$ for $\zeta \in v^p_*$, it has cardinality $\theta$ and for simplicity even order type $\theta$, of course $A^p_\zeta \subseteq u^p_*$ and for some sequence $\langle \gamma^p_{\zeta,j} : j < \theta\rangle$ listing its members with no repetitions we have
$w^p_{\zeta,i} \cap A^p_\zeta = \{\gamma^p_{\zeta,j} : j < \theta \text{ and } j \in b_i\}$

($\eta$) if $\zeta \in v^p_*, i < \theta^*$ and $\xi \in v^p_*$ <u>then</u> the set $\mathcal{U}^p_{\zeta,\xi,i}$ is an open subset (for part (2), clopen subset) of the space $Y^*$ where $\mathcal{U}^p_{\zeta,\xi,i} =: \{j < \theta : \gamma^p_{\xi,j} \in w^p_{\zeta,i}\}$.

$\bigoplus$ <u>convention</u> if $\zeta \in \lambda^* \backslash v^p_*$ we stipulate $w^p_{\zeta,i} = \emptyset$.

The <u>order</u> is: $p \le q$ iff $u^p \subseteq u^q, u^p_* = u^q_* \cap u^p, v^p \subseteq v^q, v^p_* = v^q_* \cap v^p$ and $\zeta \in v^p_* \Rightarrow w^p_{\zeta,i} = w^q_{\zeta,i} \cap u^p$.

Clearly

$(*)_0$ $\mathbb{P}$ is a partial order.

What is the desired space in $\mathbf{V}^{\mathbb{P}}$? We define a $\mathbb{P}$-name $\underset{\sim}{X}^*$ of a topological space as follows:

$\boxtimes$ set of points $\bigcup\{u^p_* : p \in \underset{\sim}{G}_{\mathbb{P}}\}$

The topology is defined by the following basis:
$\{\bigcap\limits_{\ell<n} \underset{\sim}{\mathcal{U}}_{\zeta_\ell,i_\ell} : n < \omega$ and $\zeta_\ell < \lambda^*, i_\ell < \theta^*$ for $\ell < n\}$ where

$\boxtimes_1$ $\underset{\sim}{\mathcal{U}}_{\zeta,i}[\underset{\sim}{G}_{\mathbb{P}}] = \cup\{w^p_{\zeta,i} : p \in \underset{\sim}{G}_{\mathbb{P}}$ satisfies $\zeta \in v^p_*\}$; so

$\Vdash$ "if $\zeta \in \lambda^* \setminus \cup \{v^p_* : p \in \underset{\sim}{G}_{\mathbb{P}}\}$ and $i < \theta^*$ then $\mathcal{U}_{\zeta,i}[\underset{\sim}{G}_{\mathbb{P}}] = \emptyset$"
(for part (2), also their compliments and hence their Boolean combinations).

Now we shall prove

$(*)_1$ for $\alpha < \lambda, \zeta < \lambda^*$ and $p \in \mathbb{P}$ we have
- $(i)$ $p \Vdash$ "$\alpha \in \underset{\sim}{X}^*$" iff $\alpha \in u^p_*$ and
- $(ii)$ $p \Vdash$ "$\alpha \notin \underset{\sim}{X}^*$" iff $\alpha \in u^p_\alpha \setminus u^p_*$ and
- $(iii)$ $\Vdash_{\mathbb{P}}$ "$\lambda^* = \cup\{v^p : p \in \underset{\sim}{G}_{\mathbb{P}}\}$
- $(iv)$ if $\zeta \in v^p_*$ and $i < \theta$ then $p \Vdash$ "$\underset{\sim}{\mathcal{U}}_{\zeta,i} \cap u^p_* = w^p_{\zeta,i}$"
- $(v)$ $\{p \in \mathbb{P} : \alpha \in u^p\}$ is a dense open subset of $\mathbb{P}$
- $(vi)$ $\{p \in \mathbb{P} : \zeta \in v^p\}$ is a dense open subset of $\mathbb{P}$.

[Why? Easy, e.g. let $p \in \mathbb{P}, \xi \in \lambda^* \setminus v^p$ and $\alpha \neq \beta$ are from $\lambda \setminus u^p$, we define $q \in \mathbb{P}$ by: $u^q = u^p \cup \{\alpha, \beta\}, u^q_* = u^p_* \cup \{\alpha\}, v^q = v^p \cup \{\xi\}, v^p_* = v^q_*$ and $w^q_{\zeta,i} = w^p_{\zeta,i}$ for $\zeta \in v^q_* = v^p_*$ and $i < \theta^*$. Easily $\mathbb{P} \models$ "$p \leq q$", $\beta \in u^q \setminus u^q_*, \alpha \in u^q_*$ and $\xi \in v^q \setminus v^q_*$.]

$(*)_2$ $\mathbb{P}$ is $\kappa$-complete, in fact if $\langle p_\varepsilon : \varepsilon < \delta\rangle$ is increasing in $\mathbb{P}$ and $\delta < \kappa$ <u>then</u> $p = \bigcup\limits_{\varepsilon<\delta} p_\varepsilon$ is an upper bound where $u^p = \bigcup\limits_{\varepsilon<\delta} u^{p_\varepsilon}, u^p_* = \bigcup\limits_{\varepsilon<\delta} u^{p_\varepsilon}_*, v^p = \bigcup\limits_{\varepsilon<\delta} v^{p_\varepsilon}, v^p_* = \bigcup\limits_{\varepsilon<\delta} v^{p_\varepsilon}_*$ and $w^p_{\zeta,i} = \cup\{w^{p_\varepsilon}_{\zeta,i} : \varepsilon < \delta$ satisfies $\zeta \in v^{p_\varepsilon}_*\}$ for $\zeta \in v^{p_\delta}_*$.
[Why? Straight.]

$(*)_3$ $\mathbb{P}' = \{p \in \mathbb{P} :$ if $\zeta < \lambda^*$ and $|A_\zeta \cap u^p| \geq \sigma$ then $\zeta \in v^p\}$ is a dense subset of $\mathbb{P}$

[why? for any $p \in \mathbb{P}$ we define by induction[1] on $\varepsilon \leq \sigma^+ : p_\varepsilon \in \mathbb{P}$ is increasing continuous with $\varepsilon$. Let $p_0 = p$, if $p_\varepsilon$ is defined, we define $p_{\varepsilon+1}$ by

$$v^{p_{\varepsilon+1}} = \{\zeta < \lambda^* : \zeta \in v^{p_\varepsilon} \text{ or } |A_\zeta \cap u^{p_\varepsilon}| \geq \sigma\}$$

$$v^{p_{\varepsilon+1}}_* = v^{p_\varepsilon}_* (= v^p_*)$$

$$u^{p_{\varepsilon+1}} = u^{p_\varepsilon} \cup \bigcup\{A_\zeta : \zeta \in v^{p_{\varepsilon+1}}\}$$

[1] of course, if $\delta^* < \kappa$ is a limit ordinal such that $\mathrm{cf}(\delta^*) \neq \mathrm{cf}(\sigma)$ then we may use $\langle p_\varepsilon : \varepsilon \leq \delta^*\rangle$ and $p_{\delta^*}$ is as required

$$u_*^{p_{\varepsilon+1}} = u_*^{p_\varepsilon}(= u_*^p)$$

$$w_{\zeta,i}^{p_{\varepsilon+1}} \text{ is: } w_{\zeta,i}^{p_\varepsilon}(= w_{\zeta,i}^p) \text{ if } \zeta \in v_*^{p_\varepsilon}, i < \theta^*$$

(and there are no other cases).
By assumption $(A)(ii)$, the set $v^{p_{\varepsilon+1}}$ has cardinality $< \kappa$, so $p_{\varepsilon+1}$ belongs to $\mathbb{P}$.
Clearly $p_\varepsilon \leq p_{\varepsilon+1} \in \mathbb{P}$. Now for $\varepsilon$ limit let $p_\varepsilon = \bigcup_{\xi<\xi} p_\xi$. Clearly we can carry the definition.
Now $p_{\sigma^+} = \bigcup_{\varepsilon<\sigma} p_\varepsilon$ is as required because if $A_\zeta \in \mathcal{A}, |A_\zeta \cap u^{p_{\sigma^+}}| \geq \sigma$ then for some $\varepsilon < \sigma^+$ we have $|A_\zeta \cap u^{p_\varepsilon}| \geq \sigma$ hence $\zeta \in v^{p_{\varepsilon+1}}$ hence $A_\zeta \subseteq u^{p_{\varepsilon+1}} \subseteq u^{p_{\sigma^+}}$.
Note that we use here $\sigma^+ < \kappa$ which follows from $\sigma < \theta < \kappa$.]

$(*)_4$ $\mathbb{P}$ satisfies the $\kappa^+$-c.c.
[Why? Let $p_j \in \mathbb{P}$ for $j < \kappa^+$, without loss of generality $p_j \in \mathbb{P}'$ for $j < \kappa^+$. Now by the $\Delta$-system lemma for some unbounded $S \subseteq \kappa^+$ and $v^\otimes \in [\lambda^*]^{<\kappa}, u^\otimes \in [\lambda]^{<\kappa}$ we have:
$j \in S \Rightarrow v^\otimes \subseteq v^{p_j}$ & $u^\otimes \subseteq u^{p_j}$ and $\langle v^{p_j} \backslash v^\otimes : j \in S\rangle$ are pairwise disjoint and $\langle u^{p_j} \backslash u^\otimes : j \in S\rangle$ are pairwise disjoint. Without loss of generality $\text{otp}(v^{p_j})$, $\text{otp}(u^{p_j})$ are constant for $j \in S$ and any two $p_i, p_j$ are isomorphic over $v^\otimes, u^\otimes$ (if not clear see 1.7).
Now for $j_1, j_2 \in S$ the condition $p_{j_1}, p_{j_2}$ are compatible because of the following $(*)_5$]

$(*)_5$ assume $p^1, p^2 \in \mathbb{P}$ satisfies
- $(i)$ $v_*^{p^1} \cap (v^{p^2} \backslash v_*^{p^2}) = \emptyset$ and $u_*^{p^1} \cap (u^{p^2} \backslash u_*^{p^2}) = \emptyset$
- $(ii)$ $v_*^{p^2} \cap (v^{p^1} \backslash v_*^{p^1}) = \emptyset$ and $u_*^{p^2} \cap (u^{p^1} \backslash u_*^{p^1}) = \emptyset$
- $(iii)$ if $\zeta \in v_*^{p^1} \cap v_*^{p^2}$ then $A_\zeta^{p^1} = A_\zeta^{p^2}$ and
$i < \theta^* \Rightarrow w_{\zeta,i}^{p^1} \cap (u^{p^1} \cap u^{p^2}) = w_{\zeta,i}^{p^2} \cap (u^{p^1} \cap u^{p^2})$
- $(iv)_1$ if $\zeta \in v_*^{p^1} \backslash v_*^{p^2}$ then $|A_\zeta \cap u^{p^2}| < \sigma$ or just $|A_\zeta^{p^1} \cap u^{p^2}| < \sigma$
- $(iv)_2$ similarly[2] for $\zeta \in v_*^{p^2} \backslash v_*^{p^1}$ we have $|A_\zeta^{p^2} \cap u^{p^1}| < \sigma$.

<u>Then</u> there is $q \in \mathbb{P}$ such that:
- $(a)$ $v^q = v^{p^1} \cup v^{p^2}$
- $(b)$ $v_*^q = v_*^{p^1} \cup v_*^{p^2}$
- $(c)$ $u^q = u^{p^1} \cup u^{p^2}$
- $(d)$ $u_*^q = u_*^{p^1} \cup u_*^{p^2}$
- $(e)$ $p^1 \leq q$ and $p^2 \leq q$.

[Why? To define the condition $q$ by clauses (a)-(d) above we just have to define $w_{\zeta,i}^q$ (for $\zeta \in v_*^q = v_*^{p^1} \cup v_*^{p^2}$ and $i < \theta^*$). If $\zeta \in v_*^{p^1} \cap v_*^{p^2}$ we let $w_{\zeta,i}^q = w_{\zeta,i}^{p^1} \cup w_{\zeta,i}^{p^2}$ for $i < \theta^*$ (clearly $\ell \in \{1,2\} \Rightarrow w_{\zeta,i}^q \cap u^{p^\ell} = w_{\zeta,i}^{p^\ell}$); this will be enough to guarantee $\mathbb{P} \models$ "$p^1 \leq q$ & $p^2 \leq q$"

[2]note that if $p^1, p^2 \in \mathbb{P}'$, then clauses $(iv)_1, (iv)_2$ holds automatically, but the proof of 1.7 which is very similar to the proof of 1.2, uses this version.

provided that we have $q \in \mathbb{P}$ and that for $\ell = 1, 2$ we shall define $w^q_{\zeta,i}$ for $\zeta \in v^{p^{3-\ell}}_* \backslash v^{p^\ell}_*$ such that $w^q_{\zeta,i} \cap u^{p^{3-\ell}}_* = w^{p^{3-\ell}}_{\zeta,i}$ and $w^q_{\zeta,i} \subseteq u^q_*$; so only clauses $(\delta) + (\eta)$ in the definition of membership in $\mathbb{P}$ are problematic.

Now for $\ell = 1, 2$, let $v^{p^\ell}_* \backslash v^{p^{3-\ell}}_*$ be listed as $\langle \Upsilon(\varepsilon, \ell) : \varepsilon < \varepsilon^*_\ell \rangle$ with no repetitions such that $B^\ell_\varepsilon =: A^{p^\ell}_{\Upsilon(\varepsilon,\ell)} \cap (\bigcup\limits_{\xi<\varepsilon} A^{p^\ell}_{\Upsilon(\xi,\ell)} \cup u^{p^{3-\ell}}_*)$ is of cardinality $< \sigma$ for each $\epsilon < \epsilon^*_\ell$.

[Why possible? By the assumption $(B)_2$ and clause $(iv)_\ell$ above.]

Now for each $\zeta \in v^{p^{3-\ell}}_* \backslash v^{p^\ell}_*$ we choose by induction on $\varepsilon \leq \varepsilon^*_\ell$ the sequence $\langle w^{\ell,\varepsilon}_{\zeta,i} : i < \theta^* \rangle$ such that

1) $w^{\ell,\varepsilon}_{\zeta,i} \subseteq u^{p^{3-\ell}}_* \cup \bigcup\limits_{\xi<\varepsilon} A^{p^\ell}_{\Upsilon(\xi,\ell)} (\subseteq u^{p^\ell}_* \cup u^{p^{3-\ell}}_*)$.

2) $w^{\ell,\varepsilon}_{\zeta,i}$ is increasing continuous with $\varepsilon$.

3) $w^{\ell,0}_{\zeta,i} = w^{p^{3-\ell}}_{\zeta,i}$.

4) $\varepsilon' < \varepsilon \Rightarrow w^{\ell,\varepsilon}_{\zeta,i} \cap (u^{p^{3-\ell}} \cup \bigcup\limits_{\xi<\varepsilon'} A^{p^\ell}_{\Upsilon(\xi,\ell)}) = w^{\ell,\varepsilon'}_{\zeta,i}$.

5) If $i < j < \theta^*$ and $b_i \cap b_j = \emptyset$ (hence $w^{p^{3-\ell}}_{\zeta,i} \cap w^{p^{3-\ell}}_{\zeta,j} = \emptyset$) then $w^{\ell,\varepsilon}_{\zeta,i} \cap w^{\ell,\varepsilon}_{\zeta,j} = \emptyset$.

6) For each $i < \theta^*$ the set $\{j < \theta : \gamma^{p^\ell}_{\Upsilon(\varepsilon,\ell),j} \in w^{\ell,\varepsilon+1}_{\zeta,i}\}$ is an open set in $Y^*$ (for part (2) of 1.2: clopen).]

If we succeed then we let $w^q_{\zeta,i}$ be $w^{\ell,\varepsilon^*_\ell}_{\zeta,i}$ for $\ell \in \{1, 2\}, \zeta \in v^{p^{3-\ell}}_*$; clearly by clauses (3) + (4) in the construction for $\varepsilon' = 0$ we have $w^q_{\zeta,i} \cap u^{p^{3-\ell}}_* = w^{p^{3-\ell}}_{\zeta,i}$ and by clause (1) in the construction we have $w^q_{\zeta,i} \subseteq u^q_*$ and clause $(\delta)$ in the definition of $q \in \mathbb{P}$ holds by (5), and clause $(\eta)$ by (6) in the construction. So let us carry the induction.

For $\varepsilon = 0$ use clause (3) and for limit $\varepsilon$ take unions (see clause (2)). Suppose we have defined for $\varepsilon$ and let us define for $\varepsilon + 1$. By an assumption above $B^\ell_\varepsilon = A^{p^\ell}_{\Upsilon(\varepsilon,\ell)} \cap (\bigcup\limits_{\xi<\varepsilon} A^{p^\ell}_{\Upsilon(\xi,\ell)} \cup u^{p^{3-\ell}}_*)$ has cardinality $< \sigma$ and so $Z^\ell_\varepsilon =: \{j < \theta : \gamma^{p^\ell}_{\Upsilon(\varepsilon,\ell),j} \in B^\ell_\varepsilon\}$ is a subset of $\theta$ of cardinality $< \sigma$. Hence, by assumption (E) of the theorem 1.2, we can find a sequence $\langle t_j(\varepsilon, \ell) : j \in Z^\ell_\varepsilon \rangle$ such that: $t_j(\varepsilon, \ell) < \theta^*$ and $j \in b_{t_j(\varepsilon,\ell)}$ for $j \in Z^\ell_\varepsilon$ and $\langle b_{t_j(\varepsilon,\ell)} : j \in Z^\ell_\varepsilon \rangle$ is a sequence of pairwise disjoint open subsets of $Y^*$.

Lastly, we let

$$\begin{aligned} w^{\ell,\varepsilon+1}_{\zeta,i} = w^{\ell,\varepsilon}_{\zeta,i} \cup \big\{\gamma^{p^\ell}_{\Upsilon(\varepsilon,\ell),s} :\ & \text{for some } j \in Z^\ell_\varepsilon \text{ we have} \\ & \gamma^{p^\ell}_{\Upsilon(\varepsilon,\ell),j} \in w^{\ell,\varepsilon}_{\zeta,i} \text{ and} \\ & s \in b_{t_j(\varepsilon,\ell)}\big\}. \end{aligned}$$

Clearly this is O.K. and we are done. Remember that the union of $< \sigma$ set from $\mathcal{B}$ is clopen for part (2) of 1.2.]

So we have proved $(*)_5$ hence also $(*)_4$. We sometime need a stronger version of $(*)_5$

$(*)_6$ in $(*)_5$ if in addition for $\ell = 1, 2$ we are given $Z_\ell \subseteq u^{p^\ell}_* \backslash u^{p^{3-\ell}}_*$ such that $(\forall \zeta \in v^{p^\ell}_*)[|A^{p^\ell}_\zeta \cap Z_\ell| < \sigma]$ <u>then</u> we may add to the conclusion:

$(f)$ $\ell \in \{1, 2\}, \zeta \in v^{p^{3-\ell}}_* \backslash v^{p^\ell}_*, i < \theta^* \Rightarrow w^q_{\zeta,i} \cap Z_\ell = \emptyset$.
More generally

$(f)^+$ if $g_\ell : (v_*^{p^{3-\ell}} \backslash v_*^{p^\ell}) \times \theta^* \times Z_\ell \to \{0,1\}$ satisfies $g(\zeta, j_1, \gamma) = 1 = g(\zeta, j_2, \gamma) \Rightarrow b_{j_1} \cap b_{j_2} \neq \emptyset$ for $\ell = 1, 2$, <u>then</u> we can add
$\ell \in \{1,2\}, \zeta \in v_*^{p^{3-\ell}} \backslash v_*^{p^\ell}, i < \theta^*, \gamma \in Z_\ell \Rightarrow [\gamma \in w^q_{\zeta,i} \leftrightarrow g_\ell(\zeta, i, \gamma) = 1]$.

[Why? During the proof of $(*)_5$ when for each $\zeta \in v_*^{p^{3-\ell}} \backslash v_*^{p^\ell}$, we define $\langle w^{\ell,\varepsilon}_{\zeta,i} : i < \theta^* \rangle$ by induction on $\varepsilon$ we add
(7) $i < \theta^*, \gamma \in Z_\ell \cap (u^{p^{3-\ell}} \cup \bigcup\limits_{\xi<\varepsilon} A^{p^\ell}_{\Upsilon(\xi,\ell)})$ implies $\gamma \in w^{\ell,\varepsilon}_{\zeta,i} \leftrightarrow g_\ell(\zeta, i, \gamma) = 1$.
In the proof when we use assumption (E), instead of using $B^\ell_\varepsilon = A^{p^\ell}_{\zeta(\varepsilon,\ell)} \cap (\bigcup\limits_{\xi<\varepsilon} A^{p^\ell}_{\zeta(\xi,\ell)} \cup u^{p^{3-\ell}})$ we use $B^\ell_\varepsilon = A^{p^\ell}_{\zeta(\varepsilon,\ell)} \cap (\bigcup\limits_{\xi<\varepsilon} A^{p^\ell}_{\zeta(\xi,\ell)} \cup u^{p^{3-\ell}} \cup Z_\ell)$ which still has cardinality $< \sigma$. In the end if $Z_\ell \nsubseteq \bigcup\limits_{\xi<\varepsilon^*_\ell} A^{p^\ell}_\zeta \cup u_*^{p^{3-\ell}}$ we let $w^q_{\zeta,i} =: w^{\ell,\varepsilon^*_\ell+1}_{\zeta,i} =: w^{\ell,\varepsilon^*_\ell}_{\zeta,i} \cup \{\gamma \in Z_\ell : g(\zeta, i, \gamma) = 1\}$.]

Now we come to the main point

$(*)_7$ in $\mathbf{V}^{\mathbb{P}}$, if $i(*) <$ cf$(\theta)$ and $X^* = \bigcup\limits_{i<i(*)} X_i$ <u>then</u> some closed $Y \subseteq X^*$ is homeomorphic to $Y^*$.

[Why? Toward contradiction assume $p^* \in \mathbb{P}$ and $p^* \Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{X}_i : i < i(*) \rangle$ is a counterexample to $(*)_7$". So in particular $p^* \Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{X}_i : i < i(*) \rangle$ is a partition of $\underset{\sim}{X}^*$, i.e., of $\bigcup \{u^p_* : p \in \underset{\sim}{G}_{\mathbb{P}}\}$" and let $\underset{\sim}{X}_{i(*)} =: \lambda \backslash \cup \{\underset{\sim}{X}_i : i < i(*)\}$.

For each $\alpha < \lambda$ let $\langle (p_{\alpha,j}, i_{\alpha,j}) : j < \kappa \rangle$ be such that:

(i) $\langle p_{\alpha,j} : j < \kappa \rangle$ is a maximal antichain[3] of $\mathbb{P}'$ above $p^*$

(ii) $p_{\alpha,j} \Vdash_{\mathbb{P}}$ "$\alpha \in \underset{\sim}{X}_{i_{\alpha,j}}$" and $\alpha \in u^{p_{\alpha,j}}$, so $i_{\alpha,j} \le i(*)$ and $\alpha \in u_*^{p_{\alpha,j}} \Leftrightarrow i_{\alpha,j} < i(*)$

(iii) $p^* \le p_{\alpha,j}$

(iv) $p_{\alpha,j} \in \mathbb{P}'$.

[Why can we demand $\alpha \in u^{p_{\alpha,j}}$? By clause (v) of $(*)_1$. Why not $j < j_\alpha \le \kappa$? For notational simplicity and as above any member $p$ of $\mathbb{P}$ there are $\kappa$ pairwise contradictory members (by the information on the set of points of $\underset{\sim}{X}^*$, see $(*)_1$ above).]

Now we define a function $F$, Dom$(F) = \lambda$ as follows:

$$F(\alpha) \text{ is } \bigcup \{u^{p_{\alpha,j}} : j < \kappa\} \subseteq [\lambda]^{\le\kappa}.$$

So by clause (C) of the assumption of 1.2 we can find $\zeta(*) < \lambda^*$ and $A \subseteq A_{\zeta(*)}$ of order type $\theta$ such that: if $\alpha \neq \beta$ are from $A$ then $\alpha \notin F(\beta)$. Let $A = \{\beta_\varepsilon : \varepsilon < \theta\}$ with no repetitions. We can find $\zeta(**) \in \lambda^* \backslash \cup \{v^{p_{\alpha,j}} : j < \kappa, \alpha \in A_{\zeta(*)}\} \backslash \{\zeta(*)\}$ the set $A_{\zeta(**)}$ is disjoint to $\cup \{v^{p_{\alpha,j}} : j < \kappa, \alpha \in A_{\zeta(*)}\}$, let $\langle \gamma_j : j < \theta \rangle$ be an increasing sequence of members of $A_{\zeta(*))}$ and let $p^+ \in \mathbb{P}$ be defined by: $u^{p^+} = u^{p^*} \cup A_{\zeta(**)}, u_*^{p^+} = u_*^{p^*} \cup \{\gamma_j : j < \theta\}, v^{p^+} = v^{p^*} \cup \{\zeta(**)\}, v_*^{p^+} = v_*^{p^+} \cup \{\zeta(**)\}, w^{p^+}_{\zeta,i} = w^{p^*}_{\zeta,i}$ if $w^{p^*}_{\zeta,i}$ is well defined and $w^{p^+}_{\zeta(**),i} = \{\gamma_j : j \in b_i\}$. Clearly $p \le p^+$. Now we shall choose by induction on $\varepsilon \le \theta, p_\varepsilon, g_\varepsilon$ and if $\varepsilon < \theta$ also $j_\varepsilon < \kappa$ such that:

(a) $p_\varepsilon \in \mathbb{P}, p^+ \le p_\varepsilon$

[3] if we demand only $\in \mathbb{P}$ then we should increase $F(\alpha)$ accordingly

$$u^{p_\varepsilon} = u^{p^+} \cup \bigcup_{\varepsilon(1)<\varepsilon} u^{p_{\beta_{\varepsilon(1)},j_{\varepsilon(1)}}}$$
$$u_*^{p_\varepsilon} = u_*^{p^+} \cup \bigcup_{\varepsilon(1)<\varepsilon} u_*^{p_{\beta_{\varepsilon(1)},j_{\varepsilon(1)}}}$$
$$v^{p_\varepsilon} = v^{p^+} \cup \bigcup_{\varepsilon(1)<\varepsilon} v^{p_{\beta_{\varepsilon(1)},j_{\varepsilon(1)}}}$$
$$v_*^{p_\varepsilon} = v_*^{p^+} \cup \bigcup_{\varepsilon(1)<\varepsilon} v_*^{p_{\beta_{\varepsilon(1)},j_{\varepsilon(1)}}}$$

(of course $p_0 = p^+$)

($b$) $p_{\varepsilon+1} \geq p_{\beta_\varepsilon,j_\varepsilon}$ and $j_\varepsilon =$ Min$\{j < \kappa : p_{\beta_\varepsilon,j}$ is compatible with $p_\varepsilon$ and $i_{\beta_\varepsilon,j} < i(*)\}$

($c$) $g_\varepsilon$ is a function, increasing with $\varepsilon$, from $v_*^{p_\varepsilon} \times \theta^* \times i(*)$ into the family of open subsets of $Y^*$ (for part (2), clopen)

($d$) if $b_{j_1} \cap b_{j_2} = \emptyset$ <u>then</u> $g_\varepsilon(\zeta, j_1, i) \cap g_\varepsilon(\zeta, j_2, i) = \emptyset$ when both are defined

($e$) letting $\Upsilon_\xi =$ otp$\{\xi_1 < \xi : i_{\beta_\xi,j_\xi} = i_{\beta_{\xi_1},j_{\xi_1}}\}$, for every $\zeta \in v_*^{p_\varepsilon}$ and $i < \theta^*$ and $\xi < \varepsilon$ we have:

$$\beta_\xi \in w_{\zeta,j}^{p_\varepsilon} \Leftrightarrow \Upsilon_\xi \in g_\varepsilon(\zeta, j, i_{\beta_\xi,j_\xi})$$

($f$) $p_\varepsilon$ is increasing continuous in $\mathbb{P}_\epsilon$

($g$) if $\xi < \varepsilon$ and $j < \theta^*$ then $\beta_\xi \in w_{\zeta(**),j}^{p_\varepsilon} \Leftrightarrow \Upsilon_\xi \in b_j$.

It is easy to carry the definition. For $\varepsilon = 0, p_\varepsilon = p^*$. If they are defined for $\varepsilon$ let us define for $\varepsilon + 1$ so $p_\varepsilon, j_\varepsilon, g_\varepsilon$ are well defined, hence $p_\varepsilon, p_{\beta_\varepsilon,j_\varepsilon}$ are two compatible members of $\mathbb{P}$ hence the assumptions $(i)-(iv), (iv)_2$ in $(*)_5$ holds with $p_\varepsilon, p_{\beta_\varepsilon,j_\varepsilon}$ here standing for $p^1, p^2$ there.

First we define $g_{\varepsilon+1}$ with domain $(v_*^{p_\varepsilon} \cup v_*^{p_{\beta_\varepsilon,j_\varepsilon}}) \times \theta^*$ extending $g_\varepsilon$, so we have to define $\langle g_{\varepsilon+1}(\zeta, j, i) : \zeta \in v_*^{p_{\beta_\varepsilon,j_\varepsilon}} \backslash v_*^{p_\varepsilon}$ and $j < \theta^*, i < i(*)\rangle$ and the restriction are for each $(\zeta, i)$ separately. For each $\zeta \in v_*^{p_{\beta_\varepsilon,j_\varepsilon}} \backslash v_*^{p_\varepsilon}$ the set $Z'_{\zeta,i} = \{\xi < \varepsilon : i_{\beta_\xi,j_\xi} = i$ and $\Upsilon_\xi \in u^{p_\varepsilon}\} \subseteq \theta$ has cardinality $< \sigma$ hence we can find a sequence $\langle \mathcal{U}_\xi^{\zeta,i} : \xi \in Z'_{\zeta,i}\rangle$ of pairwise disjoint open sets of $Y^*$ such that $\xi \in Z'_{\zeta,i} \Rightarrow \Upsilon_\xi \in \mathcal{U}_\xi^{\zeta,i}$.

Now we define

$$g_{\varepsilon+1}(\zeta, j, i) = \cup\{\mathcal{U}_\xi^{\zeta,i} : \xi \in Z'_\zeta \text{ and } \beta_\xi \in w_{\zeta,i}^{p_{\beta_\varepsilon,j_\varepsilon}}\}.$$

It is easy to check that $g_{\varepsilon+1}$ is as required in clauses (c) + (d).
We intend to use $(*)_6$ toward this, let

$$Z_\varepsilon^2 = \{\beta_\varepsilon\}$$

$$g_\varepsilon^1 :(v_*^{p_\varepsilon} \backslash v_*^{p_{\beta_\varepsilon,j_\varepsilon}}) \times \theta^* \to \{0,1\}$$
be defined by $g_\varepsilon^1(\zeta, j, \beta_\varepsilon) = 1 \Leftrightarrow \Upsilon_\varepsilon \in g_\varepsilon(\zeta, j, i_{\beta_\varepsilon,j_\varepsilon})$

$$Z_\varepsilon^1 = \{\beta_\xi : \xi < \varepsilon\}$$

$$g_\varepsilon^2 :(v_*^{p_{\beta_\varepsilon,j_\varepsilon}} \backslash v_*^{p_\varepsilon}) \times \theta^* \times Z_\varepsilon^1 \to \{0,1\}$$
be defined by $g_\varepsilon^2(\zeta, j, \beta_\varepsilon) = 1 \Leftrightarrow$
$\Upsilon_\xi \in g_{\varepsilon+1}(\zeta, j)_{\beta_\xi,j_\xi}$.

In limit $\varepsilon$ take union. In all cases $j_\varepsilon$ is well defined by clause $(i)$ above noting then $\beta_\varepsilon \notin u^{p_\varepsilon}$ so by $(*)_1$ we know that $p_\varepsilon \nVdash$ "$\beta_\varepsilon \notin \underset{\sim}{X}^*$, i.e., $\beta_\varepsilon \in \underset{\sim}{X}_{i(*)}$".

Having carried the induction let $i^* < i(*)$ be minimal such that the set $Z = \{\varepsilon < \theta : i_{\beta_\varepsilon, j_\varepsilon} = i^*\}$ has cardinality $\theta$; it exists as $i(*) <$ cf$(\theta)$. Note: $\zeta(*) \notin v^{p_{\beta_\varepsilon, j}}$ for $\varepsilon < \theta, j < \kappa$ as $A \cap F(\beta_\varepsilon)$ is the singleton $\{\beta_\varepsilon\}$ so $|A \cap u^{p_{\beta_\varepsilon, j}}| \leq 1$. Now we define $p$:

$$u^p = u^{p_\theta}$$

$$u^p_* = u^{p_\theta}_*$$

$$v^p = v^{p_\theta} \cup \{\zeta(*)\}$$

$$v^p_* = v^{p_\theta}_* \cup \{\zeta(*)\}$$

$A^p_{\zeta(*)} = \{\beta_\varepsilon : \varepsilon \in Z\}$ and $\gamma^p_{\zeta(*),\varepsilon}$ is the $\varepsilon$-th member of $A^p_{\zeta(*)}$, equivalently the unique $\beta_\xi$ such that $i_{\beta_\xi, j_\xi} = i^*$ & $\Upsilon_\xi = \varepsilon$ and $w^p_{\zeta,i}$ is

$(\alpha)$ $w^{p_\theta}_{\zeta,i}$ <u>if</u> $\zeta \in v^{p_\theta}$

$(\beta)$ $w^{p_\theta}_{\zeta(**),j}$ if $\zeta = \zeta(*)$.

We can easily check that $p \in \mathbb{P}$ and $p^* \leq p_{\beta_\varepsilon, j_\varepsilon}, p^+ \leq p \in \mathbb{P}$ (but we do not ask $p_\varepsilon \leq p$). Clearly $p$ forces that $\{\beta_\varepsilon : \varepsilon \in Z\}$ is included in one $\underset{\sim}{X}_i$, that is $\underset{\sim}{X}_{i^*}$.

Let $g : \theta \to \lambda$ be $g(\xi) = \beta_\varepsilon$ when $\xi < \theta, \varepsilon \in Z$, otp$(Z \cap \varepsilon) = \xi$. Now $p \geq p^*$ and we are done by $(*)_8$ below.]

$(*)_8$ if $p \in \mathbb{P}$ and $\zeta \in v^p_*$ <u>then</u>
$p \Vdash$ "the mapping $j \mapsto \gamma^p_{\zeta,j}$ for $j < \theta$ is a homeomorphism from $Y^*$ onto the closed subspace $\underset{\sim}{X} \restriction \{\gamma^p_{\zeta,j} : j < \theta\}$ of $\underset{\sim}{X}$"

[Why? Let $p \in G, G \subseteq \mathbb{P}$ be generic over $\mathbf{V}$.

$(\alpha)$ If $b \in \mathcal{B}$ (recall that $\mathcal{B}$ is a basis of $Y^*$), <u>then</u> for some open set $\mathcal{U}$ of $\underset{\sim}{X}$ [G] (clopen for part (2)) we have

$$\mathcal{U} \cap \{\gamma^p_{\zeta,j} : j < \theta\} = \{\gamma^p_{\zeta,j} : j \in b\}$$

[Why? As $b = b_i$ for some $i < \theta^*$ and $p$ forces by clause $(\varepsilon) + (\zeta)$ of the definition of $p \in \mathbb{P}$ and clause (iv) of $(*)_1$ above that $\underset{\sim}{\mathcal{U}}_{\zeta,i} \cap \{\gamma^p_{\zeta,j} : j < \theta\} = \{\gamma^p_{\zeta,j} : j \in b_i\}$, see $\boxtimes_1$ above.]

$(\beta)$ If $b$ is an open set for $Y^*$, <u>then</u> for some open subset $\mathcal{U}$ of $\underset{\sim}{X}$ we have

$$\mathcal{U} \cap \{\gamma^p_{\zeta,j} : j < \theta\} = \{\gamma^p_{\zeta,j} : j \in b\}$$

[Why? As $b = \bigcup\limits_{i \in Z} b_i$ for some $Z \subseteq \theta^*$ and apply clause $(\alpha)$]

$(\gamma)$ if $\mathcal{U}$ is an open subset of $\underset{\sim}{X}^*[G]$ and $\gamma^p_{\zeta,j(*)} \in \mathcal{U}$ (so $\zeta \in u^p_*$), then for some $i(*) < \theta^*$ we have

$$\gamma^p_{\zeta,j(*)} \in w^p_{\zeta,i(*)} \cap \{\gamma^p_{\zeta,j} : j < \theta\} \subseteq \underset{\sim}{\mathcal{U}}_{\zeta,i(*)}[G] \cap \{\gamma^p_{\zeta,j} : j < \theta\} \subseteq \mathcal{U}.$$

[Why? By the definition of the topology $\underset{\sim}{X}^*[G]$ we can find $n < \omega, \xi_\ell < \lambda^*$ and $j_\ell < \theta^*$ for $\ell < n$ such that $\gamma^p_{\zeta,j(*)} \in \bigcap\limits_{\ell<n} \underset{\sim}{\mathcal{U}}_{\xi_\ell,j_\ell}[G] \subseteq \mathcal{U}$. [Why? By $\boxtimes$.] We can find $q \in G \subseteq \mathbb{P}$ such that $p \le q$ and $\xi_\ell \in v^q_*$ for $\ell < n$. [Why? Recall $(*)_1$ and $\boxtimes_1$.] For each $\ell < n$, by clause $(\eta)$ in the definition of $\mathbb{P}$ we know that $\mathcal{U}^q_{\xi_\ell,\zeta,j_\ell} =: \{j < \theta : \gamma^q_{\zeta,j} \in \mathcal{U}^q_{\xi_\ell,j_\ell}\}$ is an open set for $Y^*$, and necessarily $j(*) \in \mathcal{U}^q_{\xi_\ell,\zeta,j_\ell}$. Let $i(*) < \theta$ be such that $j(*) \in b_{i(*)} \subseteq \bigcap\limits_{\ell<n} \mathcal{U}^q_{\zeta,\xi_\ell,j_\ell}$ hence $\gamma^p_{\zeta,j(*)} \in \underset{\sim}{\mathcal{U}}_{\zeta,i(*)}[G] \cap \{\gamma^p_{\zeta,j} : j < \theta\} \subseteq \bigcap\limits_{\ell<n} \underset{\sim}{\mathcal{U}}_{\xi_\ell,j_\ell}[G] \subseteq \mathcal{U}$ as required. So $i(*)$ is as required.]

$(\delta)$ $\{\gamma^p_{\zeta,j} : j < \theta\}$ is a closed subset of $\underset{\sim}{X}^*$

[Why? Let $\beta \in \lambda \backslash \{\gamma^p_{\zeta,j} : j < \theta\}$ and let $p \le q \in \mathbb{P}$; it suffices to find $q^+, q \le q^+ \in \mathbb{P}$ and $\xi \in v^{q^+}_*$ and $i < \theta^*$ such that $\beta \in u^{q^+} \backslash u^{q^+}_*$ or $\beta \in w^{q^+}_{\xi,i}$ and $w^{q^+}_{\xi,i} \cap \{\gamma^p_{\zeta,j} : j < \theta\} = \emptyset$. If $\beta \notin u^q_*$ define $q^+$ like $q$ except that $u^{q^+} = u^q \cup \{\beta\}$ (but $u^{q^+}_* = u^q_*$ as in clause (i) of $(*)_1$). So without loss of generality $\beta \in u^q_*$. We can find a set $u \subseteq u^q_*$ such that $\beta \in u, A^q_\zeta \cap u = \emptyset$ and $\zeta' \in v^q_* \Rightarrow \{j < \theta : \gamma^q_{\zeta',j} \in u\}$ is an open subset of $Y^*$, (why? just as in the proof of $(*)_5$; that is let $\langle \xi_\varepsilon : \varepsilon < \varepsilon^* \rangle$ be a list $v^q_*$ such that $B_\varepsilon = A_{\xi_\varepsilon} \setminus \cup\{A_{\xi_{\varepsilon(1)}} : \varepsilon(1) < \varepsilon\}$ has cardinality $< \sigma$, and as any two members of $\mathcal{A}$ has intersection of cardinality $< \sigma$ without loss of generality $\xi_0 = \zeta$, and choose $u_\varepsilon \subseteq \cup\{A_{\xi_{\varepsilon(1)}} : \varepsilon(1) < \varepsilon\}$ by induction on $\varepsilon \le \varepsilon^*$ such that $\varepsilon' < \varepsilon \Rightarrow u_{\varepsilon'} = u_\varepsilon \cap \cup\{A_{\xi_{\varepsilon(1)}} : \varepsilon(1) < \varepsilon'\}$ and $\beta \in \cup\{A_{\xi_{\varepsilon(1)}} : \varepsilon(1) < \varepsilon\} \Rightarrow \beta \in u_\varepsilon$ and $[\varepsilon(1) < \varepsilon \Rightarrow \{j : \gamma^q_{\xi_{\varepsilon(1)},j} \in u_\varepsilon\}$ is open in $Y^*]$ and $u_1 = u_0 = \emptyset$. For part (2) we ask "clopen subset of $Y^*$". In the end let $u = u_{\varepsilon^*} \cup \{\beta\}$).

By $\otimes_1$ in the beginning of the proof we can find $\xi \in \lambda^* \backslash v^q$ such that $\emptyset = A_\xi \cap u^q$ (why? apply $\otimes_1$ with $\alpha' < \beta' \in \lambda \backslash u^q$ and $B = u^q$, and then $(*)_1$) and let $\gamma_{\varepsilon,i} \in A_\xi$ for $i < \theta$ be increasing. We define $q^+$ as follows.

$$v^{q^+} = v^q \cup \{\xi\}$$

$$v^{q^+}_* = v^q_* \cup \{\xi\}$$

$$u^{q^+} = u^q \cup A_\xi$$

$$u^{q^+}_* = u^q_* \cup \{\gamma_{\xi,j} : j < \theta\}$$

$w^{q^+}_{\zeta,i}$ is $w^q_{\zeta,i}$ if $\zeta \in v^q_*$ and is $\{\gamma_{\xi,j} : j \in b_i\} \cup u$ if $\zeta = \xi\ \&\ 0 \in b_i$ and is $\{\gamma_{\xi,j} : j \in b_i\}$ if $\zeta = \xi\ \&\ 0 \notin b_i$. So $q^+$ is as required above and this suffices.]

Lastly, we would like to know that $\underset{\sim}{X}^*$ is a Hausdorff space. We prove more

$(*)_9$ In $\mathbf{V}^{\mathbb{P}}$ if $u_1 \subseteq u_2 \in [\lambda]^{<\sigma}$ then for some $\zeta, i$ we have

$$w_{\zeta,i} \cap u_2 \cap \underset{\sim}{X}^* = u_1 \cap \underset{\sim}{X}^*$$

[Why? Let $p_0 \in \mathbb{P}$ force that $\underset{\sim}{u}_1 \subseteq \underset{\sim}{u}_2$ form a counterexample, as $\mathbb{P}$ is $\kappa$-complete some $p_1 \ge p_0$ forces $\underset{\sim}{u}_1 = u_1, \underset{\sim}{u}_2 = u_2$ and without loss of generality $p_1 \in \mathbb{P}'$.

By $(*)_1$ and $\kappa$-completeness without loss of generality $u_2 \subseteq u^{p_1}$ and as by $(*)_1$ we have $p_1 \Vdash$ "$u_2 \cap \underset{\sim}{X}^* = u_2 \cap u_*^{p_1}$" we can ignore the elements of $u_2 \backslash u_*^{p_1}$ so without loss of generality $u_2 \subseteq u_*^{p_1}$.

Let $\zeta(*) \in \lambda^* \backslash v^{p_1}$ be such that $A_{\zeta(*)} \cap u^{p_1} = \emptyset$ (as in the proof of $(*)_8(\delta)$). Let $\gamma_{\zeta(*),j} \in A_{\zeta(*)}$, for $j < \theta$ be increasing. Let $u \subseteq u_*^{p_1}$ be such that $u \cap u_2 = u_1$ and $\zeta' \in v_*^{p_1} \Rightarrow \{j < \theta : \gamma_{\zeta',j}^{p_1} \in u\}$ is open (for part (1)) or is clopen (for part (2)) in $Y^*$ (exists as in the proof of $(*)_5$ and of $(*)_8(\delta)$) and define $q \in \mathbb{P}$:

$$u^q = u^{p_1} \cup A_\xi$$

$$u_*^q = u_*^{p_1} \cup \{\gamma_{\zeta(*),j} : j < \theta\}$$

$$v^q = v^p \cup \{\zeta(*)\}$$

$$v_*^q = v^{p_1} \cup \{\zeta(*)\}$$

$w_{\zeta,i}^q$ is: $w_{\zeta,i}^{p_1}$ if $\zeta \in v_*^{p_1}$, is $\{\gamma_{\zeta(*),j} : j \in b_i\} \cup u$ if $\zeta = \zeta(*)$ & $0 \in b_i$ and is $\{\gamma_{\zeta(*),j} : j \in b_i\}$ if $\zeta = \zeta(*)$ & $0 \notin b_i$. It is easy to see that $q$ is required.]

Together all is done. $\square_{1.2}$

* * *

Now when do the assumptions of 1.2 hold?

**1.4 Claim.** *1) Assume*

(a) $\mathfrak{a} \in [Reg \cap \lambda \backslash \kappa]^\theta$ *and* $J = [\mathfrak{a}]^{<\sigma}$

(b) $\Pi\mathfrak{a}/J$ *is* $(\lambda^*)^+$*-directed,*

(c) $\sigma$ *is regular,* $\lambda > \kappa^{++}, \kappa > \theta > \sigma$

(d) $\lambda^* > \lambda > \kappa^{<\kappa} = \kappa > \theta$*;*

(e) $\lambda^* < 2^\lambda$ *is regular.*

*Then*

(f) *In* $\mathbf{V}_1 = \mathbf{V}^{\mathrm{Levy}(\lambda^*, 2^\lambda)}$ *we have (a),(c),(d) and (e) and* $2^\lambda = \lambda^*$ *and*

(g) *the assumptions* $(A)(i), (B_1), (B_2), (C)$ *of Theorem 1.2 hold (recall* $(A)(i)$ *means we omit* $\theta^*$ *and the demand* $(\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)$*).*

*2) We can in* $(g)$ *above strengthen* $(C)$ *to*

$(C)^*$ *if* $\bar{x}_\beta = \langle x_{\beta,i} : i < i_\beta \rangle$ *for* $\beta < \lambda$ *and* $i_\beta < \kappa$ *then we can find* $i(*)$ *and* $A \in [\lambda]^\theta$ *such that*

(i) $\beta \in A \Rightarrow i_\beta = i(*)$

(ii) *if* $\beta_1 < \beta_2$ *are from* $A$ *and* $i_1, i_2 < i(*)$ *then* $x_{\beta_1,i_2} = x_{\beta_2,i_2} \Rightarrow x_{\beta_1,i_1} = x_{\beta_2,i_1}$ & $x_{\beta_1,i_1} = x_{\beta_1,i_2}$

(iii) *if* $i < i(*)$ *and* $\beta_1, \beta_2, \beta_3 \in A$ *are distinct then* $x_{\beta_1,i} = x_{\beta_2,i} \Rightarrow x_{\beta_1,i} = x_{\beta_3,i}$.

*Proof.* Let $\mathfrak{a} = \{\lambda_\varepsilon : \varepsilon < \theta\}$ without repetitions; without loss of generality $\lambda_\varepsilon > \kappa^{++}$. Let $J' = [\theta]^{<\sigma}$. In $\mathbf{V}$ we can find $\langle f_\alpha : \alpha < \lambda^*\rangle$ such that $\boxtimes$ below holds. We first proof below.

*1.5 Observation.* Assume $\mathfrak{a} = \{\lambda_\varepsilon : \varepsilon < \theta\}$ is a set of regular cardinals, $J'$ is an ideal on $\theta$ such that $\Pi\lambda_\varepsilon/J'$ is $(\lambda^*)^+$-directed, $\lambda^* = \mathrm{cf}(\lambda^*) > \sup(\mathfrak{a})$, and $\sigma$ regular and $\theta < \kappa \leq \mathrm{Min}(\mathfrak{a})$ (or at least for every regular $\kappa' < \kappa$ the set $\{\varepsilon < \theta : \lambda_\varepsilon \leq \kappa'\}$ belongs to $J'$ and, of course, $\mathfrak{a}$ is a set of regular cardinals $> \theta$). Then

$\boxtimes$ we can find $\langle f_\alpha : \alpha < \lambda^*\rangle$ satisfying $f_\alpha \in \prod\limits_{\varepsilon<\theta} \lambda_\varepsilon$ such that

- $(a)$ $\bar f = \langle f_\alpha : \alpha < \lambda^*\rangle$ is $<_{J'}$-increasing
- $(b)$ $\bar f$ has a $<_{J'}$-lub $f^*$
- $(c)$ if $a \in (J')^+$ then $\sup\{\mathrm{cf}(f^*(i)) : i \in a\} \geq \sup(\{\lambda_\varepsilon : \varepsilon \in a\})$
- $(d)$ if $\alpha_1 < \lambda^*$ & $\alpha_2 < \lambda^*$ & $\varepsilon_1 < \theta$ & $\varepsilon_2 < \theta$ & $f_{\alpha_1}(\varepsilon_1) = f_{\alpha_2}(\varepsilon_2)$ then $\varepsilon_1 = \varepsilon_2$
- $(e)$ for every $Z \in [\lambda^*]^{\leq\kappa}$ we can find $\bar a = \langle a_\alpha : \alpha \in Z\rangle$ such that $a_\alpha \in J'$ and $\alpha < \beta$ & $\alpha \in Z$ & $\beta \in Z$ & $\varepsilon \in \theta\backslash a_\alpha\backslash a_\beta \Rightarrow f_\alpha(\varepsilon) < f_\beta(\varepsilon)$
- $(f)$ if $\sigma \leq \theta$ and $J' = [\theta]^{<\sigma}$, then for every $Z \in [\lambda^*]^{\leq\kappa}$ for some sequence $\bar a = \langle a_\alpha : \alpha \in Z\rangle$ satisfying $a_\alpha \in J'$ for $\alpha \in Z$ and some well ordering $<^*$ of $Z$ we have $\alpha \in Z$ & $\beta \in Z$ & $\alpha <^* \beta$ & $\varepsilon \in \theta\backslash a_\beta \Rightarrow f_\alpha(\varepsilon) \neq f_\beta(\varepsilon)$.

*Proof.* By the proof of [Sh:g, Ch.II, 1.4] or see [Sh:506] we get: for some $\langle f_\alpha : \alpha < \lambda^*\rangle$ with $f_\alpha \in \Pi\mathfrak{a}$ we have (a) + (b) + (c) + (e).

Clause (d) is easy, just replace $f_\alpha$ by $f'_\alpha \in \prod\limits_{\varepsilon<\theta} \lambda_\varepsilon$ which is defined by $f'_\alpha(\varepsilon_\varepsilon) =: \theta \times f_\alpha(\varepsilon) + \varepsilon$ and replace $f^*$ by $f^{**}$, $\mathrm{Dom}(f^{**}) = \theta, f^{**}(\varepsilon) = \theta \times f^*(\varepsilon)$ recalling $\theta < \mathrm{Min}(\mathfrak{a})$. We shall prove that $\langle f'_\alpha : \alpha < \lambda^*\rangle$ is also as required in clause (f). So let $Z \in [\lambda^*]^{\leq\kappa}$ be given; let $Z = \{\gamma_\varepsilon : \varepsilon < |Z|\}$ be with no repetitions. Let $\langle a_\beta : \beta \in Z\rangle$ be as guaranteed by clause (e). We can choose by induction on $\zeta < |Z|, Z_\zeta \subseteq Z$ increasing continuous in $\zeta$ and $Z^\zeta$ such that $Z_0 = \emptyset, |Z_{\zeta+1}\backslash Z_\zeta| \leq \sigma, [Z_\zeta \neq Z \Rightarrow Z_\zeta \neq Z_{\zeta+1}]$ and $Z_{\zeta+1}\backslash Z_\zeta \subseteq Z^\zeta \subseteq Z_{\zeta+1}, |Z^\zeta| \leq \sigma, \gamma_\varepsilon \in Z_{\varepsilon+1}$ and $\alpha \in Z^\zeta$ & $\beta \in Z_\zeta$ & $\varepsilon \notin a_\beta$ & $\varepsilon \in a_\alpha$ & $f_\alpha(\varepsilon) = f_\beta(\varepsilon) \Rightarrow \beta \in Z^\zeta$ (actually the $\beta \notin Z_\zeta$, are irrelevant).

Why does such a sequence exist? The only problem is, given $Z_\zeta$ to choose $Z^\zeta$. Now as $|a_\alpha| < \sigma$ (because $a_\alpha \in J' \subseteq [\mathfrak{a}]^{<\sigma}$) and by the choice of $\langle a_\alpha : \alpha \in Z\rangle$ we have

$(*)$ $(\forall\varepsilon < \theta)(\forall\gamma < \lambda_\varepsilon)(\exists^{\leq 1}\beta \in Z)(\varepsilon \notin a_\beta$ & $f_\beta(\varepsilon) = \gamma)$

hence there are no problems, (in fact if $\sigma$ is uncountable we can ask $< \sigma$). With more details, we define $Z^\zeta_n$ by induction on $n < \omega$ as follows; $Z^\zeta_0 = \{\gamma_\varepsilon\}, Z^\zeta_{n+1} = Z^\zeta_n \cup \{\beta :$ for some $\alpha \in Z^\zeta_n$ and $\varepsilon \in a_\alpha$ we have $\varepsilon \notin a_\beta$ & $f_\beta(\varepsilon) = f_\alpha(\varepsilon)\}$.

As $\sigma$ is a regular and $\alpha \in Z \Rightarrow |a_\alpha| < \sigma$, by $(*)$ we can prove by induction on $n$ that $|Z^\zeta_n| < \sigma$, hence $Z^\zeta = \cup\{Z^\zeta_n : n < \omega\}$ is as required. Now list $Z^\zeta$ as $\langle\alpha^\zeta_\xi : \xi < \xi^*_\zeta\rangle$ such that $\xi^*_\zeta = |Z^\zeta| \leq \sigma$.

Define a well ordering $<^*$ of $Z$ as follows

$\alpha < \beta \Leftrightarrow (\exists\zeta)[\alpha \in Z_\zeta \wedge \beta \notin Z_\zeta] \vee (\exists\zeta)(\alpha \in Z^\zeta\backslash Z_\zeta \wedge \beta \in Z^\zeta\backslash Z_\zeta \wedge (\exists\xi_1, \xi_2)(\alpha = \alpha^\zeta_{\xi_1} \wedge \beta = \alpha^\zeta_{\xi_2} \wedge \xi_1 < \xi_2)$.

Now we define $a'_\alpha \in J'$ for $\alpha \in Z$ as follows: if $\alpha = \alpha^\zeta_\xi \in Z_{\zeta+1}\backslash Z_\zeta$ then $a'_\alpha = \cup\{a_{\alpha^\zeta_\varepsilon} : \varepsilon \leq \xi\}$; as $\xi < |Z^\zeta| \leq \sigma$ and $\sigma$ is regular and $\beta \in Z \Rightarrow |a_\beta| < \sigma$ clearly $|a'_\alpha| < \sigma$. Now suppose

$\langle a'_\alpha : \alpha \in Z\rangle$ fails clause (f) for $<^*$ which is a well ordering of $Z$. So there are $\varepsilon < \theta$ and $\alpha <^* \beta$ from $Z$ which exemplifies this and let $\zeta$ be such that $\alpha \in Z^\zeta \backslash Z_\zeta$ and let $\xi$ be such that $\alpha = \alpha^\zeta_\xi$; so by the definition of $<^*$ we have $\beta \notin Z_\zeta$ and $\beta \notin \{\alpha^\zeta_{\xi'} : \xi' \le \xi\}$. As $a_\alpha \subseteq a'_\alpha$ and the choice of $\langle a_\alpha : \alpha \in Z\rangle$ and as $\varepsilon \notin a_\beta$ (by the choice of $\alpha, \beta, \varepsilon$) necessarily $\varepsilon \in a_\alpha$ hence $\beta \in Z^\zeta$ (so $\beta \in Z^\zeta \backslash Z_\zeta$) but as said above $\beta \notin \{\alpha^\zeta_{\xi'} : \xi' \le \zeta\}$, so by the choice of $a'_\beta$ we get easy contradiction. $\square_{1.5}$

*Continuation of the proof of 1.4.* Clearly in $\mathbf{V}_1$ we have (a),(c),(d) of 1.4 and $\boxtimes$ of 1.5 above.

As in $\mathbf{V}, \lambda < \lambda^* = \mathrm{cf}(\lambda^*) < 2^\lambda$, clearly in $\mathbf{V}_1$ we have $2^\lambda = \lambda^*$ (and we can forget $\mathbf{V}$ and the assumption (b), recall (b) says "$\Pi\mathfrak{a}$ is $(\lambda^*)^+$-directed". More on the existence of $\bar f$ as in $\boxtimes$, see [Sh:g, Ch.VIII, §5]).

So we can in $\mathbf{V}_1$ let $\langle h_\alpha : \alpha < \lambda^*\rangle$ list the functions $h : \lambda \to [\lambda]^\kappa$. Now for each $\zeta < \lambda^*$ we define a function $g_\zeta : \kappa^{++} \to [\kappa^{++}]^{\le\kappa}$ by[4]

$$g_\zeta(\gamma) = \Big\{\beta < \kappa^{++} : \text{for some } \varepsilon_1, \varepsilon_2 < \theta \text{ we have } \theta \times \kappa^{++} \times f_\zeta(\varepsilon_1) + \theta \times \beta + \varepsilon_1 \in h_\zeta[\theta \times \kappa^{++} \times f_\zeta(\varepsilon_2) + \theta \times \gamma + \varepsilon_2]\Big\}.$$

So we can ([Ha61]) for each $\zeta < \lambda^*$ find $Z_\zeta \in [\kappa^{++}]^{\kappa^{++}}$ such that

$$\beta_1 \ne \beta_2 \in Z_\zeta \Rightarrow \beta_1 \notin g_\zeta(\beta_2).$$

For $\zeta < \lambda^*$ let

$$A_\zeta = \{\theta \times \kappa^{++} \times f_\zeta(\varepsilon) + \theta \times \beta + \varepsilon : \varepsilon < \theta \text{ and } \beta < \kappa^{++} \text{ is the } \varepsilon\text{-th member of } Z_\zeta\}.$$

Now we shall check.

Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$. Clearly

$(*)_1$ $A_\zeta \in [\lambda]^\theta$ (hence $\mathcal{A} \subseteq [\lambda]^\theta$)
[why? recall that $\theta < \kappa, \kappa^{++} < \lambda$]

$(*)_2$ $\zeta_1 \ne \zeta_2 \Rightarrow |A_{\zeta_1} \cap A_{\zeta_2}| < \sigma$
[Why? Let $\alpha \in A_{\zeta_1} \cap A_{\zeta_2}$ so for some $(\beta_1, \varepsilon_1), (\beta_2, \varepsilon_2)$, for $\ell = 1,2$ we have $\alpha = \theta \times \kappa^{++} \times f_{\zeta_\ell}(\varepsilon_\ell) + \theta \times \beta_\ell + \varepsilon_\ell$ and $\beta_\ell < \kappa^{++}, \varepsilon_\ell < \theta$. Clearly this implies $\varepsilon_1 = \varepsilon_2, \beta_1 = \beta_2, f_{\zeta_1}(\varepsilon_1) = f_{\zeta_2}(\varepsilon_2)$, and $\mathrm{otp}(\beta_\ell \cap Z_{\zeta_\ell}) = \varepsilon_\ell$, so $\beta_\ell$ is determined just by $f_{\zeta_\ell}(\varepsilon_\ell), \zeta_\ell$ and $\varepsilon_\ell$ (with no use of $\alpha$) and by clause (d) of $\boxtimes$ also $\varepsilon_\ell$ is determined by $f_{\zeta_\ell}(\varepsilon_\ell)$ hence $|A_{\zeta_1} \cap A_{\zeta_2}| \le |\{\alpha \in A_{\zeta_1}\colon$ there are $(\beta_1, \beta_2, \varepsilon_1, \varepsilon_2)$ as above$\}| \le |\{\varepsilon < \theta : f_{\zeta_1}(\varepsilon) = f_{\zeta_2}(\varepsilon)\}| < \sigma$ as $\zeta_1 < \zeta_2 \to f_{\zeta_1} <_J f_{\zeta_2}$ recalling $J = [\theta]^{<\sigma}$.]

$(*)_3$ $|\mathcal{A}| = \lambda^*$
[Why? By the choice of $\mathcal{A}$ and $(*)_1 + (*)_2$.]

[4] actually $\theta \times \kappa^{++} \times \gamma = \kappa^{++} \times \gamma$

$(*)_4$ if $F : \lambda \to [\lambda]^{\le\kappa}$, then some $A \in \mathcal{A}$ is $F$-free
[Why? For some $\alpha$ we have $F = h_\alpha$, so $Z_\alpha, A_\alpha$ were chosen to make this true.]

$(*)_5$ if $\mathcal{A}' \in [\mathcal{A}]^{<\kappa}$, then we can list $\mathcal{A}'$ as $\{A_{\zeta_i} : i < i(*)\}$ such that $|A_{\zeta_i} \cap \bigcup\limits_{j<i} A_{\zeta_j}| < \sigma$ for each $i < i(*)$ [Why? Let $\mathcal{A}' = \{A_\zeta : \zeta \in Z\}$ where $Z \subseteq \lambda^*, |Z| < \kappa$, so by clause (f) of $\boxtimes$ we can find $\langle a_\alpha : \alpha \in Z\rangle, <^*$ as there. Let $Z = \{\zeta_i : i < i(*)\}$ be $<^*$-increasing with $i$ and so

$$A_{\zeta_i} \cap \bigcup_{j<i} A_{\zeta_j} \subseteq \{\theta \times \kappa^{++} \times f_{\zeta_i}(\varepsilon) + \theta \times \beta_{\zeta_i,\varepsilon} + \varepsilon : \varepsilon \in a_{\zeta_i}\}$$

which has cardinality $< \sigma$ where $\beta_{\zeta_i,\varepsilon}$ is the $\varepsilon$-th member of $Z_\alpha$.]

So clause (A)(i) holds by clause (d) of our assumption (note, $\theta^*$ does not appear here), clause $(B)_1$ holds by $(*)_1 + (*)_2$ and $(B)_2$ holds by $(*)_5$ and lastly (C) holds by $(*)_4$.
2) Similar, just in order to get more in the proof of $(*)_4$ we let $\langle h_\alpha : \alpha < \lambda^*\rangle$ list the relevant $h$-s and choose $Z_\zeta$ accordingly. $\square_{1.4}$

*1.6 Remark.* We can get $(C)^*$ for any $\kappa$ such that $\{\theta \in \mathfrak{a} : (2^{<\kappa})^+ \le \theta\} \in J$.

**1.7 Claim.** *1) We can change the assumptions of Theorem 1.2 by omitting (A)(ii) and by replacing (E) by $(E)^-$ and (C) by $(C)'$, i.e., having:*

$(A)(i)$ *$\lambda > \kappa > \theta \ge \sigma \ge \aleph_0$ and $\kappa = \kappa^{<\kappa}$ and $\theta^* \le \kappa$*
*(i.e., this is (A)(i) without (A)(ii), i.e., omitting "$(\forall\alpha < \kappa)(|\alpha|^\sigma < \kappa), \kappa > \theta^* \ge \theta$")*

$(C)'$ *if $F : \lambda \to [\lambda]^{\le\kappa}$ then we can find $A' \in \mathcal{A}$ and $A \subseteq A'$ of order type $\theta$ such that: if $\beta \in A$ then $\beta \notin \cup\{F(\alpha) : \alpha \in A \cap \beta\}$*

$(E)^-$ *if $Y_0, Y_1$ are disjoint subsets of $Y^*$ each with $< \sigma$ points, then there are open disjoint sets $\mathcal{U}_0, \mathcal{U}_1$ of $Y^*$ such that $Y_0 \subseteq \mathcal{U}_0, Y_0 \subseteq Y_1$.*

*2) We can similarly weaken the assumptions in 1.2(2), omitting "the union of $< \sigma$ members of $\mathcal{B}$ belong to $\mathcal{B}$", but in $(E)^-$ demand $\mathcal{U}_0$ to be clopen.*

*Proof.* We indicate the changes in the proof.
We can further demand from $\langle b_i : i < \theta^*\rangle$ that

$\otimes_3$ $b_{2i} \cap b_{2i+1} = \emptyset$ and if $b_{i_0} \cap b_{i_1} = \emptyset$ then for some $j$ we have $(b_{2j}, b_{2j+1}) = (b_{i_0}, b_{i_1})$ and if $Y_0, Y_1 \in [Y^*]^{<\sigma}$ are disjoint then for some $i$ we have $Y_0 \subseteq b_{2i}, Y_1 \subseteq b_{2i+1}$.

Note that $\theta^{<\sigma} \le \kappa$ so no problem arises with the number of $b_i$'s.
In the definition of $\mathbb{P}$ we replace clause $(\delta)$ by

$(\delta)^-$ $w_{\zeta,i} \subseteq u_*$ and $w_{\zeta,2i} \cap w_{\zeta,2i+1} = \emptyset$

and in the definition of the order when $\mathbb{P} \models p \le q$ we add $\zeta \in v^q_* \backslash v^p_* \Rightarrow |A_\zeta \cap u^p| < \sigma$.
However, as we have weakened assumption $(A)$, the $\kappa^+$-c.c. may fail. So we define: the pair $(f, g)$ is an isomorphism from $p \in \mathbb{P}$ onto $q \in \mathbb{P}$ if:

$(i)$ $f$ is a one-to-one order preserving mapping from $u^p$ onto $u^q$
$(ii)$ $g$ is a one-to-one order preserving mapping from $v^p$ onto $v^q$
$(iii)$ $f$ maps $u^p_*$ onto $u^q_*$
$(iv)$ $g$ maps $v^p_*$ onto $v^q_*$
$(v)$ if $\zeta \in v^p_*$ then $A_{g(\zeta)} = \{f(\beta) : \beta \in A_\zeta\}$
$(vi)$ if $\zeta \in v^p_*$ and $j < \theta$ then $\gamma^q_{g(\zeta),j} = f(\gamma^p_{\zeta,j})$
$(vii)$ if $\zeta \in v^p_*$ and $i < \theta^*$ then

$$w^q_{g(\zeta),i} = \{f(\beta) : \beta \in w^p_{\zeta,i}\}.$$

We say $p, q$ are isomorphic if such $(f, g)$ exists. Clearly being isomorphic is an equivalent relation. Let $\chi$ be large enough and $\mathfrak{C}$ be an elementary submodel of $(\mathcal{H}(\chi), \in, <^*)$ of cardinality $\kappa$ such that $\lambda, \kappa, \theta^*, \theta, \sigma, Y^*, \langle b_i : i < \theta^*\rangle, \mathcal{A}, \mathbb{P}$ belong to $\mathfrak{C}$ and ${}^{\kappa>}\mathfrak{C} \subseteq \mathfrak{C}$. Let

$$\mathbb{Q} = \{p \in \mathbb{P} : \text{for some } q \in \mathbb{P} \cap \mathfrak{C} \text{ the conditions } p, q \text{ are isomorphic}\}.$$

In the rest of the proof $\mathbb{P}$ is replaced by $\mathbb{Q}$, each time we construct a condition we have to check if it belongs to $\mathbb{Q}$.

The only place we use $(\forall\alpha < \kappa)(|\alpha|^\sigma < \kappa)$ is in the proof of $(*)_3$. So omit $(*)_3$, and this requires us first to improve the proof of $(*)_4$ (and second $(*)_7$, see later). Let $p_j \in \mathbb{Q}$ for $j < \kappa^+$ and let $v_j = \{\zeta < \lambda^* : A_\zeta \cap u^{p_j}$ has cardinality $\geq \sigma\} \cup v^{p_j}$, so clearly $|v_j| \leq \kappa$ and $v^{p_j} \subseteq v_j$.

For some stationary $S \subseteq \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$, the conditions $p_j$ for $j \in S$ are pairwise isomorphic and $j \in S$ implies $v^{p_j} \cap (\bigcup_{i<j} v_i) = v^\otimes$ and $u^{p_j} \cap (\bigcup_{i<j}(u^{p_i} \cup \cup \bigcup_{\zeta\in v_i} A_\zeta)) = u^\otimes$. Also without loss of generality for $j_1, j_2 \in S$ the isomorphism $(f, g)$ from $p_{j_1}$ to $p_{j_2}$ satisfies $f \restriction u^\otimes = \mathrm{id}_{u^\otimes}, g \restriction v^\otimes = \mathrm{id}_{v^\otimes}$. We would like to apply $(*)_5$ from the proof of 1.2 to $p_i, p_j$ for any $j > i$ from $S$, so we have to verify clauses $(i), (ii), (iii), (iv)_1, (iv)_2$ there. Now only clause $(iv)_2$ is problematic. Now if $i \neq j$ are from $S$ and $\zeta \in v^{p_i}_* \backslash v^{p_j}_*$ and $|A^{p_i}_\zeta \cap u^{p_j}| \leq \sigma$, note that $A^{p_i}_\zeta \subseteq u^{p_i}$ hence $A^{p_i}_\zeta \cap u^{p_j} \subseteq u^{p_i} \cap u^{p_j} = u^\otimes \subseteq v_{\min(S)+1}$. So for $i, j \in S \backslash \{\min(S), p_i, p_j$ are compatible by $(*)_5$ in the proof of 1.2.

In the proof of $(*)_5$ and $(*)_6$ (hence $(*)_7$), clause $(E)^-$ gives us less but the change in the definition of $\mathbb{P}$ (weakening $(\delta)$ to $(\delta)^-$) demands less and they fit, e.g., during the proof of $(*)_5$ we can deal with each pair $(w_{\zeta,2i}, w_{\zeta,2i+1})$ separately.

The second place in which we use $(*)_3$ is during the proof of $(*)_7$. The proof is similar but:

- $\circledast$ $F(\alpha)$ now is a subset of $\lambda$ which includes $\cup\{u^{p_{\alpha,j}} : j < \kappa\}$ and satisfies $A \in \mathcal{A}$ & $|A \cap F(\alpha)| \geq \sigma \Rightarrow A \subseteq F(\alpha)$.

[Why such $F(\alpha)$ exists? As in the proof of $(*)_3$.] Then first, we apply clause $(C)'$ from 1.7 to find $\zeta(*) < \lambda^*$ and $A \subseteq A_{\zeta(*)}$ of order type $\theta$ such that $A$ satisfying the demands of $(C)'$. Second, choosing $p_\varepsilon$ by induction on $\varepsilon$, choosing $p_{\varepsilon+1}$, verifying the conditions in $(*)_5$ they hold because of the change in the definition of the order of $\mathbb{P}$.

Lastly, for proving "$\underset{\sim}{X}$ is Hausdorff", clause $(\delta)^-$ is weaker but as $Y^*$ is Hausdorff (and the choice of $\langle b_i : i < \theta^*\rangle$) there is no problem. $\square_{1.7}$

*1.8 Concluding Remarks.* 1) We could make in 1.2 only some of the changes from 1.7, i.e.,

- $(\alpha)$ in 1.2 we replace (A),(C) by (A)(i),$(C)^*$ of 1.7
- $(\beta)$ in 1.2 replace (E) by $(E)^-$.

2) In 1.2(1) can we make the space regular $(T_3)$?

In view of 1.2(2) this may be not so interesting, still needed for a regular $X^* \to (\mathbb{R})^1_{\aleph_0}$. Note that for $X$ to be a $T_3$-space it is enough that there is a family $\mathcal{B}$ of open subsets such that their finite intersections forms a basis, $(\forall x \neq y \in X)(\exists \mathcal{U} \in \mathcal{B})(x \in \mathcal{U}$ & $y \notin \mathcal{U})$ and $x \in \mathcal{U}_0 \in \mathcal{B} \Rightarrow (\exists \mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B})(x \in \mathcal{U}_{i_1} \subseteq \mathcal{U}_{i_0}$ & $\mathcal{U}_{i_0} \cup \mathcal{U}_{i_2} = X$ & $\mathcal{U}_{i_1} \cap \mathcal{U}_{i_2} = \emptyset)$. Let $R_0 \subseteq \{(i,j) : b_i \cap b_j = \emptyset\}$ (so to include generalization, as in 1.7 we chose $R_0 \subseteq \{(2i, 2i+1) : i < \theta^*\})$ and $R_1 \subseteq \{(i,j) : b_i \cup b_j = Y^*\}, R_2 \subseteq \{(i,j) : b_i \subseteq b_j\})$.

We need: for $i_0 < \theta^*, j < \theta$ such that $j \in b_{i_0}$ there are $i_1, i_2 < \theta^*$ such that $j \in b_{i_1}, b_{i_1} \subseteq b_{i_0}, b_{i_0} \cup b_{i_2} = Y^*, b_{i_1} \cap b_{i_2} = \emptyset$ and moreover $(i_1, i_0) \in R_2, (i_0, i_2) \in R_1, (i_1, i_2) \in R_0$. If $Y^*$ is a $T_3$-space with a basis of cardinality $\leq \theta^*$ then there is no problem to find such $\bar{b}$.
Then we should change the definition of $\mathbb{P}$, clause $(\delta)$ to

$(\delta)^-$ $(a)$ $w^p_{\zeta,i} \subseteq u^p_*$

$(b)$ $(i,j) \in R_0 \Rightarrow w^p_{\zeta,i} \cap w^p_{\zeta,j} = \emptyset$;

$(c)$ $(i,j) \in R_1 \Rightarrow w^p_{\zeta,i} \cup w^p_{\zeta,j} = u_*$

$(d)$ $(i,j) \in R_2 \Rightarrow w^p_{\zeta,i} \subseteq w^p_{\zeta,j}$

$(\theta)$ if $\alpha \in w^p_{\zeta,i_0}$ then for some $i_1, i_2$ we have $\alpha \in w^\zeta_{\zeta,i_1}, (i_1, i_0) \in R_2, (i_0, i_2) \in R_1, (i_1, i_2) \in R_0$ [follows from the next]
(or use a three place relation $R$).

So there is no problem to generalize the proof of 1.2.
3) In 1.4, as indicated in the proof, we can replace in the assumption (b) + (e), i.e. "$\Pi\mathfrak{a}/J$ is $(\lambda^*)^+$-directed, $\lambda^* < 2^\lambda$ is regular" by: $\lambda^* = 2^\lambda$ and

$(*)$ there is $\bar{f} = \langle f_\alpha : \alpha < \lambda^* \rangle, f_\alpha \in \Pi\mathfrak{a}$ such that for every $Z \in [\lambda^*]^{<\kappa}$ we can find $\langle a_\alpha : \alpha \in Z \rangle, a_\alpha \in J$ such that $\alpha \neq \beta \in Z \ \& \ \varepsilon \in \theta \backslash a_\alpha \backslash a_\beta \Rightarrow f_\alpha(\varepsilon) \neq f_\beta(\varepsilon)$.

4) By the proof of 1.5, if $\mathfrak{a}, J = [\mathfrak{a}]^{<\sigma}$ and $\bar{f}$ are as in $(*)$ of 1.8(3) then

$(*)'$ there is $\bar{f}' = \langle f'_\alpha : \alpha < \lambda^* \rangle, f'_\alpha \in \Pi\mathfrak{a}$ such that for every $Z \in [\lambda^*]^{<\kappa}$ we can find $\langle a_\alpha : \alpha \in Z \rangle, a_\alpha \in J$ and well ordering $<^*$ of $Z$ such that $\alpha <^* \beta \in Z \ \& \ \varepsilon \in \theta \backslash a_\beta \Rightarrow f'_\alpha(\varepsilon) \neq f'_\beta(\varepsilon)$ (in fact $\bar{f}' = \bar{f}$).

5) See more 4.17: for more colours.
6) If CON(ZFC $+\exists$ supercompact) <u>then</u> CON(CH + there is a $T_3$-topological space with $\aleph_{\omega+1}$ needs such that $X \rightarrow (\mathbb{R})^1_{\aleph_0}$).
[Why? Similar to the proof of 2.6 below, using 1.8(2).]

## §2 Consistency from supercompact

In the first section we got consistency results concerning Cantor discontinuum partition problem but using pcf statement of unclear consistency status (they come from 1.4); this is very helpful toward finding the consistency strength, and unavoidable if e.g. we like CH to fail (see §3), but it does not give a well grounded consistency result. Here relying on Theorem 1.2 of the first section, we get consistency results using "only" supercompact cardinals. First we give a sufficient condition for clause (C) of Theorem 1.2 which is reasonable under instances of G.C.H. We then (2.2) quote Hajnal Juhasz Shelah [HJSh 249], [HJSh 697] (for $\sigma = \aleph_0, \sigma > \aleph_0$, respectively) and from it (in claim 2.3), in the natural cases, prove that the assumptions of 1.2 hold deducing (in 2.6) the consistency of CH + there is a $T_3$-space $X$ with clopen basis with $\aleph_{\omega+1}$ point such that $X \rightarrow (\text{Cantor set})^1_{\aleph_0}$ starting with a supercompact cardinal. This gives a (consistent) negative answer to the Cantor discontinuum partition problem. We can even make it compact. We also try to clarify the relations between such properties of, e.g., $\aleph_{\omega+1}$.

2.1 Observation: If clauses[5] $(A)(i) + (B)_1$ of Theorem 1.2 holds, then clause (C) there follows from

$(C)^+$ if $\langle Y_i : i < \kappa^+\rangle$ is a partition of $\lambda$ then for some $A \in \mathcal{A}$ and $i < \kappa^+$ we have $A \subseteq Y_i$.

*Proof.* Let $F : \lambda \rightarrow [\lambda]^{\leq\kappa}$ be given. Choose by induction on $\zeta \leq \lambda$ a set $U_\zeta \subseteq \lambda$ and $g_\zeta : U_\zeta \rightarrow \kappa^+$, both increasing continuous with $\zeta$ such that:

$(*)(i)$ if $\alpha \in U_\zeta$ then $F(\alpha) \subseteq U_\zeta$ and

$(ii)$ if $\alpha \in U_\zeta$ then $F(\alpha)\backslash\{\alpha\} \subseteq \{\beta \in U_\zeta : g_\zeta(\beta) \neq g_\zeta(\alpha)\}$.

For $\zeta = 0$ let $U_\zeta = \emptyset = g_\zeta$, for $\zeta$ limit take unions. If $U_\zeta = \lambda$, let $U_{\zeta+1} = U_\zeta$ and $g_{\zeta+1} = g_\zeta$, otherwise let $\alpha_\zeta = \text{Min}\{\lambda\backslash U_\zeta\}$ and let $W_\zeta \in [\lambda]^{\leq\kappa}$ be such that $\alpha_\zeta \in W_\zeta$ and $(\forall\alpha \in W_\zeta)[F(\alpha) \subseteq W_\zeta]$. Let $\varepsilon_\zeta = \sup\{g_\zeta(\beta) : \beta \in U_\zeta \cap W_\zeta\}$ so $\varepsilon_\zeta < \kappa^+$ and let $U_{\zeta+1} = U_\zeta \cup W_\zeta$ and let $g_{\zeta+1}$ extend $g_\zeta$ such that $g_{\zeta+1} \restriction (W_\zeta\backslash U_\zeta)$ is one to one with range $\subseteq [\varepsilon_\zeta, \varepsilon_\zeta + \kappa)$.
Clearly $\zeta \subseteq U_\zeta = \text{Dom}(g_\zeta)$ so $g = \cup\{g_\zeta : \zeta < \lambda\}$ is a function with domain $\lambda$.
Now applying $(C)^+$ to the partition which $g$ defines, we get some $A \in \mathcal{A}$ on which $g$ is constant so by $(*)(ii)$ we are done. $\square_{2.1}$

By [HJSh 249], [HJSh 697], or see more below in 2.7 – 2.9, (toward equiconsistency) we have:

**2.2 Claim.** *Assume* $\mathbf{V} \models GCH$ *(for simplicity) and* $\sigma < \chi < \chi_0^{<\chi} \leq \kappa < \mu < \mu^+ = \lambda$ *and* $\sigma, \chi, \chi_0, \kappa, \lambda$ *are regular,* $\text{cf}(\mu) = \sigma$ *and* $\chi$ *is a supercompact cardinal (or just* $\lambda$*-supercompact), e.g.* $\mu = \chi_0^{+\sigma}$.
*Then for some forcing notion* $\mathbb{P}$*, which is* $\sigma$*-complete of cardinality* $\chi_0$*, in* $\mathbf{V}^{\mathbb{P}}, 2^\sigma = \sigma^+, 2^{\sigma^+} = \chi_0 = \sigma^{++}$ *(and GCH holds) and some* $\bar{B} = \langle B_\delta : \delta \in S\rangle$ *satisfies for any regular* $\kappa \in (\chi_0, \mu)$*:*

$(*)(i)$ $S \subseteq \{\delta < \lambda : cf(\delta) = \sigma^+\}$ *is stationary,*
$B_\delta \subseteq \delta$, $\text{otp}(B_\delta) = \sigma^+$ *and* $\delta_1 \neq \delta_2 \in S \Rightarrow |B_{\delta_1} \cap B_{\delta_2}| < \sigma$

[5] actually from $(B)_1$, only "$(B)_1^- \mathcal{A} \subseteq [\lambda]^\theta$" is used; as we do not change $\mathcal{A}$ and the cardinals this is O.K.

(ii) $\lambda = \mu^+ = 2^\mu, \mu$ *strong limit and letting* $\theta = \sigma^+$ *we have* $\sigma < \theta < \kappa = \kappa^{<\kappa} < \mu$*; note that if* $\mu = \chi_0^{+\omega}$ *then* $\lambda = \aleph_{\omega+1}$

(iii) *cf*$(\mu) = \sigma$ *(this actually follows) by (i) and (ii)*

What we need is getting in such model, condition $(C)^+$ of 2.1 which also is from [HJSh 697] but for completeness we shall prove what we use.

**2.3 Claim.** *Assume*

(a) $\bar{B} = \langle B_\delta : \delta \in S\rangle, \sigma, \kappa, \mu, \lambda$ *are as in the conclusion* $(*)$ *of the previous claim 2.2 and*

(b) $S$ *reflects in no ordinal of cofinality* $\leq \kappa$ *(holds automatically if* $\kappa < \sigma^{+\sigma}$*, see [Sh 108], [Sh 88a]), but see 2.7, 2.8.*

*Then without loss of generality* $\sigma, \theta =: \sigma^+, \lambda, \mathcal{A} = \{B_\delta : \delta \in S\}$ *(and* $\theta^* = \theta$*) satisfies the set theoretic requirements* $(A), (B)_1, (B)_2, (C)$ *in Theorem 1.2 and even* $(C)^*$ *of 1.4.*

*Proof.* Without loss of generality "$\delta \in S \Rightarrow \mu^\omega$ divides $\delta$", and as we are assuming $\mu$ is strong limit of cofinality $\sigma$ and $\lambda = \mu^+ = 2^\mu$ and $\delta \in S \Rightarrow \text{cf}(\delta) = \sigma^+ \neq \sigma = \text{cf}(\mu)$ we have $\diamondsuit_S$ ([Sh:108]). So let $\langle f_\delta : \delta \in S\rangle$ be such that $f_\delta : \delta \to [\delta]^\kappa$ satisfy $(\forall f : \lambda \to [\lambda]^\kappa)(\exists^{\text{stat}}\delta \in S)(f_\delta = f \restriction \delta)$. For each $\delta \in S$, let $B_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \sigma^+\}$ be increasing with $\varepsilon$ and let $g_\delta : \kappa^{++} \to [\kappa^{++}]^{\leq\kappa}$ be defined by

$$g_\delta(\beta) = \big\{\gamma < \kappa^{++} : \text{for some } \varepsilon_1, \varepsilon_2 < \sigma^+ \text{ we have } \kappa^{++} \times \alpha_{\delta,\varepsilon_1} + \gamma \in f_\delta(\kappa^{++} \times \alpha_{\delta,\varepsilon_2} + \beta)\big\}.$$

So by the free subset lemma (Hajnal [Ha61]) there is $Z_\delta \in [\kappa^{++}]^{\kappa^{++}}$ such that $\gamma_1 \neq \gamma_2 \in Z_\delta \Rightarrow \gamma_1 \notin g_\delta(\gamma_2)$. Let $\gamma_{\delta,\varepsilon} \in Z_\delta$ be strictly increasing with $\varepsilon < \sigma^+$ and let $B'_\delta = \{\kappa^{++} \times \alpha_{\delta,\varepsilon} + \gamma_{\delta,\varepsilon} : \varepsilon < \sigma^+\}$. So clauses $(A), (B)_1$ are immediate. Now clearly (C) of 1.2 holds and lastly $(B)_2$ of 1.2 follow from the assumption (b) on $S$ (see [Sh 108]). In order to get $(C)^*$ of 1.7 we should shrink $Z_\delta$ further.

Now $\mathcal{A} = \{B'_\delta : \delta \in S\}$ are as required in Theorem 1.2. $\square_{2.3}$

In similar spirit, we do further analysis.

**2.4 Claim.** *Assume*

(a) $\lambda$ *is regular,* $\theta^\sigma < \lambda$

(b) $\bar{B} = \langle B_\delta : \delta \in S\rangle$*, where* $S \subseteq \lambda$ *is stationary*

(c) $B_\delta \subseteq \delta$ *and* $B_\delta$ *has cardinality* $\theta$

(d) *if* $\delta_1 \neq \delta_2$ *are from* $S$ *then* $\sigma > |B_{\delta_1} \cap B_{\delta_2}|$

(e) $\diamondsuit_S$.

*Then for some* $\langle B'_\delta : \delta \in S'\rangle$ *we have*

(α) $S' \subseteq S$

(β) $B'_\delta \subseteq \delta$ *has order type* $\theta$

(γ) *for* $\delta_1 \neq \delta_2$ *from* $S'$ *we have* $\sigma > |B'_{\delta_1} \cap B'_{\delta_2}|$

(δ) *if* $Z \subseteq \lambda$ *is unbounded in* $\lambda$ *then for stationarily many* $\delta \in S'$ *we have* $B_\delta \subseteq Z$

($\varepsilon$) *$S'$ is stationary*

($\zeta$) *if $F : \lambda \to [\lambda]^{\leq\kappa}, \kappa = \mathrm{cf}(\kappa), \kappa^{++} < \lambda$ <u>then</u> for stationarily many $\delta \in S'$ the set $B_\delta$ is $F$-free.*

*2.5 Remark.* 1) If (a)-(d) of 2.4 hold in $\mathbf{V}$, then (a)-(e) holds after we force with Levy$(\lambda, 2^{<\lambda})$, note that if $\lambda = 2^{<\lambda}$ this is equivalent to adding a Cohen subset to $\lambda$.
2) We can add (in 2.4 see proof below):

$(\delta)^+$ if $Z_\varepsilon \subseteq \lambda = \sup(Z_\varepsilon)$ for $\varepsilon < \theta$ then for stationarily many $\delta \in S'$ we have: for every $\varepsilon < \theta$, the $\varepsilon$-th member of $B_\delta$ belong to $Z_\varepsilon$

$(\beta)_1$ if $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \mathrm{cf}(\theta)$ then $B_\delta$ is unbounded in $\delta$

$(\beta)_2$ if $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \theta_1 \neq \mathrm{cf}(\theta)$ so $\theta_1 < \theta$ then $B_\delta$ has order type $\theta \times \theta_1$ and is unbounded in $\delta$.

*Proof.* Without loss of generality $\mathrm{otp}(B_\delta) = \theta$ and $\delta = \sup(B_\delta)$.

Let $\bar{Z} = \langle Z_\delta : \delta \in S\rangle$ be such that $Z_\delta \subseteq \delta$ and for every $Z \subseteq \lambda$ the set $\{\delta \in S : Z \cap \delta = Z_\delta\}$ is stationary, such a sequence exists as $\diamondsuit_S$ holds.
Now we choose $B'_\delta \subseteq \delta$ by induction on $\delta$ such that $B'_\delta \neq \emptyset \Rightarrow \mathrm{otp}(B'_\delta) = \theta$. We let $B'_\delta$ be $Z^*_\delta = \{\alpha \in Z_\delta : \mathrm{otp}(Z_\delta \cap \alpha) \in B_\delta\}$ when $\mathrm{otp}(Z^*_\delta) = \theta$ & $(\forall \delta' \in S \cap \delta)[|Z^*_\delta \cap B'_{\delta'}| < \sigma]$ and let $B'_\delta$ be $\emptyset$ otherwise. Let $S' = \{\delta \in S : B'_\delta \neq \emptyset\}$ and we shall prove that $\langle B'_\delta : \delta \in S'\rangle$ is as required.

Clauses $(\alpha), (\beta), (\gamma)$ are obvious and clause $(\varepsilon)$ follows from clause $(\delta)$, so let us prove clause $(\delta)$.

Let $Z \subseteq \lambda$ be unbounded. So $C_Z = \{\delta < \lambda : \delta = \sup(Z \cap \delta) = \mathrm{otp}(Z \cap \delta)\}$ is a club of $\lambda$ and let $S_Z = \{\delta \in S : Z \cap \delta = Z_\delta\}$. By the choice of $\bar{Z}$ clearly $S_Z$ is a stationary subset of $\lambda$, so also $S_Z \cap C_Z$ is a stationary subset of $\lambda$. Let $S'_Z = \{\delta \in S_Z \cap C_Z : B'_\delta = Z_\delta\}$, so it is enough to prove that $S'_Z$ is a stationary subset of $\lambda$, we shall prove more:

$(*)$ $S^*_Z = S_Z \cap C \backslash S'_Z$ is not a stationary subset of $\lambda$.

Toward contradiction assume $S^*_Z$ is stationary.
Now for every $\delta \in S^*_Z$, clearly $Z^*_\delta$ is a subset of $Z_\delta = Z \cap \delta$ of order type $\theta$, but $B'_\delta \neq Z^*_\delta$ hence $(\exists \alpha_1 \in S \cap \delta)(|Z^*_\delta \cap B'_{\alpha_1}| \geq \sigma)$, so we choose such $\alpha_\delta \in S \cap \delta$. So for some stationary $S^{**}_Z \subseteq S^*_Z$ and $\alpha^*$ we have $(\forall \delta \in S^{**}_Z)[\alpha_\delta = \alpha^*]$. Now $\delta \in S^{**}_Z$ implies $\sigma \leq |Z^*_\delta \cap B'_{\alpha^*}|$ hence for some $A^*_\delta \in [B'_{\alpha^*}]^\sigma$ we have $A^*_\delta \subseteq Z^*_\delta$. As $|[B'_{\alpha^*}]^\sigma| \leq \theta^\sigma < \lambda = \mathrm{cf}(\lambda)$, possibly shrinking $S^{**}_Z$ for some $A^*$ we have $\delta \in S^{**}_Z \Rightarrow A^*_\delta = A^*$. Now easily $\delta \in S^{**}_Z \Rightarrow B_\delta \supseteq \{\mathrm{otp}(\gamma \cap \mathbb{Z}) : \gamma \in A^*\}$ which has cardinality $\sigma$, so $\delta_1 \neq \delta_2 \in S^{**}_Z \Rightarrow \sigma \leq |B_{\delta_1} \cap B_{\delta_2}|$, contradiction.
Lastly, clause $(\zeta)$ follows from clause $(\delta)$ by 2.1 as $\lambda$ is regular <u>or</u> alternatively if $F : \lambda \to [\lambda]^{\leq\kappa}$, by [Ha61] some unbounded $Z \subseteq \lambda$ is $F$-free so by clause $(\delta)$ there are stationarily many $\delta \in S'$ such that $B'_\delta$ is $F$-free. $\square_{2.4}$

*Proof of 2.5(2).* Without loss of generality $(\forall \delta \in S)(\mathrm{cf}(\delta) = \theta_1)$ for some $\theta_1$.

Let $\delta^* \in [\theta, \theta^+)$, $\mathrm{cf}(\delta^*) = \theta_1$ for what we state in 2.5 we have $\delta^*$ is $\theta$ if $\theta_1 = \theta$ and is $\theta \times \theta_1$ if $\theta_1 < \theta$. Let $h : \delta^* \to \theta$ be one to one onto. Let $\langle\langle Z_{\delta,\varepsilon} : \varepsilon \leq \delta^*\rangle : \delta \in S\rangle$ be such that for every sequence $\bar{Z} = \langle Z_\varepsilon : \varepsilon \leq \delta^*\rangle$ satisfying $Z_\varepsilon \subseteq \lambda$ the set $\{\delta \in S : (\forall \varepsilon \leq \delta^*)(Z_\varepsilon \cap \delta = Z_{\delta,\varepsilon})\}$ is stationary, it exists as $\diamondsuit_S$ holds. Moreover, we can find $\langle C_\delta : \delta \in S\rangle$ such that $C_\delta$ is a closed unbounded subset of order type $\delta^*$, let $C_\delta = \{\gamma_{\delta,\varepsilon} : \varepsilon < \delta^*\}$ and if $Z_\varepsilon \subseteq \lambda$ for $\varepsilon < \delta^*$ then the following subset of $S$ is stationary

$$\begin{aligned}\{\delta \in S :& Z_{\delta,\varepsilon} = Z_\varepsilon \cap \delta \text{ for } \varepsilon < \delta^* \text{ and}\\ &\operatorname{otp}(Z_\varepsilon \cap \gamma_{\delta,\zeta}) = \gamma_{\delta,\zeta} \text{ for } \varepsilon < \zeta < \delta^* \text{ and}\\ &\gamma_{\delta,\varepsilon} \text{ is closed under pr (a pairing function)}\}.\end{aligned}$$

Let

$$\begin{aligned}Z^*_\delta = \{\alpha :& \text{for some } \varepsilon < \delta^* \text{ and } \beta \text{ we have}\\ &\gamma_{\delta,\varepsilon} < \alpha < \gamma_{\delta,\varepsilon+1}, \beta \in A_\delta,\ \operatorname{otp}(\beta \cap A_\delta) = h(\varepsilon)\\ &\text{and } \alpha \text{ is the } \operatorname{pr}(\gamma_{\delta,\varepsilon}, \beta)\text{-th member of } Z_{\delta,\varepsilon}\}.\end{aligned}$$

Now check. $\square_{2.5}$

<u>2.6 Conclusion</u>: If CON(ZFC $+\exists$ supercompact), <u>then</u> CON(CH + there is a $T_3$-topological space $X$ with clopen basis, even compact, with $\aleph_{\omega+1}$ nodes such that if we divide $X$ to countably many parts, at least one contains a closed copy of the Cantor discontinuum $\omega_2$).

*Proof.* By 2.2 + 2.3 we get a universe with GCH and $\sigma = \aleph_0, \theta = \aleph_1, \kappa = \aleph_2, \lambda = \aleph_{\omega+1}$ satisfying the set theoretic requirements of 1.2. So as the Cantor discontinuum satisfies clauses (D), (E) of 1.2 and the demand in 1.2(2) we are done by 1.2. $\square_{2.6}$

* * *

Lastly, we start to resolve the connection between the various statements around. Now [HJSh 249] continue and strengthen [Sh 108], [Sh 88a] (and [HJSh 697] continue them). We show that by a "small nice forcing" (not involving extra large cardinals assumption) we can get the result of [HJSh 249] used above from the one in [Sh 108], [Sh 88a]. (See also [Sh 652, §5] on the semi-additive colouring involved, i.e. it is proved that consistently there is a colouring of the kind appearing in the analysis (there, or see the proof of 2.7 below)). On $I[\lambda]$ see [Sh 108], [Sh 88a], [Sh 420, §1]. However, there is a price, our "small nice forcing" has to violate G.C.H. quite strongly.

**2.7 Claim.** *Assume*

(*a*) $\operatorname{cf}(\mu) = \kappa < \mu$ *and* $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$ *and* $\lambda = \mu^+$

(*b*) $S \subseteq \{\delta < \lambda : \operatorname{cf}(\delta) = \kappa^+\}$ *is stationary,* $S \notin I[\lambda]$ *and*

(*c*) $2^{\kappa^+} < \lambda$ *and* $\kappa = \kappa^{<\kappa}$.

*<u>Then</u> for some forcing notion* $\mathbb{Q}$ *we have:*

($\alpha$) $\mathbb{Q}$ *is* $(<\kappa)$*-complete,* $|\mathbb{Q}| = \kappa^+$ *and* $\mathbb{Q}$ *is* $\kappa^+$*-c.c.*

($\beta$) *in* $\mathbf{V}^{\mathbb{Q}}$*, for some stationary* $S' \subseteq S$ *there is a sequence* $\langle A_\delta : \delta \in S'\rangle$ *such that each* $A_\delta$ *is an unbounded subset of* $\delta$ *of order type* $\kappa^+$ *and* $\delta_1 \neq \delta_2 \in S' \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \kappa$

*Proof.* Let $\mu = \sum_{i<\kappa} \lambda_i$ where $\lambda_i < \mu$ is increasing continuous with $i, \lambda_0 > \kappa$. Choose $\bar{A} = \langle A_\alpha : \alpha \in S\rangle$, with $A_\alpha = \{\gamma_{\alpha,\varepsilon} : \varepsilon < \kappa^+\}$ being any unbounded subset of $\alpha$ of order type $\kappa^+$ and $\gamma_{\alpha,\varepsilon}$ increasing with $\varepsilon$.

We can find $\bar{a}^\alpha = \langle a^\alpha_i : i < \kappa\rangle$ for $\alpha < \mu^+$ such that

$(*)_1$ $\alpha = \bigcup\limits_{i<\kappa} a^\alpha_i, a^\alpha_i$ is increasing continuous in $i, |a^\alpha_i| \le \lambda_i$

$(*)_2$ if $\alpha \in a^\beta_i$ then $a^\alpha_i \subseteq a^\beta_i$.

Without loss of generality

$(*)_3$ $A_\alpha \subseteq a^\alpha_0$.

Let $\mathbf{c} : [\mu^+]^2 \to \kappa$ be $\mathbf{c}\{\alpha,\beta\} = \text{Min}\{i : \alpha \in a^\beta_i\}$ for $\alpha < \beta < \lambda^+$ so

$\boxtimes$ $\alpha < \beta < \gamma \Rightarrow \mathbf{c}\{\alpha,\gamma\} \le \text{Max}\{\mathbf{c}\{\alpha,\beta\}, \mathbf{c}\{\beta,\gamma\}\}$.

For $\alpha \in S$ let $c_\alpha : [\kappa^+]^2 \to \kappa$ be defined by:
for $\varepsilon < \zeta < \kappa^+$ we let

$$c_\alpha\{\varepsilon,\zeta\} = \mathbf{c}\{\gamma_{\alpha,\varepsilon}, \gamma_{\alpha,\zeta}\}.$$

Let $\mathcal{C} = \{c_\alpha : \alpha \in S\}$ so $c_\alpha \in {}^{([\kappa^+]^2)}\kappa$, hence $|\mathcal{C}| \le 2^{\kappa^+}$. Let for $c \in \mathcal{C}, S_c = \{\alpha \in S : c_\alpha = c\}$, so $\langle S_c : c \in \mathcal{C}\rangle$ is a partition of $S$ to $\le 2^{\kappa^+} < \mu^+$ sets hence necessarily for some $c_* \in \mathcal{C}$ we have

$(*)_4$ $S_{c_*} \notin I[\lambda]$ and in particular is stationary.

We fix $c_*$. We define a forcing notion $\mathbb{Q}$:

$(A)$ $p \in \mathbb{Q}$ iff $p = (u^p, \xi^p)$ where $u^p \in [\kappa^+]^{<\kappa}$ and $\xi^p < \kappa$ and $\text{Rang}(c^* \restriction [u^p]^2) \subseteq \xi^p$

$(B)$ $\mathbb{Q} \models p \le q$ <u>iff</u>: $(p, q \in \mathbb{Q}$ and)
  $(i)$ $u^p \subseteq u^q$
  $(ii)$ $\xi^p \le \xi^q$
  $(iii)$ for every $\beta \in u^p$ and $\alpha \in (u^q \backslash u^p) \cap \beta$ we have $c_*\{\alpha,\beta\} \ge \xi^p$.

Now

$(*)_5$ $(a)$ $\mathbb{Q}$ is a $(<\kappa)$-complete partial order of cardinality $\kappa^+$
  $(b)$ $\mathbb{Q}' =: \{p \in \mathbb{Q} : u^p$ has a maximal element$\}$ is a dense subset of $\mathbb{Q}'$
  $(c)$ if $\alpha < \kappa^+$ then $\mathbb{Q}'_\alpha = \{p \in \mathbb{Q} : u^p$ has a maximal element and $\max(u') > \alpha\}$ is a dense subset of $\mathbb{Q}'$.

[Why? As $\kappa = \kappa^{<\kappa}$ clearly $|\mathbb{Q}| = \kappa^+$ and $\mathbb{Q}$ is closed under union of length $<\kappa$, together we have Clause (a), as for clause (b), for any $p \in \mathbb{Q}$ choose $j \in (\sup(u^p) + 1, \kappa^+)$ and define $q = (u^q, \xi^q)$ by $u^q = u^p \cup \{j\}$ and $\xi^q = \sup(\{\xi^p\} \cup \text{Rang}(c_* \restriction [u^q]^2)) + 1$ clearly $p \le q \in \mathbb{Q}$ (clause (iii) of (B) is empty) and $u^q$ has a last member $j$. Clause (c) has the same proof except that we choose $j > \alpha$]

$(*)_6$ $\mathbb{Q}$ satisfies the $\kappa^+$-c.c.

[Why? Assume toward contradiction that $\langle p_i : i < \kappa^+\rangle$ are pairwise incompatible. Without loss of generality $p_i \in \mathbb{Q}'$. As $\kappa = \kappa^{<\kappa}$ without loss of generality $\langle u^{p_i} : i < \kappa^+\rangle$ is a $\Delta$-system with heart $u^*$. Also without loss of generality $\xi^{p_i} = \xi^*$. So $C = \{\delta < \kappa^+ : u^{p_\delta} \backslash u^*$ is disjoint to $\delta$ and $(\forall j < \delta)(u^{p_j} \subseteq \delta)\}$ is a club of $\kappa^+$. For $\delta \in C$ let $\varepsilon_\delta = \text{Min}(u^{p_\delta} \backslash \delta)$ and $\zeta_\delta = \max(u^{p_\delta})$ so $\delta \le \varepsilon_\delta \le \zeta_\delta$. Now assume $\alpha < \beta$ are from $C$, and $p_\alpha, p_\beta$ is incompatible. Why is $q = (u^{p_\alpha} \cup u^{p_\beta}, \xi)$ not a common upper bound where we let $\xi = \sup(\{\xi^*\} \cup \text{Rang}(\mathbf{c} \restriction [u^{p_\alpha} \cup u^{p_\beta}]^2)) + 1$? As $q \in \mathbb{Q}$ and as $u^{p_\alpha} \cap \alpha = u^{p_\beta} \cap \beta, u^{p_\alpha} \subseteq \beta$ and $\xi^* = \xi^{p_\alpha} = \xi^{p_\beta} \le \xi^q$ clearly $p_\alpha \le q$, hence necessarily $\neg(p_\beta \le q)$ so clause (iii) of (B) fails, i.e. for some

$\gamma_2 \in u^{p_\beta}$ and $\gamma_1 \in u^q \cap \gamma_2 \backslash u^{p_\beta}$ (hence $\gamma_1 \in u^{p_\alpha} \backslash \alpha$ and $\gamma_2 \in u^{p_\beta} \backslash \beta$) we have $c_*\{\gamma_1, \gamma_2\} < \xi^{p_\beta} = \xi^*$. But $\varepsilon_\alpha \le \gamma_1$ and $\varepsilon_\alpha < \gamma_1 \Rightarrow c_*\{\varepsilon_\alpha, \gamma_1\} < \xi^{p_\alpha} = \xi^*$ and $\gamma_2 \le \zeta_\beta$ and $\gamma_2 < \zeta_\beta \Rightarrow c_*\{\gamma_2, \zeta_\beta\} < \xi^{p_\beta} = \xi^*$. Hence by $\boxtimes$ above necessarily $c_*\{\varepsilon_\alpha, \zeta_\beta\} < \xi^*$.
So for $\delta \in S_{c_*}, \langle \gamma_{\delta, \varepsilon_i} : i \in C \rangle$ is strictly increasing hence with limit $\delta$ and for each $i \in C, \gamma_{\delta, \zeta_i}$ is above $\{\gamma_{\delta, \varepsilon_j} : j < i\}$ but $< \delta$ and

$$j < i \Rightarrow \mathbf{c}\{\gamma_{\delta,\varepsilon_j}, \gamma_{\delta,\zeta_i}\} < \xi^* \Rightarrow \gamma_{\delta,\varepsilon_j} \in a_{\xi^*}^{\gamma_{\delta,\zeta_i}}.$$

By [Sh:108] it follows that $S \in I[\lambda]$ (or directly, for every $\gamma < \lambda, |\{\langle \gamma_{\delta,\varepsilon_j} : j \in C \cap i^* \rangle : \delta \in S, i^* \in C_\zeta, \gamma_{\delta,\zeta_i^*} = \gamma\}| < \lambda$ as for each $i < \kappa^+$ (and $\gamma$) we have $\le |a_{\xi^*}^\gamma|^{|i^*|} \le (\lambda_{\xi^*})^{|i|} \le \mu$ possibilities); contradiction to $(*)_4$. So $\mathbb{Q}$ satisfies the $\kappa^+$-c.c.]

Now clearly for every $i < \kappa^+$ there is $p_i \in \mathbb{Q}'$ such that $i < \max(u^{p_i})$, hence (by $(*)_6$), for some $i(*) < \kappa^+$ we have $p_{i(*)} \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{W}_1 = \{i : p_i \in G \text{ and} \operatorname{cf}(i) = \kappa\}$ is stationary in $\kappa^+$". Let $p_{i(*)} \in G \subseteq \mathbb{Q}$ with $G$ generic over $\mathbf{V}$ and $W_1 = \underset{\sim}{W}_1[G]$. Let $C = \{\delta < \kappa^+ : (\forall i < \delta)[\sup(u^{p_i}) < \delta]\}$, it is a club of $\kappa^+$. Let $W_2 = C \cap W_1$ and for $i \in S_2$ let $\varepsilon_i = \operatorname{Min}(u^{p_i} \backslash i), \zeta_i = \max(u^{p_i})$. Now

$(*)_7$ if $i \in W_2$ and $\xi < \kappa$, <u>then</u> $\{j \in W_1 \cap i : c_*(\varepsilon_j, \varepsilon_i) < \xi\}$ has cardinality $< \kappa$.

[Why? By density argument for some $q \in G$ we have $p_i \le q$ and $\xi^q > \xi$. Now if $j \in W_1 \cap i \backslash u^q$ then $p_j \in G$ hence for some $q^+ \in G \subseteq \mathbb{Q}$ we have $q \le q^+$ & $p_j \le q^+$, so $\varepsilon_j \in u^{q^+} \cap \varepsilon_i$ and as $q \le q^+$ by the definition of $\le^{\mathbb{Q}}$, necessarily $c_*(\varepsilon_i, \varepsilon_j) \ge \xi^q > \xi$, as asserted.]

Now for $\delta \in S_{c_*}$ define $A'_\delta = \{\gamma_{\delta,\varepsilon} : \varepsilon \in W_2\}$. So $A'_\delta$ is an unbounded subset of $\delta$ of order type $\kappa^+$.

$(*)_8$ if $\delta_1 \ne \delta_2$ are from $S_{c_*}$ then $A'_{\delta_1} \cap A'_{\delta_2}$ has cardinality $< \kappa$.

[Why? Without loss of generality $\delta_1 < \delta_2$, let $\varepsilon(*) \in S_2$ be such that $\delta_1 < \gamma_{\delta_2, \varepsilon(*)}$. Assume toward contradiction that $A = A'_{\delta_1} \cap A'_{\delta_2}$ has cardinality $\ge \kappa$. Recall (by $(*)_3$) that $\beta \in A \Rightarrow \mathbf{c}\{\beta, \delta_1\} = 0$; now letting $\xi^* = \mathbf{c}\{\delta_1, \gamma_{\delta_2,\varepsilon(*)}\} < \kappa$ we get by $\boxtimes$ that $\beta \in A \Rightarrow \mathbf{c}\{\beta, \gamma_{\delta_2,\varepsilon(*)}\} \le \max\{\mathbf{c}\{\beta, \delta_1\}, \mathbf{c}\{\delta_1, \gamma_{\delta_1,\varepsilon(*)}\} = \max\{0, \xi^*\} = \xi^*$.
So $A^- = \{\varepsilon < \kappa^+ : \gamma_{\delta_2,\varepsilon} \in A\}$ has cardinality $\kappa$ and $\varepsilon \in A^- \Rightarrow c_*\{\varepsilon, \varepsilon(*)\} \le \xi^*$, contradicting $(*)_7$.]

So we are done. $\square_{2.7}$

**2.8 Claim.** *Assume*

$(A)(i)$ $\lambda > \kappa > \theta > \sigma \ge \aleph_0$ *and* $\kappa = \kappa^{<\kappa}$

$(B)_1$ $\mathcal{A} \subseteq [\lambda]^\theta$ *and* $A_1 \ne A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma$.

<u>Then</u> *for some forcing notion* $\mathbb{Q}$ *and* $\mathbb{Q}$*-name* $\underset{\sim}{\mathcal{A}}'$ *of a subset of* $\mathcal{A}$ *we have:*

- $(a)$ $\mathbb{Q}$ *is a strategically* $(< \kappa)$*-complete forcing notion (hence add no new sequence of length* $< \kappa$*)*
- $(b)$ $\mathbb{Q}$ *is* $\kappa^+$*-c.c. forcing notion of cardinality* $\lambda^{<\kappa}$
- $(c)$ *in* $\mathbf{V}^{\mathbb{Q}}$*, clauses* $(A)(i), (B)_1$ *above still hold for* $\mathcal{A}$ *hence for* $\underset{\sim}{\mathcal{A}}'$ *and* $\underset{\sim}{\mathcal{A}}'$ *satisfies also* $(B)_2$ *from 1.2, i.e.*
  $\mathcal{A}$ *is* $(< \kappa)$*-free*

(*d*) *if* $\lambda, \kappa, \mathcal{A}$ *satisfies clause* (*C*) *of 1.2 in* $\mathbf{V}$, *then* $\lambda, \kappa, \underset{\sim}{\mathcal{A}}$ *satisfies clause* (*C*) *in* $\mathbf{V}^{\mathbb{Q}}$

(*e*) *like clause* (*d*) *for* $(C)^*$ *from 1.4*

*Proof.* Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$ with no repetitions.
Let $\mathbb{Q}$ be the set of $p = (v, v_*) = (v^p, v^p_*)$ such that:

(*a*) $v_* \subseteq v \in [\lambda^*]^{<\kappa}$

(*b*) there is a list $\bar\zeta = \langle \zeta(\varepsilon) : \varepsilon < \varepsilon^*\rangle$ of $v_*$ such that for every $\varepsilon < \varepsilon^*$ we have $A_{\zeta(\varepsilon)} \cap \bigcup\limits_{\xi<\varepsilon} A_{\zeta(\xi)}$ has cardinality $< \sigma$; we call $\langle \zeta(\varepsilon) : \varepsilon < \varepsilon^*\rangle$ a witness, also the list $\bar\zeta$ and the the well ordering on $v^p_*$ it induces are called witnesses.

The order is defined by

$$
\begin{array}{lll}
p \le q \ \underline{\text{iff}}\ (\alpha) & & v^p_* \subseteq v^q_* \text{ and} \\
(\beta) & & v^p \backslash v^p_* \subseteq v^q \backslash v^q_* \\
(\gamma) & & \text{every } \bar\zeta \text{ witnessing } p \in \mathbb{Q} \text{ can be end-extended to } \bar\zeta' \\
& & \text{witnessing } q \in \mathbb{Q}.
\end{array}
$$

Define $\mathbb{Q}$-names $\underset{\sim}{Y} = \cup\{v^p_* : p \in \underset{\sim}{G}_{\mathbb{Q}}\}$ and $\underset{\sim}{\mathcal{A}}' = \{A_\zeta : \zeta \in \underset{\sim}{Y}\}$. Now

$(*)_1$ $\mathbb{Q}$ is a partial order

$(*)_2$ $|\mathbb{Q}| = (\lambda^*)^{<\kappa} \le (\lambda^{<\theta})^{<\kappa} = \lambda^{<\kappa}$

$(*)_3$ any increasing continuous sequence of members of $\mathbb{Q}$ of length $< \kappa$ has a least upper bound.
Hence

$(*)_4$ $\mathbb{Q}$ is strategically $(< \kappa)$-complete.

For $p \in \mathbb{Q}$ let $u^p = \cup\{A_\zeta : \zeta \in v^p\}$

$(*)_5$ for $p \in \mathbb{Q}$ we have $u^p \in [\lambda]^{<\kappa}$ and $p \le q \Rightarrow u^p \subseteq u^q$.

Let $\mathbb{Q}' = \{p \in \mathbb{Q} :$ if $\zeta < \lambda^*$ and $|A_\zeta \cap u^p| \ge \sigma$ then $\zeta \in v^p\}$; compare with the proof of 1.7. For $p \in \mathbb{Q}$ let $v^p_\otimes = \{\zeta < \lambda^* : |A_\zeta \cap u^p| \ge \sigma\}$, so:

$(*)_6$ (*a*) $v^p \subseteq v^p_\otimes$ and $p \in \mathbb{Q} \Rightarrow |v^p_\otimes| \le \kappa$ and

(*b*) if $(\forall \alpha < \kappa)[|\alpha|^\sigma < \kappa]$ <u>then</u> $p \in \mathbb{Q} \Rightarrow |v^p_\otimes| < \kappa$, and

(*c*) if $p \in \mathbb{Q}$ <u>then</u> $(p \in \mathbb{Q}') \Rightarrow (v^p_\otimes = v^p)$ and

(*d*) $\mathbb{Q}'$ is a dense subset of $\mathbb{Q}$ if $(\forall \alpha < \kappa)[|\alpha|^\sigma < \kappa]$

[Why? E.g. for clause (d), let $p \in \mathbb{Q}$ we choose by induction on $\varepsilon \le \sigma^+ (< \kappa)$ a condition $p_\varepsilon$ such that: $p_0 = p, v^{p_\varepsilon}_* = v^p_*, p_\varepsilon$ is increasing continuous with $\varepsilon$ and $v^{p_{\varepsilon+1}} = \{\zeta < \lambda^* : \zeta \in v^{p_\varepsilon}$ or just $|A_\zeta \cap u^{p_\varepsilon}| \ge \sigma\}$. There are no problems and $p_{\sigma^+}$ is as required as $|A_\zeta \cup u^{p_{\sigma^+}}| \ge \sigma \Rightarrow$ for some $\varepsilon < \sigma^+, |A_\zeta \cap u^{p_\varepsilon}| \ge \sigma \Rightarrow$ for some $\varepsilon < \sigma^+, \zeta \in v^{p_{\varepsilon+1}} \subseteq v^{p_{\sigma^+}}$.]

$(*)_7$ if $p \in \mathbb{Q}', \zeta \in \lambda^* \backslash v^p$ or just $p \in \mathbb{Q}, \zeta \in \lambda^* \backslash v^p_\otimes$ <u>then</u> $p' = (v^p \cup \{\zeta\}, v^p_* \cup \{\zeta\})$ and $p'' = (v^p \cup \{\zeta\}, v^p_*)$ are in $\mathbb{Q}$ (even $p \in \mathbb{Q}' \Rightarrow p' \in \mathbb{Q}'$) and are $\ge p$.

We say $p_0, p_1 \in \mathbb{Q}$ are isomorphic if $\mathrm{otp}(v^{p_0}) = \mathrm{otp}(v^{p_1})$, $\mathrm{otp}(u^{p_0}) = \mathrm{otp}(u^{p_1})$, and $\mathrm{OP}_{v^{p_1}, v^{p_0}}$ maps $v^{p_0}_*$ onto $v^{p_1}_*$, $OP_{u^{p_1}, u^{p_0}}$ maps $u^{p_0}$ onto $u^{p_1}$ and for $\zeta \in v^{p_0}, \alpha \in u^{p_0}$ we have $\alpha \in A_\zeta \Leftrightarrow \mathrm{OP}_{u^{p_1}, u^{p_0}}(\alpha) \in A_{\mathrm{OP}_{v^{p_1}, v^{p_0}}(\zeta)}$

$(*)_8$ $\mathbb{Q}$ satisfies the $\kappa^+$-c.c.

[Why? Let $p_\alpha \in \mathbb{Q}$ for $\alpha < \kappa^+$. Let $v_\alpha = \bigcup\limits_{\beta<\alpha} v^{p_\beta}_\otimes$ and $u_\alpha = \cup\{A_\zeta : \zeta \in v_\alpha\}$ so $u^{p_\beta} \subseteq u_\alpha$ for $\beta < \alpha$ and $\langle v_\alpha : \alpha < \kappa^+\rangle, \langle u_\alpha : \alpha < \kappa^+\rangle$ are increasing continuous. As $v^{p_\alpha} \in [\lambda^*]^{<\kappa}$, we can find stationary $S \subseteq \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$ and $v$ such that $\alpha \in S \Rightarrow v^{p_\alpha} \cap v_\alpha = v$. Similarly without loss of generality $\alpha \in S \Rightarrow u^{p_\alpha} \cap u_\alpha = u$. Without loss of generality for $\alpha, \beta \in S$ the conditions $p_\alpha, p_\beta$ are isomorphic, the isomorphisms being the identity $v$ and $u$. So $v^{p_\alpha}_* \cap v = v_*$ for some $v_* \subseteq v$. Let $<^*_\alpha$ be a well ordering of $v^{p_\alpha}_*$ which witnesses $p_\alpha \in \mathbb{Q}$, so without loss of generality $<^*_\alpha \restriction v_* = <^*$. Let $\alpha < \beta$ be in $S$ and define $q = (v^{p_\alpha} \cup v^{p_\beta}, v^{p_\alpha}_* \cup v^{p_\alpha}_*)$. Clearly $v^q_* \subseteq v^q \in [\lambda^*]^{<\kappa}$, also $\zeta \in v^{p_\alpha} \backslash v^{p_\beta}$ or $\zeta \in v^{p_\beta} \backslash v^{p_\alpha}$ implies $|A_\zeta \cap u| < \sigma$.

(Why? If not by the isomorphism of $p_\alpha$ and $p_\beta$ we can find $\zeta_1 \in v^{p_\alpha} \backslash v^{p_\beta}, \zeta_2 \in v^{p_\beta} \backslash v^{p_\alpha}$ such that $\zeta_2 = \mathrm{OP}_{v^{p_\beta}, v^{p_\alpha}}(\zeta_1)$ and $\zeta \in \{\zeta_1, \zeta_2\}$ and $A_{\zeta_1} \cap u = A_{\zeta_2} \cap u$, so $|A_{\zeta_\ell} \cap u| \geq |A_{\zeta_1} \cap A_{\zeta_2}| \geq \sigma$ hence $\zeta_2 \in v^{p_{\zeta_1}}_\otimes$ hence $\zeta_2 \in v$ so $\zeta_2 \in v^{p_\alpha} \cap v^{p_\beta}$ hence $\zeta_1 = \zeta_2 \in v^{p_\alpha} \cap v^{p_\beta}$, contradiction.

Hence $\zeta \in v^{p_\alpha} \backslash v^{p_\beta} \Rightarrow |A_\zeta \cap u^{p_\beta}| < \sigma$ (otherwise $A_\zeta \cap u^{p_\beta} \subseteq u_\beta \cap u^{p_\beta} = u$ hence $|A_\zeta \cap u| \geq \sigma$ and get a contradiction by the previous statement) and $\zeta \in v^{p_\beta} \backslash v^{p_\alpha} \Rightarrow |A_\zeta \cap v^{p_\alpha}| < \sigma$ (similar proof). Now define a two-place relation $<^*$ on $v^q_*$:

$$\begin{aligned} \zeta_1 <^* \zeta_2 \ \underline{\text{iff}}\ & \zeta_1 <^*_\alpha \zeta_2 \ (\text{so } \zeta_1, \zeta_2 \in v^{p_\alpha}) \\ & \text{or } \zeta_1 \in v^{p_\alpha}_* \ \ \&\ \ \zeta_2 \in v^{p_\beta}_* \backslash v^{p_\alpha}_* \\ & \text{or } \{\zeta_1, \zeta_2\} \subseteq v^{p_\beta}_* \backslash v^{p_\alpha}_* \ \ \&\ \ \zeta_1 <^*_\beta \zeta_2 \end{aligned}$$

Easily $<^*$ is a well order of $v^q_*$ (as $\zeta \in v^{p_\beta}_* \backslash v^{p_\alpha}_* \Rightarrow |A_\zeta \cap u^{p_\alpha}| < \sigma$), and it is a witness. So $q \in \mathbb{Q}$. Does $p_\alpha \leq q$? Clauses $(\alpha), (\beta)$ are very straight and for clause $(\gamma)$, as $p_\alpha, p_\beta$ are isomorphic for any given witness $<^1$, a well ordering of $v^{p_\alpha}_*$, we can find $<^2$, a witness for $p_\beta$ which is a well ordering of $v^{p_\beta}_*$, and is conjugate to $<^1$; now use $<^1, <^2$ as we use $<^*_\alpha, <^*_\beta$ above. So really $p_\alpha \leq q$. Similarly $p_\beta \leq q$.]

$(*)_9$ $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathcal{A}}' = \{A_\zeta : \zeta \in \cup\{v^p_* : p \in \underset{\sim}{G}_{\mathbb{Q}}\}\}$ is $(< \kappa)$-free".

[Why? Read the definitions of $\mathbb{Q}$ and of being $(< \kappa)$-free, remembering that forcing with $\mathbb{Q}$ add no new sets of ordinals $< \kappa$ as it is strategically $(< \kappa)$-complete.]

$(*)_{10}$ if $p, q \in \mathbb{Q}$ are compatible, <u>then</u> they have an upper bound $r \in \mathbb{Q}$ such that $v^r = v^p \cup v^q$

$(*)_{11}$ if $\mathcal{A}$ satisfies clause (C) of 1.2 then $\underset{\sim}{\mathcal{A}}'$ satisfies it in $\mathbf{V}^{\mathbb{Q}}$.

[Why? Assume $p^* \in \mathbb{Q}, p^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{F} : \lambda \to [\lambda]^{\leq\kappa}$ is a counterexample". As $\mathbb{Q}$ satisfies the $\kappa^+$-c.c. and as increasing the $\underset{\sim}{F}(\alpha)$ is O.K., without loss of generality each $\underset{\sim}{F}(\alpha)$ is an object from $\mathbf{V}$ so for some function $F : \lambda \to [\lambda]^{\leq\kappa}$ from $\mathbf{V}$ we have $\underset{\sim}{F} = F$. As we can increase each $F(\alpha)$, without loss of generality $\zeta \in v^{p^*}_\otimes \Rightarrow A_\zeta \subseteq \bigcap\limits_\alpha F(\alpha)$. As $\mathbf{V}, \mathcal{A}$ satisfies clause (C) there are $\zeta$ and $A \in [A_\zeta]^\theta$ which is $F$-free, by the previous sentence $\zeta \notin v^{p^*}_\otimes$. Define $q = (v^q, v^q_*), v^q = v^{p^*} \cup \{\zeta\}, v^q_* = v^{p^*}_* \cup \{\zeta\}$. It is easy to prove $p^* \leq q \in \mathbb{Q}$, the point being $|A_\zeta \cap \cup\{A_\xi : \xi \in v^{p^*}_*\}| < \sigma$ which holds as $\zeta \notin v^{p^*}_\otimes$, and $q$ forces that $A \in [A_\zeta]^\theta$ is as required concerning $F$.] $\square_{2.8}$

*2.9 Observation.* Assume that $\kappa = \kappa^{<\kappa} < \lambda$ and $S \subseteq \lambda$ stationary. <u>Then</u> for some $\kappa^+$-c.c., strategically $\kappa$-complete forcing notion $\mathbb{Q}$ of cardinality $\lambda^{<\kappa}$, we have $\Vdash_{\mathbb{Q}}$ "$S$ is the union of $\leq \kappa$ sets each not reflecting any $\delta$ of cofinality $< \kappa$".

*Proof.* Straightforward. [Used in $(C) \Rightarrow (D)$ of the proof of 3.2 below.] $\square_{2.9}$

So putting together the claims above we can conclude, e.g.
<u>2.10 Conclusion</u> If $(*)$ below holds, <u>then</u> there is a forcing notion $\mathbb{P}$ of cardinality $2^\mu = \mu^\sigma$ not adding sequences of length $< \kappa$, not collapsing cardinals $\leq \mu^+$ (or $> 2^\mu$), not changing cofinalities such that in $\mathbf{V}^{\mathbb{P}}$ the cardinals $(\sigma < \theta < \kappa = \kappa^{<\kappa}, 2^\kappa \leq \mu)$ satisfies the assumption of 1.2; also its conclusion and $(C)^*$ of 1.7 where

- $(*)$ $\sigma = \operatorname{cf}(\sigma), \theta = \sigma^+ < \kappa = \kappa^{<\kappa} < \mu, \mu$ strong limit singular of cofinality $\sigma$ such that $\{\delta < \mu^+ : \operatorname{cf}(\delta) = \sigma^+\} \notin I[\lambda]$.

## §3 Equi-consistency

Let ${}^\omega 2$ denote here the Cantor discontinuum.

The following theorem clarifies the consistency strength of the problem to a large extent. We can hardly expect a stronger kind of result as long as inner models for supercompacts have not been discovered. Concentrating on ${}^\omega 2$ is for historical reason; we can replace $\aleph_0$ by $\mu$. Also, using the same claims we can replace $\lambda > \beth_2$ by other restrictions. Note that 3.7 continues [Sh 460, §3], [HJSh 249]. The claims will give more, naturally. However, a real problem is:

3.1 Problem: What occurs if we demand GCH?

**3.2 Theorem.** *The following are equi-consistent with ZFC + $\kappa = \mathrm{cf}(\kappa) > 2^{\aleph_0}$. (In fact we get more than equiconsistency: the model for one statement is gotten from another by (set) forcing. Moreover, the forcing notions we use are from a very restricted family where $\kappa$ is involved in its definition. We use only forcing notions which preserves the cardinals and cofinalities $\le (2^{\aleph_0})^+$ and even $\le \kappa$ and do not change the value of $2^{\aleph_0}$, in fact finite composition of $\kappa$-complete ones and c.c.c. of cardinality $\le 2^{\aleph_0}$ ones; so we can add $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} = \aleph_2$ or $2^{\aleph_0} = \aleph_{\omega^3+\omega+3}$ or whatever, to all clauses simultaneously)*

$(A)[{}^\omega 2] = (A)_{(\omega_2)}$ *there is a compact Hausdorff space $X$ such that $X \to_w ({}^\omega 2)^1_2$ but no subspace with $\le 2^{<\kappa}$ points has this property*[6] *(on $\to_w$ see 1.1(2) and ${}^\omega 2$ is the Cantor discontinuum)*

$(A)^+$ *like $(A)_{({}^\omega 2)}$ replacing ${}^\omega 2$ by "for any Hausdorff space $Y^*$ with $\le 2^{\aleph_0}$ points"*

$(B)[{}^\omega 2] = (B)_{(\omega_2)}$ *there is a compact Hausdorff space $X$ with clopen basis such that $X \to ({}^\omega 2)^1_{<\mathrm{cf}(2^{\aleph_0})}$ but no subspace with $\le 2^{<\kappa}$ points has this property*

$(B)^+$ *like $(B)[{}^\omega 2]$ replacing ${}^\omega 2$ by "for any Hausdorff space $Y^*$ with $\le 2^{\aleph_0}$ points" and demand $X$ has a clopen basis only if $Y$ has; i.e. for every Hausdorff space $Y^*$ with $\le 2^{\aleph_0}$ points <u>there is</u> a Hausdorff space $X$ with clopen basis if $Y^*$ has, such that $X \to (Y^*)^1_{<\mathrm{cf}(2^{\aleph_0})}$ but no subspace of $X$ with $\le 2^{<\kappa}$ points has this property*

$(C)$ *there are $\lambda, S, \bar f$ such that*

(1) $(a)$ *$S \subseteq \lambda$ is stationary, $\lambda > 2^{<\kappa}$ is regular*

(2) $(b)$ *$\bar f = \langle f_\delta : \delta \in S\rangle$*

(3) $(c)$ *$f_\delta$ is a one-to-one function from $A \subseteq {}^\omega 2$ of cardinality $2^{\aleph_0}$ to $\delta$*

(4) $(d)$ *if $\delta_1 \ne \delta_2$ then $\{\eta \in {}^\omega 2 : f_{\delta_1}(\eta) = f_{\delta_2}(\eta)\}$ has scattered closure (in the topological space ${}^\omega 2$)*

$(D)$ *there are $\lambda, S, \bar A$ such that*

(1) $(a)$ *$S \subseteq \lambda$ is stationary, $\lambda > 2^{<\kappa}$ is regular*

(2) $(b)$ *$\bar A = \langle A_\delta : \delta \in S\rangle$*

(3) $(c)$ *$A_\delta$ is a subset of $\delta$ of cardinality $2^{\aleph_0}$*

(4) $(d)$ *for $\delta_1 \ne \delta_2$ from $S$ we have $A_{\delta_2} \cap A_{\delta_1}$ is finite*

(5) $(e)$ *$\{A_\delta : \delta \in S\}$ is $\kappa$-free, that is, for any $u \in [S]^{<\kappa}$ there is a sequence $\langle B_\delta : \delta \in u\rangle$ such that $B_\delta \in [A_\delta]^{<\aleph_0}$ and $\langle A_\delta \backslash B_\delta : \delta \in u\rangle$ are pairwise disjoint*

(6) $(f)$ *if $F : \lambda \to [\lambda]^{\le\kappa}$ then for some $\delta \in S$ the set $A_\delta$ is $F$-free.*

---

[6] of coures, if, e.g., $\kappa = (2^{\aleph_0})^+$ this holds

*3.3 Remark.* 1) Note that we can easily add clauses sandwiched between two existing ones. We can also add the parallel statement on $X \to [Y]^1_{\text{cf}(2^{\aleph_0})}$, see 3.12, 3.13, 3.8.
2) We can add the case of regular spaces (i.e. $T_3$) or work as in 1.7.
3) Clearly most of the proof of most arrows in the proof (of 3.2) have little to do with the properties of the topological space ${}^\omega 2$; still mainly 3.14 does, so

3.4 Question: With what can we replace the space ${}^\omega 2$ (but see 4.17(2))?

We make some definitions and prove some claims before proving 3.2. One of them (3.13) depends on §4, also 3.7, 3.8 which are not explicitly needed. The following definition is used in 3.7. To see the point of this definition look at Example 3.6 below and part (2) of the definition.

**3.5 Definition.** 1) For a cardinal $\kappa$ and $I_0, I_1$ such that $I_\ell \subseteq \{(a,b) : a, b \subseteq \kappa$ are disjoint (normally $\kappa = \cup\{a \cup b : (a,b) \in I_0 \cup I_1\}$) so we may forget to mention $\kappa$) and cardinal $\theta$ we say that a cardinal $\lambda$ is $(I_0, I_1, \theta)$-approximate or $(\kappa, I_0, I_1, \theta)$-approximate if we can find $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in C \rangle$ such that

- $(i)$ $C$ a club of $\lambda$
- $(ii)$ $\mathcal{P}_\alpha \subseteq [\alpha]^{<\theta}$ for $\alpha \in C$ and $|\mathcal{P}_\alpha| \leq \text{Min}(C \backslash (\alpha+1))$
- $(iii)$ for any 1-to-1 function $f$ from $\kappa$ to $\lambda$, for some $\alpha \in C$ at least one of the following holds
  - $(a)$ for some $c \in \mathcal{P}_\alpha$ and $(a,b) \in I_1$ we have $(\forall i \in a)(f(i) \in c)$ and $(\forall i \in b)[f(i) \geq \alpha]$
  - $(b)$ for some $(a,b) \in I_0$ we have
    - $(\alpha)$ $(\forall i < \kappa)(f(i) < \alpha \to i \in a)$
    - $(\beta)$ $(\forall i < \kappa)[i \in b \to \alpha \leq f(i) < \text{Min}(C \backslash (\alpha+1))]$.

2) If $c\ell$ is a function from $\mathcal{P}(\kappa)$ to $\mathcal{P}(\kappa)$ and $K \subseteq \mathcal{P}(\kappa)$ and

$$I_1[\kappa, c\ell, K] = I_1[c\ell, K] = I_1 = \{(a,b) : a \subseteq \kappa, b \in K \text{ and } b \subseteq c\ell(a)\}$$

$$I_0[\kappa, c\ell, K] = I_0[c\ell, K] = I_0 = \{(a,b) : a \subseteq \kappa, b \in K \text{ and } a \cap b = \emptyset\}$$

then we may say $\lambda$ is $(K, c\ell, \theta)$-approximate or $(\kappa, K, c\ell, \theta)$-approximate instead of $\lambda$ is $(I_0, I_1, \theta)$-aproximate.
3) We may replace $\kappa$ by another set of this kind call the domain of the tuple (understood from $I_0, I_1$). We may write this set before $I_0$, i.e. in the place of $\kappa$ for clarification.
4) We may replace $(I_0, I_1, \theta)$ by $(\mathbf{I}, \theta)$ if $\mathbf{I}$ is a set of pairs $(I_0, I_1)$ such that $\langle \mathcal{P}_\alpha : \alpha \in C \rangle$ satisfies the requirement above for all the triples $(I_0, I_1, \theta)$ such that $(I_0, I_1) \in \mathbf{I}$ (not necessarily all pairs have the same domain $A$).
Similarly, $\mathbf{K}$ stands for a set of tuples $(\kappa, K, c\ell, \theta)$ or in short $(\kappa, K, c\ell)$ when $\theta$ is understood from the context or even $(K, c\ell)$ as in part (2). (We may even vary $\theta$).

Concerning 4.4 below
3.6 Examples: 1) Let $\mathbf{C}$ be a Cantor set (say ${}^\omega 2$),
$c\ell^{\mathbf{C}}$ is the (topological) closure operation on subsets of $\mathbf{C}$
$K^{\mathbf{C}} = \{A \subseteq \mathbf{C} : A \text{ is closed perfect hence uncountable}\}$ and $I^{\mathbf{C}}_\ell = I_\ell[\mathbf{C}, c\ell, K^{\mathbf{C}}]$ for $\ell = 0,1$; see Definition 3.5(2).
2) Let $\mathbb{R}$ be the real line, $c\ell^{\mathbb{R}}$ be the (topological) closure operation on subsets of $\mathbb{R}$ and $K^{\mathbb{R}} = \{A \subseteq \mathbb{R} : A \text{ is closed perfect uncountable, bounded (from below and above)}\}$ and $I^{\mathbb{R}}_\ell = I_\ell(\mathbb{R}, c\ell^{\mathbb{R}}, K^{\mathbb{R}})$ for $\ell = 0, 1$.

**3.7 Lemma.** *Assume*

(a) $\lambda > \chi \geq \kappa \geq \theta$ *and* $\sigma$ *are infinite cardinals,*

(b) $c\ell$ *is a partial function from* $[\lambda]^{<\theta}$ *to* $K \subseteq [\lambda]^{\leq\kappa}$

(c) $\mathbf{K}$ *is a set of triples* $(\kappa, K^*, c\ell^*)$ *with* $K^* \subseteq \mathcal{P}(\kappa), c\ell^*$ *a function from* $[\kappa]^{<\theta}$ *to* $\mathcal{P}(\kappa)$ *as in Definition 3.5(2) above (for* $\theta$*)*

(d) *if* $b \in K$, *then for some* $(\kappa, K^*, c\ell^*) \in \mathbf{K}$ *and one to one function* $f$ *from* $\kappa$ *into* $b$, *we have:*

  ($\alpha$) $b' \in K^* \Rightarrow \{f(\alpha) : \alpha \in b'\} \in K$

  ($\beta$) $a', b' \subseteq \kappa \;\&\; c\ell^*(a') = b' \Rightarrow c\ell\{f(\alpha) : \alpha \in a'\} \supseteq \{f(\alpha) : \alpha \in b'\}$

(e) *for every* $A \in [\lambda]^{\leq\chi}$ *we can find a* $[K, \sigma]$*-colouring* $\mathbf{c}$ *of* $A$, *where:*

  $\boxtimes$ *for any* $A \subseteq \lambda$, $\mathbf{c}$ *is a* $[K, \sigma]$*-colouring of* $A$ *means that* $\mathbf{c}$ *is a function from* $A$ *to* $\sigma$ *such that* $a \in K \;\&\; a \subseteq A \Rightarrow$ *Rang*$(\mathbf{c} \restriction a) = \sigma$

(f) *for every* $\mu$, *if* $\chi < \mu \leq \lambda$ *then* $\mu$ *is* $(\mathbf{K}, \theta)$*-approximate.*

*Then there is* $[K, \sigma]$*-colouring* $\mathbf{c}$ *of* $\lambda$.

*Proof.* See after the proof of 4.14 below. (The reader may prefer to read first §4 up to the proof of 3.7, 3.13).

3.8 Conclusion: 1) Assume

(a) every cardinal $\mu, 2^{\aleph_0} < \mu \leq \lambda$ is $(\mathbf{C}, K^{\mathbf{C}}, c\ell^{\mathbf{C}}, \aleph_1)$-approximate (using the notation of 3.6(1))

(b) $X$ is a Hausdorff topological space.

Then $X \nrightarrow (\text{Cantor set})^1_2$ moreover $X \nrightarrow [\text{Cantor set}]^1_{2^{\aleph_0}}$, see Definition 3.12 below.
2) We can replace in part (1), $\mathbf{C}$ by $\mathbb{R}$.

*Proof.* By 3.7 (and 3.6). $\square_{3.8}$

**3.9 Claim.** *The forcing notions in 1.2 and in 2.8 satisfies, e.g., the condition* $*^{\sigma^+}_{\kappa^+}$; *see below Definition 3.10(1A).*

*Proof.* Included in the proof of 1.2, 2.8, respectively. $\square_{3.9}$

**3.10 Definition.** 1) Let $D$ be a normal filter on $\mu^+$ to which $\{\delta < \mu^+ : \text{cf}(\delta) = \mu\}$ belongs. A forcing notion $\mathbb{Q}$ satisfies $*^\epsilon_D$ where $\epsilon$ is a limit ordinal $< \mu$, if player I has a winning strategy in the following game $*^\epsilon_D[\mathbb{Q}]$ defined as follows:
Playing: the play finishes after $\epsilon$ moves.

In the $\zeta$-th move:

Player I — if $\zeta \neq 0$ he chooses $\langle q^\zeta_i : i < \mu^+\rangle$ such that $q^\zeta_i \in \mathbb{Q}$
and $(\forall \xi < \zeta)(\forall^D i < \mu^+) p^\xi_i \leq q^\zeta_i$ and he chooses a function $f_\zeta : \mu^+ \to \mu^+$ such that for the $D$-majority of $i < \mu^+, f_\zeta(i) < i$;
if $\zeta = 0$ let $q^\zeta_i = \emptyset_{\mathbb{Q}}$, $f_\zeta =$ is identically zero.

Player II — he chooses $\langle p^\zeta_i : i < \mu^+\rangle$ such that $(\forall^D i) q^\zeta_i \le p^\zeta_i$ and $p^\zeta_i \in \mathbb{Q}$.

The Outcome: Player I wins provided that for some $E \in D$: if
$\mu < i < j < \mu^+, i, j \in E$ and $\bigwedge_{\xi < \epsilon} f_\xi(i) = f_\xi(j)$ then the set $\{p^\zeta_i : \zeta < \epsilon\} \cup \{p^\zeta_j : \zeta < \epsilon\}$ has an upper bound in $\mathbb{Q}$.
1A) If $D$ is $D^*_{\mu^+} =: \{A \subseteq \mu^+ :$ for some club $E$ of $\mu^+$ we have $i \in E \;\&\; \mathrm{cf}(i) = \mu \Rightarrow i \in A\}$ we may write $\mu$ instead of $D$ (in $*^\varepsilon_D$ and in the related notions defined below and above). Usually we assume $D^*_{\mu^+} \subseteq D$.
2) We may allow the strategy to be non-deterministic, e.g. choose not $f_\zeta$ just $f_\zeta/D$.
3) We say a forcing notion $\mathbb{Q}$ is $\varepsilon$-strategically complete if for the following game, $\bigotimes^\varepsilon_{\mathbb{Q}}$ player I has a winning strategy.
A play last $\varepsilon$ moves. In the $\zeta$-th move:
Player I - if $\zeta \ne 0$ he chooses $q_\zeta \in \mathbb{Q}$ such that $(\forall \xi < \zeta) p_\xi \le q_\zeta$ if $\zeta = 0$ let $q_\zeta = \emptyset_{\mathbb{Q}}$.
Player II - he chooses $p_\zeta \in \mathbb{Q}$ such that $q_\zeta \le p_\zeta$.

The Outcome: In the end Player I wins provided that he always has a legal move.

**3.11 Lemma.** *If $\mu = \mu^{<\mu}$, $\epsilon$ a limit ordinal $< \mu$, then the property "$\mathbb{Q}$ is $(< \mu)$-strategically complete and has $*^\varepsilon_\mu$" is preserved by $(< \mu)$-support iteration.*

*Proof.* See [Sh 546] and history there; in each coordinate we preserve that the sequence of conditions is increasingly continuous and on each stationary $S \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mu\}$ on which the pressing down function is constant the conditions form a $\Delta$-system.

$\square_{3.11}$

We can also consider

**3.12 Definition.** 1) We say $X^* \to [Y^*]^n_\theta$ if $X^*, Y^*$ are topological spaces and for every $h : [X^*]^n \to \theta$ there is a closed subspace $Y$ of $X^*$ homeomorphic to $Y^*$ such that for some $\alpha < \theta, \alpha \notin \mathrm{Rang}(h \restriction [Y]^n)$ is not $\theta$.
2) If we omit the "closed" we shall write $\to_w$ instead of $\to$ and $\nrightarrow, \nrightarrow_w$ denote the negations. [Compare with 3.2, 3.7.]

**3.13 Claim.** *1) Assume $X$ is a Hausdorff space with $\lambda$ points. Assume further $X \to [{}^\omega 2]^1_\theta$ and $\mu \ge 2^{\aleph_0}$ but no subspace $X^*$ of $X$ with $\le \mu$ points satisfy $X^* \to [{}^\omega 2]^1_\theta$ and $\mu = \mu^{\aleph_0}$. Then*

- $(*)$ *we can find a regular $\kappa \in (\mu, \lambda]$, a stationary $S \subseteq \kappa$ and a sequence $\bar f = \langle f_\alpha : \alpha \in S\rangle$ such that:*
  - $(i)$ $\mathrm{Dom}(f_\alpha) \subseteq {}^\omega 2$ *has cardinality* $2^{\aleph_0}$
  - $(ii)$ *$f_\alpha$ is one-to-one and is a homeomorphism from ${}^\omega 2 \restriction \mathrm{Dom}(f_\alpha)$ onto $X \restriction \mathrm{Rang}(f_\alpha)$*
  - $(iii)$ *if $\alpha \ne \beta$ are from $S$, then $\{\eta \in \mathrm{Dom}(f_\alpha) : f_\alpha(\eta) \in \mathrm{Rang}(f_\beta)\}$ has scattered closure in ${}^\omega 2$*
  - $(iv)$ *for a club of $\delta \in S$ we have $\mathrm{Rang}(f_\alpha) \subseteq \bigcup_{\beta \in \alpha \cap S} \mathrm{Rang}(f_\beta)$.*

*2) Similarly for* $\rightarrow_w$ *and/or for* $\mathbb{R}$ *instead* ${}^{\omega}2$.

We shall prove it later (after the proof of 4.14).
3.14 Observation: There is a c.c.c. forcing notion $\mathbb{Q}$ of cardinality $2^{\aleph_0}$ such that:

$$\begin{array}{l}\Vdash_{\mathbb{Q}} \text{“there is } h : {}^{\omega}2 \rightarrow \omega \text{ such that :}\\ \quad (\alpha) \quad \text{if } C (\in \mathbf{V}) \text{ is closed scattered then each}\\ \qquad\quad C \cap h^{-1}\{n\} \text{ is finite, and}\\ \quad (\beta) \quad \text{if } A \subseteq ({}^{\omega}2) \text{ is uncountable (and from } \mathbf{V})\\ \qquad\quad \text{then } |A \cap h^{-1}\{n\}| = |A| \text{ for each } n\text{”.}\end{array}$$

*Proof.* Let $p \in \mathbb{Q}$ be $(f^p, \mathcal{C}^p)$ where $f^p$ is a finite function from ${}^{\omega}2$ to $\omega$ and $\mathcal{C}^p$ is a finite family of closed scattered subsets of ${}^{\omega}2$.
The order is:

$$\begin{aligned} p \leq q \text{ \underline{iff} } & f^p \subseteq f^q, \mathcal{C}^p \subseteq \mathcal{C}^q \text{ and } C \in \mathcal{C}^p \ \& \ \eta \in C \cap \text{ Dom}(f^q) \ \& \\ & \eta \neq \nu \ \& \ \nu \in C \cap \text{ Dom}(f^q) \backslash \text{Dom}(f^p) \Rightarrow f^q(\eta) \neq f^q(\nu). \end{aligned}$$

Clearly

$(*)_1$ $\mathbb{Q}$ is a forcing notion of cardinality $2^{\aleph_0}$

$(*)_2$ $\mathbb{Q}$ satisfies the c.c.c.
[why? let $p_\alpha \in \mathbb{Q}$ for $\alpha < \omega_1$, let $\text{Dom}(f_\alpha) = \{\eta_{\alpha,\ell} : \ell < \ell_\alpha\}, \mathcal{C}^{p_\alpha} = \{C_{\alpha,k} : k < k_\alpha\}$ both lists with no repetitions and let $m_\alpha = \text{ Min}\{m : \langle \eta_{\alpha,\ell} \restriction m : \ell < \ell_\alpha\rangle$ is with no repetitions$\}$. Without loss of generality $m_\alpha = m(*), \ell_\alpha = \ell(*), k_\alpha = k(*), \eta_{\alpha,\ell} \restriction m(*) = \nu_\ell$. By $\Delta$-system lemma without loss of generality for some $\ell(**) \leq \ell(*)$ we have:

- $(\alpha)$ $\ell < \ell(**) \Rightarrow \langle \eta_{\alpha,\ell} : \alpha < \omega_1\rangle$ is with no repetitions
- $(\beta)$ $\alpha < \omega_1 \ \& \ \ell \in [\ell(**), \ell(*)) \Rightarrow \eta_{\alpha,\ell} = \eta_\ell$
- $(\gamma)$ $\{\eta_{\alpha,\ell} : \alpha < \omega_1, \ell < \ell(**)\}$ is with no repetitions.

Now as each $C_{\alpha,k}$ is closed and scattered it is necessarily countable so without loss of generality

$$\alpha < \beta < \omega_1 \ \& \ \ell < \ell(**) \Rightarrow \eta_{\beta,\ell} \notin \bigcup_{k<k(*)} C_{\alpha,k}.$$

We now choose by induction on $\ell \leq \ell(**)$ sets $A_\ell, B_\ell \in [\omega_1]^{\aleph_1}$, decreasing with $n$ such that

$$\alpha \in A_{\ell+1} \ \& \ \beta \in B_{\ell+1} \ \& \ \alpha < \beta \rightarrow \eta_{\alpha,\ell} \notin \bigcup_{k<k(*)} C_{\beta,k}.]$$

This is straight: let $A_0 = \omega_1 = B_0$; let $A_\ell, B_\ell$ be given. Clearly for some $\alpha^*_\ell \in A_\ell$ the set $\{\eta_{\alpha,\ell} : \alpha \in A_\ell \backslash \alpha^*_\ell\}$ is $\aleph_1$-dense in itself, i.e. $(\forall \alpha \in A_\ell \backslash \alpha^*_\ell)(\forall n < \omega)(\exists^{\aleph_1} \beta \in A_\ell)(\eta_{\beta,\ell} \restriction n = \eta_{\alpha,\ell} \restriction n)$. Let $T_\ell = \{\eta_{\alpha,\ell} \restriction n : \alpha \in A_\ell \backslash \alpha^*_\ell$ and $n < \omega\}$, it is a subtree

of ${}^{\omega>}2$ and $\lim(T_\ell)$ is a perfect subset of ${}^{\omega}2$. So for each $\beta \in B_\ell$ for some $\nu^\ell_\beta \in T_\ell$ we have $(\forall \rho \in \bigcup_k C_{\beta,k})(\neg \nu^\ell_\beta \triangleleft \rho)$ so for some $\nu_\ell \in T_\ell$ we have $B_{\ell+1} =: \{\beta \in B_\ell : \nu^\ell_\beta = \nu_\ell\}$ is uncountable and let $A_{\ell+1} =: \{\alpha \in A_\ell : \nu_\ell \triangleleft \eta_{\alpha,\ell}\}$.
For $\alpha < \beta, \alpha \in A_{\ell(**)}, \beta \in B_{\ell(**)}$, we have $p_\alpha, p_\beta$ are compatible.]

$(*)_3$ if $A \subseteq {}^{\omega}2$ is uncountable and $n < \omega$ then $\mathcal{I}_{A,n} =: \{p : \text{for some } \eta \in A, f^p(\eta) = n\}$ is dense open.
[Why? Let $p \in \mathbb{Q}$. Now as $C^* =: \cup\{C : C \in \mathcal{C}^p\}$ is closed and scattered hence countable clearly for some $\eta \in A$ we have $\eta \notin C^*$ so $q = (f^p \cup \{(\eta, n)\}, \mathcal{C}^p)$ satisfies $p \le q \in \mathbb{Q} \cap \mathcal{I}_{A,n}$.]

$(*)_4$ for each $\eta \in {}^{\omega}2$ the set
$\mathcal{I}_\eta = \{p : \eta \in \text{Dom}(f^p)\}$ is dense open
[why? being open is trivial; as for density for $p \in \mathbb{Q}$ let $n = \sup(\text{Rang}(f^p)) + 1$ and without loss of generality $p \notin \mathcal{I}_\eta$ hence $\eta \notin \text{Dom}(f^p)$, now letting $q = (f^p \cup \{(\eta, n)\}, \mathcal{C}^p)$ we have $p \le q \in \mathcal{I}_\eta$.]

$(*)_5$ for each closed scattered $C$, the set $\mathcal{I}_C = \{p : C \in \mathcal{C}^p\}$ is dense open
[why? immediate as $p \in \mathbb{Q} \Rightarrow p \le (f^p, \mathcal{C}^p \cup \{C\}) \in \mathbb{Q}$.]

Let $\underset{\sim}{f} = \cup\{f^p : p \in \underset{\sim}{G}\}$, it is a $\mathbb{Q}$-name.

$(*)_6$ $\underset{\sim}{f}$ is a function from $({}^{\omega}2)^{\mathbf{V}}$ to $\omega$ and for each closed scattered $C \in \mathbf{V}, f \restriction C$ is one to one except on a finite set.
[Why? For any $p \in \mathbb{Q}$ there is $q$ such that $p \le q \in \mathbb{Q}$ & $C \in \mathcal{C}^q$, now $[q \le r \in \mathbb{Q} \Rightarrow f^r \restriction (C \setminus \text{Dom}(f^q))$ is one to one]; so $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{f} \restriction (C \setminus \text{Dom}(f^q))$ is one to one, so as $\text{Dom}(f^q)$ is finite we are done.]

$(*)_7$ $\Vdash_{\mathbb{Q}}$ "$A \cap \underset{\sim}{f}^{-1}\{n\}$ has cardinality $|A|$ for $A \in \mathbf{V}, A \subseteq {}^{\omega}2, A$ uncountable".
[Why? As in $\mathbf{V}$ we can find pairwise disjoint $A_i \subseteq A$ for $i < |A|, |A_i| = |A|$ and apply $(*)_3$.]

Together we are done. $\square_{3.14}$

*Proof of Theorem 3.2.*

$(B)^+ \Rightarrow (B)[{}^{\omega}2]$
Trivial (special case).

$(A)^+ \Rightarrow (A)[{}^{\omega}2]$
Trivial (a special case).

$(B)^+ \Rightarrow (A)^+$
Trivial (stronger demands).

$(B)[{}^{\omega}2] \Rightarrow (A)[{}^{\omega}2]$
Trivial (stronger demands).

$(A)[{}^{\omega}2] \Rightarrow (C)$
By 3.13 for $\theta = 2, \mu = 2^{<\kappa}$.

$(C) \Rightarrow (D)$
Forcing by $\text{Levy}(\kappa, 2^{<\kappa})$ change nothing so without loss of generality $\kappa = \kappa^{<\kappa}$. Let $\lambda, S, \bar{f} = \langle f_\alpha : \alpha \in S\rangle$ be as in clause (C) of 3.2. Next let $\mathbb{Q}$ be the forcing notion

from 3.14 which is a c.c.c. forcing notion of cardinality $2^{\aleph_0}$, so we get the conclusion of 3.14 so let $h$ be as there and let $g : ({}^\omega 2)^{\mathbf{V}} \to 2^{\aleph_0}$ be one to one. $W\log \alpha \in S \Rightarrow (2^{\alpha_0})^\omega | \alpha$. For each $\alpha \in S$ let

$$A_\alpha =: \{(2^{\aleph_0}) \times f(\eta) + g(\eta) : \eta \in \text{ Dom}(f_\alpha) \text{ and } h(\eta) = 0\}.$$

we get: $A_\alpha \subseteq {}^\omega 2, |A_\alpha| = 2^{\aleph_0}$ and for $\alpha \neq \beta$ from $S, A_\alpha \cap A_\beta$ is finite. So clauses (a)-(d) of clause (D) from Theorem 3.2 holds. Then we force by Levy$(\lambda, 2^{<\lambda})$, nothing changes but we get $\diamondsuit_S$. By 2.4 without loss of generality clause (f) of (D) of 3.2 holds. By 2.8 without loss of generality we have the $\kappa$-freeness (i.e., clause (e) of $(D)$ of 3.2 which is equivalent to $(B)_2$ of 1.2) while clause (f) of (D) of 3.2 (= clause (C) of 1.2) is preserved by clause (d) of the conclusion of 2.8.

<u>$(D) \Rightarrow (B)^+$ and $(D) \Rightarrow (A)^+[2^\omega]$</u>

We do it by forcing but for the proof any $\kappa$ such that $\aleph_1 \leq \text{ cf}(\kappa) = \kappa, 2^{<\kappa} < \lambda$ can serve. We can force by Levy$(\kappa, 2^{<\kappa})$, so without loss of generality $\kappa = \kappa^{<\kappa}$.
<u>First</u> assume that $\kappa > 2^{\aleph_0}$ (as in the main case) and we restrict ourselves to spaces $Y^*$ with a basis of cardinality $< \kappa$ which is no restriction if $\kappa > \beth_2$ <u>or</u> if we are proving just $(D) \Rightarrow (A)^+[2^\omega]$, <u>then</u> we can use a product of forcing instead of iteration. Now any strategically $(< \kappa)$-complete $\kappa^+$-c.c. forcing notion preserves (D), we do not use this in this first case, but still note it. By forcing by Levy$(\lambda, 2^{<\lambda})$ (see 4.3) without loss of generality $\diamondsuit_S$ for the $S$ of clause (D), this will be preserved for any forcing notion $\mathbb{P}$ if $\mathbb{P}$ has density $\leq \lambda$, which holds in our case.
Let $\langle Y^*_i : i < i^* \rangle$ list the topological spaces as in clause $(B)^+$ with set of points $2^{\aleph_0}$ or $i^* = 1$ & $Y^*_0 = {}^\omega 2$, depending on what we are proving. For each $i < i^*$, let $\mathbb{Q}_i$ be the forcing from 1.2, the assumption of 1.2 holds by (D) and the assumptions on $Y^*_i$ and $\underset{\sim}{X}^*_i$ be the $\mathbb{Q}_i$-name of the topological space which $\mathbb{Q}_i$ produces. Let $\mathbb{Q}$ be the product of $\{\mathbb{Q}_i : i < i^*\}$ with support $< \kappa$. Now $\mathbb{Q}$ is $\kappa$-complete. Hence by our present assumption no new relevant space $Y^*$ is added by forcing by $\mathbb{Q}$.
Why is $\underset{\sim}{X}^*_i$ as required, i.e., $\underset{\sim}{X}^*_i \to (Y^*_i)^1_{<\text{cf}(2^{\aleph_0})}$ also in $\mathbf{V}^{\mathbb{Q}}$? Forcing by $\mathbb{Q}$ add more "colouring" $\underset{\sim}{c}$, i.e., functions from $\underset{\sim}{X}^*_i$, i.e., $\lambda$ into some ordinal $< 2^{\aleph_0}$. However, the proof of 1.2 can be repeated for this case.

<u>Second</u>, consider the general case.
Now we use iterated forcing $\langle \mathbb{P}_j, \mathbb{Q}_i : j \leq i(*), i < i(*) \rangle$ with $(< \kappa)$-support, each satisfying the $*^{\sigma^+}_{\kappa^+}$ version of $\kappa^+$-c.c. and for simplicity $(< \kappa)$-strategically complete (see 3.9). Now let each $\underset{\sim}{\mathbb{Q}}_i$ be as in 1.2 for some $\underset{\sim}{Y}^*_i$ (a $\mathbb{P}_i$-name of a topological space as in 1.2) and it forces an example $\underset{\sim}{X}^*_i$. With suitable bookkeeping (if $\kappa > 2^{\aleph_0}$ is easier) we finish as those iterations preserve "$(< \kappa)$-strategic completeness hence no new set of ordinals of cardinality $< \kappa$ and (the strong version of) $\kappa^+$-c.c." is preserved, see 3.11.
Still we have to prove that the example $X^*_i$ we force to satisfy "$X^*_i \to (Y^*_i)^1_\sigma$ if $\sigma < \text{ cf}(2^{\aleph_0})$" has this property not only in $\mathbf{V}^{\mathbb{P}_{i+1}}$ but also in $\mathbf{V}^{\mathbb{P}_{i(*)}}$. For this we repeat the relevant part of the proof of 1.2 noting the explicit way the $\mathbb{Q}_i$'s; this will be presented in full in [Sh:F567]. $\square_{3.2}$

For self-containment we recall (really [Sh:g, II,2.2] and see [Sh 108], [Sh 88a]).

**3.15 Claim.** *Assume $\kappa$ is strongly compact and $\chi = \text{ cf}(\chi) \leq \text{ cf}(\mu) < \kappa < \lambda = \text{ cf}(\lambda) = \mu^+$ (so $\lambda = \lambda^{<\kappa}$ and $(\forall \alpha < \mu)(|\alpha|^{<\kappa} < \mu)$) and $\mathfrak{a} \subseteq \text{ Reg} \cap \mu \backslash \kappa, \mu = \sup(\mathfrak{a}), |\mathfrak{a}| = \text{ cf}(\mu)$ and $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is a $<_{J^{\text{bd}}_{\mathfrak{a}}}$-increasing cofinal sequence in $\Pi\mathfrak{a}$.*

*Then for some* $\mu_0 \in (\chi, \kappa)$, *we have* $\mathrm{cf}(\mu_0) = \mathrm{cf}(\mu) < \mu, (\forall \alpha < \mu_0)(|\alpha|^\chi < \mu_0)$ *and if* $\mathbb{P} = \mathrm{Levy}(\chi, \mu_0) * \underset{\sim}{\mathrm{Levy}}(\mu_0^{++}, \kappa)$, *in* $\mathbf{V}^{\mathbb{P}}$ *we have*

(a) $S^{\mu^+}_{\mu_0^+} = \{\delta < \mu^+ : \mathbf{V} \models \mathrm{cf}(\delta) = \mu_0^+\} \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \chi^+\}$ *does not belong to* $I[\mu^+]$,

(b) $\mathrm{bad}(\bar f) \supseteq S^* = \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mu_0^+, \bar f \restriction \delta$ *has a* $<_{J^{\mathrm{bd}}{}_{\mathfrak{a}}}$*-lub* $f \in \Pi\mathfrak{a}$ *such that* $\theta \in \mathfrak{a} \Rightarrow \mathrm{cf}(f(\theta)) < \mathrm{cf}(\delta)\}$ *is a stationary subset of* $\mu^+$

(c) *forcing by* $\mathbb{P}$, *preserve* $\mu_0^+$ *is a cardinal and the stationarity of subsets of* $S^{\mu^+}_{\mu_0^+}$ *(from* $\mathbf{V}$*) and preserve "*$\delta \in S^*$ *is not a good point in* $\bar f$*" and* $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ *is* $<_{J^{\mathrm{bd}}_{\mathfrak{a}}}$*-increasing cofinal in* $\Pi\mathfrak{a}$*; if* $\mu = \kappa^{+\omega}, \mathfrak{a} = \{\mu^{+n} : n \in (0, \omega)\}$ *we get the result for* $\prod\limits_{n<\omega} \aleph_n / J^{bd}_\omega$.

*Proof.* The choice of $\mu_0$ (and clause (a)) is a main point in [Sh 108], [Sh 88a]. Now $S^* = \cup\{S^*_{\mu_0^+} : \mu_0^+ < \kappa, \mathrm{cf}(\mu_0) = \mathrm{cf}(\mu) < \mu_0\}$ is stationary by [Sh:g, 2.2, 5.6] using $(*)'$ not $(*)$, so for some $\mu_0^+, S_{\mu_0^+}$ is stationary, and (c) is obvious. Of course, (b) $\Rightarrow$ (a) by [Sh:g, I]. $\square_{3.15}$

Remark: We can joint 2.7 to 3.2; we will return to this elsewhere.

## §4 Decomposing families of almost disjoint functions

Let $(I, J)$ be a pair of ideals on say $\theta =$ Dom$(I, J)$ such that $I \subseteq J$ and we consider a family $\mathcal{F}$ of functions each from some $A \in J^+$ to $\lambda$ which are almost disjoint in the sense that

$\circledast$ if $f \neq g$ are from $\mathcal{F}$ then $\{i < \theta : i \in \text{Dom}(f) \cap \text{Dom}(g) \text{ and } f(i) = g(i)\} \in I$.

A decomposition is a representation of $\mathcal{F}$ as $\bigcup\limits_{\alpha} \mathcal{F}_\alpha$ such that the $\mathcal{F}_\alpha$ are pairwise disjoint, "small" and $f \in \mathcal{F}_\alpha \Rightarrow \{i \in \text{Dom}(f) : f(i) \notin \bigcup\{\text{Rang}(g) : g \in \mathcal{F}_j \text{ for some } j < \alpha\}\}$ is "small". We try to prove that if such decomposition does not exist, then there are "transparent" counterexamples.

This helps the equiconsistency in §3 and continue [Sh 161], [HJSh 249], [Sh:g, II,§6], [Sh 460, 3.9].

The reader can concentrate on the case that $\mathcal{Y}$ is a singleton.

**4.1 Definition.** 1) Let $\mathcal{Y}$ denote a set of pairs of the form $(I, J)$ where

- $(a)$ $I \subseteq J$ are ideals over a common set called Dom$(I, J) =$ Dom$(I) =$ Dom$(J)$ or just
- $(b)$ $\emptyset \in I \subseteq J \subseteq \mathcal{P}(\text{Dom}(I, J))$ and $[A \subseteq B \in I \Rightarrow A \in I], [A \subseteq B \in J \Rightarrow A \in J]$, Dom$(I, J) \notin J$.

Let $I^+ = \mathcal{P}(\text{Dom}(I)) \backslash I$ and $J^+ = \mathcal{P}(\text{Dom}(J)) \backslash J$.

Let $\kappa(\mathcal{Y}) = \sup\{|\text{Dom}(I, J)| : (I, J) \in \mathcal{Y}\}$. We call $\mathcal{Y}$ standard if for each $(I, J) \in \mathcal{Y}$, the set Dom$(I, J)$ is a cardinal; we call $\mathcal{Y}$ disjoint if $\langle \text{Dom}(I, J) : (I, J) \in \mathcal{Y}\rangle$ is a sequence of pairwise disjoint sets.

2) NFr$_1(\lambda, \mathcal{Y})$ means that for some $\lambda^* > \lambda$ we have NFr$_1(\lambda^*, \lambda, \mathcal{Y})$ which means that $\lambda \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$ and there are $\langle \mathcal{F}_{(I,J)} : (I, J) \in \mathcal{Y}\rangle$ exemplifying it which means:

- $(a)$ $\mathcal{F}_{(I,J)} \subseteq \{f : f \text{ a function, Dom}(f) \in J^+\}$
- $(b)$ if $f \neq g \in \mathcal{F}_{(I,J)}$ then $\{x : x \in \text{Dom}(f) \cap \text{Dom}(g) \text{ but } f(x) = g(x)\}$ belongs to $I$
- $(c)$ $\lambda \geq |\cup\{\text{Rang}(f) : f \in \mathcal{F}_{(I,J)} \text{ and } (I, J) \in \mathcal{Y}\}|$
- $(d)$ $\lambda < \lambda^* = \sum\{|\mathcal{F}_{(I,J)}| : (I, J) \in \mathcal{Y}\}$.

2) NFr$_2(\lambda, \mathcal{Y})$ means that $\lambda$ is regular $> |\mathcal{Y}| + \kappa(\mathcal{Y})$ and there is $\langle f_\delta : \delta \in S\rangle$ such that

- $(a)$ $S \subseteq \lambda$ is stationary and is the disjoint union of $\langle S_{(I,J)} : (I, J) \in \mathcal{Y}\rangle$
- $(b)$ Dom$(f_\delta) \in J^+$ and Rang$(f_\delta) \subseteq \delta$ for each $\delta \in S_{(I,J)}$
- $(c)$ $\delta_1 \neq \delta_2 \in S_{(I,J)} \Rightarrow \{x : x \in \text{Dom}(f_{\delta_1}) \cap \text{Dom}(f_{\delta_2}) \text{ and } f_{\delta_1}(x) = f_{\delta_2}(x)\} \in I$.

3) We omit $N$ from NFr in parts (1) and (2) for the negation. If $\mathcal{Y} = \{(I, J)\}$ we may write just $(I, J)$.

<u>4.2 Fact</u>: 1) NFr$_1(\lambda, \mathcal{Y})$ is preserved by increasing $\mathcal{Y}$ to $\mathcal{Y}'$ when $|\mathcal{Y}'| + \kappa(\mathcal{Y}') \leq \lambda$. Also NFr$_2(\lambda, \mathcal{Y})$ is preserved by increasing $\mathcal{Y}$ to $\mathcal{Y}'$ if $|\mathcal{Y}'| + \kappa(Y') < \lambda$. Similarly if NFr$_1(\lambda^*, \lambda, \mathcal{Y})$, $\lambda^* \geq \lambda^*_1 > \lambda_1 \geq \lambda, \lambda_1 \geq |\mathcal{Y}_1| + \kappa(\mathcal{Y}_1)$ and $\mathcal{Y}_1 \supseteq \mathcal{Y}$ <u>then</u> NFr$_1(\lambda^*_1, \lambda_1, \mathcal{Y}_1)$.

2) NFr$_1(\lambda, \mathcal{Y})$ is equivalent to NFr$_1(\lambda^+, \lambda, \mathcal{Y})$ which is equivalent to $(\exists (I, J) \in \mathcal{Y})$ NFr$_1(\lambda^+, \lambda, (I, J))$.

3) If $\lambda^*$ is regular or at least cf$(\lambda^*) > |\mathcal{Y}|$ <u>then</u> NFr$_1(\lambda^*, \lambda, \mathcal{Y})$ <u>iff</u> there is $(I, J) \in \mathcal{Y}$ such that NFr$_1(\lambda^*, \lambda, \{(I, J)\})$.

4) NFr$_1(\lambda, \mathcal{Y})$ implies NFr$_2(\lambda^+, \mathcal{Y})$.

5) NFr$_2(\lambda, \mathcal{Y})$ <u>iff</u> there is $(I, J) \in \mathcal{Y}$ such that NFr$_2(\lambda, \{(I, J)\})$ and $|\mathcal{Y}| < \lambda$.

*Proof.* Check.

**4.3 Claim.** *1) Assume that* $\mathrm{NFr}_2(\lambda, \{(I,J)\})$ *and let* $\bar f = \langle f_\delta : \delta \in S\rangle$ *exemplifies it and* $\tau^{++} < \lambda, \tau \ge |\mathrm{Dom}(I,J)|$ *and for simplicity* $\kappa = \mathrm{Dom}(I,J)$. *If* $\diamondsuit_S$ *then we can find* $\langle f'_\delta : \delta \in S \cap E\rangle$ *exemplifying* $\mathrm{NFr}_2(\lambda, \{(I,J)\})$ *and* $E$ *a club of* $\lambda$ *such that*

(∗) *if* $F : \lambda \to [\lambda]^{\le\tau}$ *then for some* $\delta \in S \cap E$ *the set* $\mathrm{Rang}(f'_\delta)$ *is* $F$*-free (i.e.* $\alpha \neq \beta \in \mathrm{Rang}(f'_\delta) \Rightarrow \beta \notin F(\alpha)$*).*

*2) The forcing of adding a Cohen subset of* $\lambda$ *(i.e.* $({}^{\lambda >}2, \triangleleft)$*) preserve "*$\bar A$ *exemplifies* $\mathrm{NFr}_2(\lambda, \{(I,J)\})$*" (as it preserves "S is stationary"), add no bounded subsets to* $\lambda$ *and forces* $\diamondsuit_S$.

*Proof.* 1) As in the proof of 2.3.

Let $\bar h = \langle h_\delta : \delta \in S\rangle$ be such that $h_\delta : \delta \to [\delta]^{\le\tau}$ and for every $h : \lambda \to [\lambda]^{\le\tau}$ the set $\{\delta \in S : h_\delta = h \restriction \delta\}$ is a stationary subset of $\lambda$; such $\bar g$ exists as we assume $\diamondsuit_S$. Let $E = \{\delta < \lambda : \tau^{++} \times \omega \text{ divide } \delta\}$ it is a club of $\lambda$ and for $\delta \in S \cap E$ we define the function $g_\delta : \tau^{++} \to [\tau^{++}]^{\le\tau}$ by

$$\begin{aligned} g_\delta(\beta) = \{\gamma < \tau^{++} :&\text{for some } \varepsilon_1, \varepsilon_2 < \mathrm{Dom}(I,J) \text{ we have} \\ &\tau^{++} \times f_\delta(\varepsilon_1) + \gamma \in h_\delta(\tau^{++} \times f_\delta(\varepsilon_2) + \beta\}. \end{aligned}$$

Note that $|g_\delta(\beta)| \le \tau$ as $h_\delta(\tau^{++} \times f_\delta(\varepsilon_2) + \beta)$ has cardinality $\le \tau$ and the number relevant of $\varepsilon_1, \varepsilon_2$ is $\le |\mathrm{Dom}(I,J)| = \kappa \le \tau$. So by [Ha61] there is an unbounded subset $Z_\delta$ of $\kappa^{++}$ such that $\beta_1 \neq \beta_2 \in Z_\delta \Rightarrow \beta_1 \notin g_\delta(\beta_2)$.
Let $Z_\delta = \{\gamma_{\delta,\varepsilon} : \varepsilon < \tau^{++}\}$, with $\gamma_{\delta,\varepsilon}$ increasing with $\varepsilon$. Now for $\delta \in S \cap E$ we define $f'_\delta : \mathrm{Dom}(I,J) \to \delta$ by

$$f'_\delta(\varepsilon) = \tau^{++} \times f_\delta(\varepsilon) + \gamma_{\delta,\varepsilon}.$$

Now clearly $f'_\delta$ is a function from $\kappa = \mathrm{Dom}(I,J)$ into $\lambda$, in fact, it is into $\delta$ as $\mathrm{Rang}(f_\delta) \subseteq \delta$ & $(\tau^{++} \times \omega) | \delta$. Also for $\delta_1 \neq \delta_2$ from $S \cap E$

$$\begin{aligned} \{\varepsilon < \kappa : f'_{\delta_1}(\varepsilon) = f'_{\delta_2}(\varepsilon)\} = \{\varepsilon < \kappa : &\tau^{++} \times f_{\delta_1}(\varepsilon) + \gamma_{\delta_1,\varepsilon} \\ &= \tau^{++} \times f_{\delta_2}(\varepsilon) + \gamma_{\delta_2,\varepsilon}\} \subseteq \{\varepsilon < \kappa : f_{\delta_1}(\varepsilon) = f_{\delta_2}(\varepsilon)\} \in I. \end{aligned}$$

Lastly, if $\delta \in S \cap E$ and $\varepsilon_1 \neq \varepsilon_2 < \kappa$ and $f'_\delta(\varepsilon_1) \in h_\delta(f'_\delta(\varepsilon_2))$ then

$$\begin{aligned} \tau^{++} \times f_\delta(\varepsilon_1) + \gamma_{\delta,\varepsilon_1} = f'_\delta(\varepsilon_1) \in h_\delta(f'_\delta(\varepsilon_1)) &= h_\delta(\tau^{++} \times f_\delta(\varepsilon_2) + \gamma_{\delta,\varepsilon_2}) \\ &\subseteq \bigcup_{\varepsilon<\kappa} h_\delta(\tau^{++} \times f_\delta(\varepsilon) + \gamma_{\delta,\varepsilon_2}) \end{aligned}$$

so $\gamma_{\delta,\varepsilon_1} \in g_\delta(\gamma_{\delta,\varepsilon_2})$ by the definition of $g_\delta$, but $\gamma_{\delta,\varepsilon_1}, \gamma_{\delta,\varepsilon_2}$ are distinct members of $Z_\delta$, contradiction to its choice. By the choice of $\langle h_\delta : \delta \in S\rangle$, for every $F : [\lambda] \to [\lambda]^{\le\tau}$ for stationary many $\delta \in S$ (hence $\delta \in S \cap E0$ we have $h_\delta = h \restriction \delta$ & $\delta \in S \cap E$ hence $\mathrm{Rang}(f'_\delta)$

is $F$-free.
2) Straight. $\square_{4.3}$

* * *

Now we give sufficient conditions for the existence of decomposition which implies easily (in the cases needed see later) the existence of suitable colouring. The reader may concentrate on the case $\mathcal{Y}$ is a singleton.

**4.4 The Decomposition Claim.** *Assume:*

(a) *$\mathcal{Y}$ is as in Definition 4.1(1)*

(b) $\lambda > \mu \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$

(c) *for no regular $\kappa \in (\mu, \lambda]$ do we have* $\mathrm{NFr}_2(\kappa, \mathcal{Y})$

(d) *$c\ell$ is a function from $[\lambda]^{\leq\mu}$ to $[\lambda]^{\leq\mu}$*

(e) *for $A, B \in [\lambda]^{\leq\mu}$ we have $A \subseteq c\ell(A)$ and*[7] $A \subseteq B \Rightarrow c\ell(A) \subseteq c\ell(B)$

(f) *$\mathcal{P} \subseteq [\lambda]^{\leq\mu}$ has cardinality*[8] $\leq \lambda$ *or at least has a dense*[9] *such subfamily and satisfies: for every $A \in \mathcal{P}$ there are a pair $(I, J) \in \mathcal{Y}$, a set $\mathcal{U} \in J^+$ and a one to one function $f : \mathcal{U} \to A$ such that:*

$(\alpha)_{f,A,I}$ *if $\mathcal{U}' \subseteq \mathcal{U}$ & $\mathcal{U}' \in I^+$ then for some $A' \in \mathcal{P}$ we have $A' \subseteq A \cap c\ell(\{f(i) : i \in \mathcal{U}'\})$*

$(\beta)_{f,A,I}$ *there are $\mathcal{U}'_\alpha \subseteq \mathcal{U}, \mathcal{U}'_\alpha \in I^+$ for $\alpha < \alpha^*$ for some $\alpha^* \leq \mu$ such that for any $\mathcal{U}' \subseteq \mathcal{U}, \mathcal{U}' \in I^+$ for some $\alpha < \alpha^*$ we have $\mathcal{U}'_\alpha \subseteq \mathcal{U}'$ or at least $A \cap c\ell(\{f(i) : i \in \mathcal{U}'_\alpha\}) \subseteq A \cap c\ell\{f(i) : i \in \mathcal{U}\}$.*

(g) *if $A \in \mathcal{P}$ then*[10] $c\ell(A) = A$.

<u>Then</u>

*$Dec(\lambda, \mathcal{P}, \mu, \mathcal{Y})$: for every $\chi > \lambda$ and $x \in \mathcal{H}(\chi)$ there is a sequence $\langle M_\alpha : \alpha < \lambda\rangle$ such that:*

(i) $M_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$

(ii) $\mu \cup \{\mathcal{Y}, \lambda, \mu, x\} \subseteq M_\alpha$ *and* $\|M_\alpha\| = \mu$

(iii) $\bigcup\limits_{\alpha<\lambda} M_\alpha$ *includes* $\lambda$

(iv) *Assume $A \in \mathcal{P}$ and define $\alpha(A) = Min\{\alpha \leq \lambda :$ if $\alpha < \lambda$ then for some $(I, J) \in \mathcal{Y}$ and $\mathcal{U} \in J^+$ and $f : \mathcal{U} \to A$ which is one-to-one, we have $\{f, \mathcal{U}\} \in M_\alpha$ hence* $\mathrm{Rang}(f) \subseteq M_\alpha$ *and $\{i \in \mathcal{U} : f(i) \in \bigcup\limits_{\beta\leq\alpha} M_\beta\} \in J^+\}\}$. <u>Then</u> $\alpha(A) < \lambda$ and for some $(I, J), \mathcal{U}, f$ which are witnesses to $\alpha(A) = \alpha$ we have: $\{i \in \mathcal{U} : f(i) \in \bigcup\limits_{\beta<\alpha} M_\beta\} \in J$, [we could have added here and in 4.5: moreover for some $X \in M_\alpha$ of cardinality $\leq \mu$ (so $X \subseteq M_\alpha$) we have $\{i \in \mathcal{U} : f(i) \in X \setminus \bigcup\limits_{\beta<\alpha} M_\beta\} \in J^+$.]*

[7] no real harm in adding $c\ell(A) = c\ell(c\ell(A))$
[8] see 4.5 below
[9] i.e. there is $\mathcal{P}' \subseteq \mathcal{P}$ such that $(\forall A \in \mathcal{P})(\exists B \in \mathcal{P}')[B \subseteq A]$ and $|\mathcal{P}'| \leq \lambda$
[10] used in 4.7

(v) *For any pregiven* $\sigma = cf(\sigma) \le \mu$ *we can demand* $M_\alpha = \bigcup_{\varepsilon<\sigma} M_{\alpha,\varepsilon}, M_{\alpha,\varepsilon}$ *increasing with* $\varepsilon$ *and* $\langle M_{\alpha,\zeta} : \zeta \le \varepsilon\rangle \in M_{\alpha,\varepsilon+1}$ *and* $\mu \cup \{\mathcal{Y}, \lambda, \mu, x\} \subseteq M_{\alpha,\varepsilon}$.

Now below we shall prove that Claim 4.4 follows from the following variant (we change (d), (e), (f)).

**4.5 Claim.** *Assume*

$(a)'$ $\mathcal{Y}$ *is as in Definition 4.1(1)*

$(b)'$ $|X| = \lambda > \mu \ge |\mathcal{Y}| + \kappa(\mathcal{Y})$

$(c)'$ *for no regular* $\kappa \in (\mu, \lambda]$ *do we have* $\mathrm{NFr}_2(\kappa, \mathcal{Y})$

$(d)'$ $\bar{\mathcal{F}} = \langle \mathcal{F}_t : t \in T\rangle, T$ *is a partial order of cardinality*[11] $\le \lambda$ *or at least density*[12] $\le \lambda$*; we consider the* $\mathcal{F}_t$*'s as indexed sets such that* $t \ne s \Rightarrow \mathcal{F}_s \cap \mathcal{F}_t = \emptyset$ *though they may have common members, so* $f \in \mathcal{F}_{t(f)}$

$(e)'$ *for each* $t \in T$ *each member* $f \in \mathcal{F}_t$ *is a one-to-one function such that for some* $(I, J) = (I_f, J_f) \in \mathcal{Y}$ *we have* $\mathrm{Dom}(f) \in J^+, \mathrm{Rang}(f) \subseteq X$

$(f)'$ *if* $t \in T$ *and* $f \in \mathcal{F}_t$, *<u>then</u> there is a subset* $T[f]$ *of* $T$ *of cardinality* $\le \mu$ *such that* $T[f]$ *is a cover of* $T_{<f>}$ *which means* $(\forall s \in T_{<f>})(\exists t \in T[f])[s \le_T t]$ *<u>where</u> for* $f \in \mathcal{F}_t$ *we let*

$$T_{<f>} =: \{r \in T : \text{and for some } g \in \mathcal{F}_r \text{ we have } (I_g, J_g) = (I_f, J_f) \text{ and } \{i : i \in \mathrm{Dom}(f), i \in \mathrm{Dom}(g) \text{ and } f(i) = g(i)\} \in I_f^+ = I_g^+\}.$$

<u>*THEN*</u>

*Dec*$(\lambda, \bar{\mathcal{F}}, \mu, \mathcal{Y})$*: for every* $\chi > \lambda$ *and* $x \in \mathcal{H}(\chi)$ *there is a sequence* $\langle M_\alpha : \alpha < \lambda\rangle$ *such that:*

(i) $M_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$

(ii) $\mu \cup \{\mathcal{Y}, \lambda, \mu, x\} \subseteq M_\alpha$ *and* $\|M_\alpha\| = \mu$

(iii) $\bigcup_{\alpha<\lambda} M_\alpha$ *includes* $\lambda$

(iv) *if* $s \in T$, *<u>then</u> for some* $t, s \le_T t \in T$ *and for some* $\alpha < \lambda$ *and* $g \in \mathcal{F}_t$ *we have*

 (α) $\{i \in \mathrm{Dom}(g) : g(i) \in \bigcup_{\beta<\alpha} M_\beta\} \in J_g$

 (β) $t, g \in M_\alpha$ *hence* $\mathrm{Rang}(g) \subseteq M_\alpha$

(v) *for any pregiven* $\sigma = \mathrm{cf}(\sigma) \le \mu$ *we can demand* $M_\alpha = \bigcup_{\varepsilon<\sigma} M_{\alpha,\varepsilon}$ *where* $\langle M_{\alpha,\varepsilon} : \varepsilon < \sigma\rangle$ *is increasing,* $\mu \cup \{Y, \lambda, \mu, M, *\} \subseteq M_{\alpha,\varepsilon}, M_\alpha = \bigcup_{\varepsilon<\sigma} M_{\alpha,\varepsilon}, \langle M_{\alpha,\zeta} : \zeta \le \varepsilon\rangle \in M_{\alpha,\varepsilon+1}$ *and* $\langle M_\beta : \beta < \alpha\rangle \in M_{\alpha,\varepsilon}$.

Before proving 4.5 we deduce 4.4 from it and prepare the ground.

*Proof of 4.4 from 4.5.* Clearly without loss of generality $\mathcal{Y}$ is disjoint and let the set of elements of $T$ be $\mathcal{P}$ and for $A \in T$ we let

[11] light assumption by 4.2(4)

[12] i.e., there is $T' \subseteq T$ satisfying $|T'| \le \lambda$ and $(\forall s \in T)(\exists t \in T')(s \le_T t)$

$$\mathcal{F}_A = \Big\{ f : \text{for some } (I,J) \in \mathcal{Y}, \text{ we have } \mathrm{Dom}(f) \in J^+ \text{ and,}$$
$$f \text{ is one to one into } A \text{ and clauses } (\alpha)_{f,A,I}, (\beta)_{f,A,I} \text{ of (f) of 4.4 hold} \Big\}$$

and for any $f \in \mathcal{F}_A$ let $(I_f, g_f)$ be as in the definition of $\mathcal{F}_A$.

We define the partial order $\leq_T$ on $T$ by: $A_1 \leq_T A_2$ iff $A_2 \subseteq A_1$. We have to check that the assumptions in 4.5 holds, now clauses $(a)', (b)', (c)'$ are the same as $(a), (b), (c)$ of 4.4, and clauses $(d)', (e)'$ are obvious. As for clause $(f)'$ it follows by clause $(f)$ and the definition of $\langle \mathcal{F}_A : A \in \mathcal{P} \rangle$.

[Why? We are given in $(f)'$ the objects $t \in T, f \in \mathcal{F}_t$ so $t = A \in \mathcal{P}$ and let $(I,J) = (I_f, J_f) \in \mathcal{Y}$ and $\mathcal{U} = \mathrm{Dom}(f) \in J^+$. Now as $f \in \mathcal{F}_A$ by the definition of $\mathcal{F}_A$ let $\langle \mathcal{U}'_\alpha : \alpha < \alpha^* \rangle$ be as in subclause $(\beta)$ of clause (f) of 4.4. For each $\alpha < \alpha^*$ choose $A'_\alpha$ as in subclause $(\alpha)_{f,A,I}$ of clause (f) of 4.4 for $\mathcal{U}' = \mathcal{U}'_\alpha$ so in particular $A'_\alpha \in \mathcal{P}$ and $A'_\alpha \subseteq A \cap c\ell\{f(i) : i \in \mathcal{U}'_\alpha\}$. Let us choose $T[f] =: \{A'_\alpha : \alpha < \alpha^*\}$, so obviously $T[f] \in [\mathcal{P}]^{\leq \mu} = [T]^{\leq \mu}$ as $\alpha^* \leq \mu$. Let us check that $T[f]$ is as required in clause $(f)'$ of 4.5. For being a cover: let $r \in T_{\langle f \rangle}$, i.e., let $r = A'$ and (by the definition of $T_{\langle f \rangle}$ inside 4.5), there is $g \in \mathcal{F}_r$ such that $(I_g, J_g) = (I_f, J_f)$ satisfying $\mathcal{U}' =: \{i : i \in \mathrm{Dom}(f) \text{ and } i \in \mathrm{Dom}(g) \text{ and } f(i) = g(i)\} \in I_f^+$, so $\mathcal{U}' \in I^+$. So (by clause $(\beta)_{f,A,I}$ from (f) from 4.4) for some $\alpha < \alpha^*$ we have $A \cap c\ell\{f(i) : i \in \mathcal{U}'_\alpha\} \subseteq A \cap c\ell\{f(i) : i \in \mathcal{U}'\}$; by the choice of $T[f]$ we have $A'_\alpha \in T[f]$, let $s = A'_\alpha$.
Now $s \in T_{[f]}$ and (by the choice of $A'_\alpha$) clearly $s = A'_\alpha \subseteq A$ which means $r \leq s$. Lastly, $T[f]$ has cardinality $\leq |\alpha^*| \leq \mu$. So clause $(f)'$ of 4.5 holds.]

Finally let $\chi$ be large enough and $x \in \mathcal{H}(\chi)$. So by 4.5 there is a sequence $\langle M_\alpha : \alpha < \lambda \rangle$ for our $\langle \mathcal{F}_A : A \in T \rangle, x, \chi$ as required there. It is enough to show that $\langle M_\alpha : \alpha < \lambda \rangle$ is as required in the conclusion of 4.4. Now clauses (i), (ii), (iii) and (v) of the conclusion of 4.4 are just like clauses (i), (ii), (iii) and (v) of the conclusion of 4.5, so we should check only clause (iv). So assume $A \in \mathcal{P}$ and let $\alpha(A)$ be as defined there. By clause (iv) of the conclusion of 4.5 applied to $s = A$ there are $t, \alpha, g$ as there, i.e. $s \leq_T t, \alpha < \lambda, g \in \mathcal{F}_t$ and $\{i \in \mathrm{Dom}(g) : g(i) \in \bigcup_{\beta < \alpha} M_\beta\} \in J_g$, $\mathrm{Dom}(g) \in J_g^+$ and $t, g \in M_\alpha$. So $t \in \mathcal{P}, t \subseteq A$, $\mathrm{Rang}(g) \subseteq t \subseteq s = A$, $\mathrm{Dom}(g) \subseteq M_\alpha$ and $\mathrm{Rang}(g) \in M_\alpha$. So $\alpha$ is as required.

$\square_{4.4}$

*4.6 Observation.* 1) If in Claim 4.4 we add as an assumption clause (g) stated below and $\theta \leq \mu$, <u>then</u> we can find a function $h$ from $\lambda$ to $\mu$ such that for every $A \in \mathcal{P}$ we have $\theta = \mathrm{Rang}(f \restriction A)$; where

- $(h)$ if $(I,J) \in \mathcal{Y}$ and $\mathcal{U} \in J^+$ then[13] $|\mathcal{U}| = \mu$.

2) Assume

- $(a)''$ $\mathcal{Y}$ as in Definition 4.1(1)
- $(b)''$ $\lambda > \mu \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$
- $(c)''$ $\mathcal{P} \subseteq [\lambda]^{\leq \mu}$
- $(d)''$ the conclusion of 4.4 holds
- $(e)''$ as (h) above.

[13] if $J$ is not an ideal we should say: if $(I,J) \in \mathcal{Y}, \mathcal{U}_1 \in J^+, \mathcal{U}_0 \in J$ then $|\mathcal{U}_1 \backslash \mathcal{U}_0| = \mu$

Then we can find $h : \lambda \to \mu$ such that $A \in \mathcal{P} \Rightarrow \mu = \text{Rang}(f \restriction A)$.

*Proof.* 1) For each $\alpha < \lambda$ let $B_\alpha = M_\alpha \cap \lambda \backslash \cup \{M_\beta : \beta < \alpha\}$, and choose $h_\alpha : B_\alpha \to \mu$ such that

$$A \in M_\alpha \wedge A \in [\lambda]^\mu \wedge |A \cap B_\alpha| = \mu \Rightarrow \text{Rang}(h_\alpha \restriction (A \cap B_\alpha)) = \mu.$$

Let $h : \lambda \to \mu$ extend every $h_\alpha$.
2) Similarly. $\square_{4.6}$

Below we think of the functions from $\mathcal{F}$ as say continuous embedding.

**4.7 Claim.** *1) In 4.4 (i.e. if its assumption so its conclusion holds), we have $(A)_\theta \Rightarrow (B)_\theta$ where*

- $(A)_\theta$ *if $\mathcal{P}' \subseteq \mathcal{P}$ has cardinality $\leq \mu$, then we can find $h : \cup\{A : A \in \mathcal{P}'\}$ to $\theta$ such that $A \in \mathcal{P}' \Rightarrow \theta = \text{Rang}(h \restriction A) = \theta$*
- $(B)_\theta$ *we can find $h : \lambda \to \theta$ such that $A \in \mathcal{P} \Rightarrow \theta = \text{Rang}(h \restriction A)$ provided that we add in clause (f) of 4.4 the statement*
- $(\gamma)_\theta$ *in $(\beta)$ we can add: if $\mathcal{U}_1 \in J$ then for some $\alpha < \alpha^*, c\ell\{f(i) : i \in \mathcal{U}'_\alpha\}$ is disjoint to $\{f(i) : i \in \mathcal{U}_1\}$.*

*2) In 4.5 we can conclude $(A)_\theta \Rightarrow (B)_\theta$ when*

- $(A)_\theta$ *if $T' \subseteq T, |T'| \leq \mu$ and $G$ is a function with domain $\cup\{\mathcal{F}_t : (\exists s \in T')(s \leq_T t)\}$ such that $G(f) \in J_f$, then we can find a function $h$ and $\langle (t_s, f_s) : s \in T' \rangle$ such that $s \leq_T t_s$ and $f_s \in \mathcal{F}_{t_s}$ and $s \in T' \Rightarrow \theta = \{(h(f_s(i)) : i \in \text{Dom}(f_s) \backslash G(f_s)\}$*
- $(B)_\theta$ *we can find a function $h : \lambda \to \theta$ as in $(A)_\theta$ for $T' = T$.*

*provided that e.g. ${(\gamma)'}_\theta$ below holds.*

*3) In part (1) we can replace $(\gamma)_\theta$ by*

- $(\gamma)'_\theta$ *if $\mathcal{U} \in J^+$ and $\mathcal{U}_1 \in J$ then $|\mathcal{U} \backslash \mathcal{U}_1| = \mu$.*

*Proof.* 1) Recall that $A \in \mathcal{P} \Rightarrow c\ell(A) = A$. Let $\{A^*_{\alpha,\zeta} : \zeta < \zeta_\alpha \leq \mu\}$ list $\{A \in \mathcal{P} : \alpha(A) = \alpha\}$ and let $(I^\alpha_\zeta, J^\alpha_\zeta), \mathcal{U}^\alpha_\zeta, f^\alpha_\zeta$ witness $\alpha(A_i) = \alpha$. So by the assumption of 4.4, clause $(f)(\gamma)$ appearing only in 4.7(1) there is $A'_{\alpha,\zeta} \in \mathcal{P} \cap M_\alpha$ such that $A'_{\alpha,\zeta} \subseteq c\ell\{f^\alpha_\zeta(i) : i \in \mathcal{U}_\zeta$ and $f_\zeta(i) \in M_\alpha \backslash \bigcup\limits_{\beta<\alpha} M_\beta\} \subseteq A^*_{\alpha,\zeta}$. Clearly $A'_{\alpha,\zeta} \in \mathcal{P}$ and $\alpha(A'_{\alpha,\zeta}) = \alpha$ and we apply clause $(A)_\theta$ to $\mathcal{P}_\zeta = \{A'_{\alpha,\zeta} : \zeta < \zeta_\alpha\}$ getting $h_\alpha : \bigcup\limits_{\zeta<\zeta_\alpha} A'_{\alpha,\zeta} \to \theta$ so without loss of generality $h_\alpha : \lambda \cap M_\alpha \backslash \bigcup\limits_{\beta<\alpha} M_\beta \to \theta$. Now $h = \bigcup\limits_{\alpha<\lambda} h_\alpha$ is as required.
2), 3) Similar. $\square_{4.7}$

*4.8 Remark.* In part (1) of 4.7 we can omit clause $(\gamma)$ if we replace $(A)_\theta$ by

- $(A)^1_\theta$ if $\mathcal{P}' \subseteq \mathcal{P}$ has cardinality $\leq \mu$ and $\langle (A^*_{\alpha,\zeta}, I^\alpha_\zeta, J^\alpha_\zeta, \mathcal{U}^\alpha_\zeta, f^\alpha_\zeta) : \zeta < \zeta^*_\alpha \rangle$ is as in the proof of 4.7, then for some function $h$ with domain $\cup\{A_{\alpha,\zeta} : \zeta < \zeta_\alpha\} \backslash \cup \{B_{\alpha,\zeta} : \zeta < \zeta_\alpha\}$ we have $B_{\alpha,\zeta} \subseteq \mathcal{U}^\alpha_\zeta, B_{\alpha,\zeta} \in J_\alpha$, $\text{Rang}(h \restriction (\text{Rang}(f^\alpha_\zeta \restriction (\mathcal{U}^\alpha_\zeta) \cap \text{Dom}(h))) = \theta$.

The following is close to [Sh 161, §3] (or see [Sh 523, §3] or [EM]).

**4.9 Definition.** 1) We say $\Gamma = (S, \bar{\lambda})$ is a full $(\lambda, \mu)$-set <u>if</u>:

- $(a)$ $S \neq \emptyset$ is a set of finite sequences of ordinals
- $(b)$ $S$ is closed under initial segments
- $(c)$ $\bar{\lambda} = \langle \lambda_\eta : \eta \in S \rangle$ and $\lambda_{<>} = \lambda$
- $(d)$ for each $\eta \in S$, the set $\{\alpha : \eta\hat{\ }\langle \alpha \rangle \in S\}$ is empty or the regular $\mathrm{cf}(\lambda_\eta)$
- $(e)$ $\lambda_\eta > \mu$ <u>iff</u> $\lambda_\eta \neq \mu$ <u>iff</u> $(\exists \alpha)(\eta\hat{\ }\langle \alpha \rangle \in S)$ <u>iff</u> $\eta \in S \backslash S^{\mathrm{mx}}$ where $S^{\mathrm{mx}}$ is the set of $\triangleleft$-maximal $\eta \in S$
- $(f)$ $(\alpha)$ if $\lambda_\eta > \mu$ is a successor cardinal then $\alpha < \lambda_\eta \Rightarrow \lambda^+_{\eta\hat{\ }\langle \alpha \rangle} = \lambda_\eta$
  - $(\beta)$ if $\lambda_\eta > \mu$ is a limit cardinal then $\langle \lambda_{\eta\hat{\ }\langle \alpha \rangle} : \alpha < \mathrm{cf}(\lambda_\eta) \rangle$ is (strictly) increasing with limit $\lambda_\eta$.

<u>4.10 Observation/Definition</u>: 1) If $\Gamma = (S, \bar{\lambda})$ is a full $(\lambda, \mu)$-set, <u>then</u> from $S$ and $\lambda$ we can reconstruct $\bar{\lambda}$ hence $\Gamma$, so we may say "$S$ is a full $(\lambda, \mu)$-set" or "$\bar{\lambda} = \bar{\lambda}^{[S]}$". Also if $S \neq \{<>\}$, from $S$ we can reconstruct $\lambda$ and $\mu$.
2) Let $S^{\mathrm{mx}} = \{\eta \in S : \lambda_\eta = \mu\}$.
3) If $\eta \in S$ and $\lambda_\eta \neq \mu$ <u>then</u> for every ordinal $\alpha$ we have $\alpha < \mathrm{cf}(\lambda_\eta) \Leftrightarrow \eta\hat{\ }\langle \alpha \rangle \in S$.

<u>4.11 Fact/Definition</u>: 1) If $S$ is a full $(\lambda, \mu)$-set and $\eta \in S$ let $S^{<\eta>} =: \{\nu : \eta\hat{\ }\nu \in S\}$, it is a full $(\lambda_\eta, \mu)$-set.
2) If for each $\alpha < \mathrm{cf}(\lambda), S_\alpha$ is a full $(\lambda_\alpha, \mu)$ set and $(\lambda_\alpha = \lambda_0 \ \& \ \lambda = \lambda_0^+)$ or $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ is (strictly) increasing with limit $\lambda, \lambda_0 \geq \mu$, <u>then</u>
$S = \{<>\} \cup \bigcup\limits_{\alpha < \mathrm{cf}(\lambda)} \{\langle \alpha \rangle\hat{\ }\eta : \eta \in S_\alpha\}$ is a full $(\lambda, \mu)$-set.
3) For a full $(\lambda, \mu)$-set $S$ and $\eta \in S$, let $\eta^+ = \langle \eta(\ell) : \ell < k \rangle\hat{\ }\langle \eta(k) + 1 \rangle$ if $\ell g(\eta) = k + 1$ but $<>^+$ will be used though not well defined.

*Proof.* Straightforward.

**4.12 Definition.** 1) We define by induction on $\lambda$ the following. For a set $X$ of cardinality $\lambda, \chi$ large enough and $x \in \mathcal{H}(\chi)$ we say $\bar{N}$ is a full $\mu$-decomposition of $X$ for $\mathcal{H}(\chi), x$ (or $(\lambda, \mu)$-decomposition) <u>if</u> for some full $(\lambda, \mu)$-set $S$

$(*)$ $\bar{N}$ is an $S$-decomposition of $X$ inside $\mathcal{H}(\chi)$ for $x$, which means that for the uniquely determined $\langle \lambda_\eta : \eta \in S \rangle$, letting $\lambda_{<>^+} = \lambda$ we have:

- $(a)$ $\bar{N} = \langle (N_\eta, N^+_\eta) : \eta \in S \rangle$
- $(b)$ $N_\eta \prec N^+_\eta \prec (\mathcal{H}(\chi), \in, <^*)$
- $(c)$ $\{X, x\} \in N^+_\eta$ and $\ell < \ell g(\eta) \Rightarrow \{N_{\eta \restriction \ell}, N^+_{\eta \restriction \ell}\} \in N^+_\eta$
- $(d)$ $\|N^+_\eta\| = \lambda_{\eta^+} = |(N^+_\eta \backslash N_\eta) \cap X|$ and $\lambda_{\eta^+} \subseteq N^+_\eta$
- $(e)$ if $\lambda_{<>} > \mu$, <u>then</u> $\langle N_{<\alpha>} : \alpha < \mathrm{cf}(\lambda_{<>}) \rangle$ is $\prec$-increasing continuous with union containing $N^+_{<>}$
- $(f)$ $N^+_{<\alpha>} = N_{<\alpha+1>}$ for $\alpha < \mathrm{cf}(\lambda_{<>})$ and $N_{<0>} = N_{<>}$ has cardinality $\mu$
- $(g)$ for each $\alpha < \mathrm{cf}(\lambda_{<>}(S))$ the sequence $\langle (N_{<\alpha>\hat{\ }\eta}, N^+_{<\alpha>\hat{\ }\eta}) : \eta \in S^{<\alpha>} \rangle$ is a $(\lambda_\eta, \mu)$-decomposition of $X \cap N^+_{<\alpha>} \backslash N_{<>}$ for $\mathcal{H}(\chi)$ and $x' =: \langle x, N_\alpha, N^+_\alpha \rangle$.

2) We say $\bar N$ is a full $(\lambda,\mu,\sigma)$-decomposition of $X$ for $\mathcal{H}(\chi), x$ if $\sigma = \text{cf}(\sigma) \le \mu$ and in addition

(*h*) for each $\eta \in S \backslash S^{\max}$ there is a sequence $\langle N_{\eta,\varepsilon} : \varepsilon \le \sigma\rangle$ which is $\prec$-increasing continuous, $N_{\eta,0} = N_\eta$, for each $\varepsilon < \sigma$ we have $\langle N_{\eta,\zeta} : \zeta \le \varepsilon\rangle \in N_{\eta,\varepsilon+1}$ and $N^+_\eta = N_{\eta,\sigma}$ [alternatively the objects we demand $\in N^+_\eta$ belong to $N_{\eta,\sigma}$ (in clauses (c) and (h))] and in (c) we add $\ell < \ell g(\eta) \Rightarrow \langle N_{\eta\restriction\ell,\varepsilon} : \varepsilon \le \sigma\rangle \in N^+_\eta$.

3) We can write $\langle N_\eta : \eta \in S \cup \{<>^+\}$ instead of $\langle (N_\eta, N^+_\eta) : \eta \in S\rangle$ by clause (f) so $N_{\langle\rangle^+} = N^+_{\langle\rangle}$.

**4.13 Definition.** 1) Let $X, \lambda, \mu, \mathcal{Y}, \bar{\mathcal{F}}$ be as in 4.5 so $T = \text{Dom}(\bar{\mathcal{F}})$.
We say $\bar N$ is good for $(x, X, \mathcal{Y}, T, \bar{\mathcal{F}})$ if:

(*a*) $\bar N$ is a full $(\lambda,\mu)$-decomposition of $X$ for $\mathcal{H}(\chi)$ and $x' =: \langle x, X, \lambda, \mu, \mathcal{Y}, T, \bar{\mathcal{F}}\rangle$; let $\bar N = \langle (N_\eta, N^+_\eta) : \eta \in S\rangle$ and $\bar\lambda = \bar\lambda^{[S]}$

(*b*) if $s \in T$, <u>then</u> for some $t \in T, s \le_T t$ and for some $\eta \in S^{\text{mx}}$ (i.e., $\lambda_\eta = \mu$) there is $f \in \mathcal{F}_t$ and so $(I_f, J_f) \in \mathcal{Y}, \mathcal{U}_f \in J^+_f$ and $f : \mathcal{U}_f \to \text{Rang}(g)$ witnessing it, such that:

$(*)_1$ $\{i \in \mathcal{U}_f : f(i) \in \cup\{N_\nu : \nu \le_{\ell x} \eta \text{ and } \nu \in S^{\text{mx}}\}\}$ belongs to $J_f$

$(*)_2$ $t, f$ belong to $\cap\{N^+_{\eta\restriction\ell} : \ell < \ell g(\eta)\} \cap N^+_\eta$
hence

$(*)_3$ $\{i \in \mathcal{U}_f : f(i) \in \cap\{N^+_{\eta\restriction\ell} : \ell < \ell g(\eta)\} \cap N^+_\eta \backslash \cup \{N_\nu : \nu \le_{\ell x} \eta \text{ and } \nu \in S^{\text{mx}}\}\}$ belongs to $J^+_f$.

2) We may omit $x$ if clear from the context.

**4.14 The Main Claim.** *Under the assumption of 4.5, for $x \in \mathcal{H}(\chi), \sigma = \text{cf}(\sigma) \le \mu$ and $\chi$ large enough there is a full $(\lambda,\mu,\sigma)$-decomposition of $X$ for $\chi, x$ good for $(X, \mathcal{Y}, T, \bar{\mathcal{F}})$.*

*Proof.* By induction on $\lambda = |X|$ for all possible $(T, \bar{\mathcal{F}})$ without loss of generality $|T| = \lambda$.

<u>Case 1</u>: $\lambda = \mu$.
Trivial.

<u>Case 2</u>: $\lambda = \text{cf}(\lambda) > \mu$.
Choose $\langle N_\alpha : \alpha < \text{cf}(\lambda)\rangle$ such that the set $x^* =: \{x, X, \bar{\mathcal{F}}, \mu, \lambda, f \mapsto T_{<f>}, f \mapsto T[f]\}$ belongs to $N_0, N_\alpha \prec (\mathcal{H}(\chi), \in <^*_\chi), N_\alpha$ is $\prec$-increasingly continuous, $\langle N_\beta : \beta \le \alpha\rangle \in N_{\alpha+1}$, each $N_\alpha$ has cardinality $< \lambda$ and $N_\alpha \cap \lambda$ is an initial segment if $\alpha > 0$ and $\|N_0\| = \mu, \mu \subseteq N_0$. For $t \in T$ let $\alpha(t) =: \text{Min}\{\alpha :$ for some $f \in \bigcup\limits_{s \ge t} \mathcal{F}_s$ and $(I_f, J_f) \in \mathcal{Y}$ (as in 4.5 clause $(e)'$) we have $\{i : i \in \text{Dom}(f) \text{ and } f(i) \in N_\alpha\} \in J^+\}$. By renaming $X = \lambda$.
Let $S = \{\beta < \lambda : \text{ for some } t \in T \text{ we have } \beta = \alpha(t)\} \subseteq \lambda$. For each $\beta \in S$ choose $t_\beta \in T$ and $s_\beta$ satisfying $t_\beta \le_T s_\beta$ such that $\beta = \alpha(t_\beta)$ and $f_\beta \in \mathcal{F}_{s_\beta}$ witness this. Let $\mathcal{U}_\beta = \{i \in \text{Dom}(f_\beta) : f_\beta(i) \in N_\beta\}$ and let $f'_\beta =: f_\beta \restriction \mathcal{U}_\beta$ and let $(I_\beta, J_\beta) = (I_{f_\beta}, J_{f_\beta})$ so $\mathcal{U}_\beta \in J^+_\beta$ and $\text{Rang}(f_\beta) \subseteq N_\beta \cap \lambda$. Now without loss of generality $f_\beta \in N_{\beta+1}$ (hence $s_\beta, I_\beta, J_\beta \in N_{\beta+1}$) as all the requirements on $f_\beta$ have parameters in $N_{\beta+1}$ so we could have chosen $f_\beta$ in $N_{\beta+1}$.

Assume toward contradiction that $S$ is stationary. Now as $\mathcal{Y} \in N_0, |\mathcal{Y}| \leq \mu < \lambda$ clearly $\mathcal{Y} \subseteq N_0$ hence for some $y \in \mathcal{Y}$ the set $S_y = \{\beta \in S : (I_\beta, J_\beta) = y\}$ is stationary. Let $y = (I^*, J^*)$ and $S'_y = \{\beta \in S_y : N_\beta \cap \lambda = \beta\}$, clearly it is stationary. It suffices to show that $\langle f'_\delta : \delta \in S'_y \rangle$ exemplifies $\mathrm{NFr}_2(\lambda, \mathcal{Y})$ contradicting assumption $(c)'$ from 4.5. Clearly $S'_y \subseteq \lambda$ and $\delta \in S'_y \Rightarrow \mathrm{Dom}(f'_\delta) \in J^+$ and $\mathrm{Rang}(f'_\delta) \subseteq \delta$, i.e., let us prove clause (c) of Definition 4.1(2) holds. If not, for some $\delta_1 < \delta_2$ in $S'_y$ we have $B =: \{i : i \in \mathrm{Dom}(f'_{\delta_1}), i \in \mathrm{Dom}(f'_{\delta_2})$ and $f_{\delta_1}(i) = f_{\delta_2}(i)\} \in I^+$, hence $t_{\delta_2} \in T_{\langle f_{\delta_1} \rangle}$ (see 4.5, clause $(f)'$) hence by an assumption there is $t'_{\delta_2}$ such that $t_{\delta_2} \leq_T t'_{\delta_2} \in T[f_{\delta_1}]$. But $x^*, \bar{\mathcal{F}}, f_{\delta_1}$ belong to $N_{\delta_1+1} \prec N_{\delta_2}$ hence $T[f_{\delta_1}] \in N_{\delta_1+1}$ but $T[f_{\delta_1}]$ has cardinality $\leq \mu$ (see clause $(f)'$ of 4.5) hence $T[f_{\delta_1}] \subseteq N_{\delta_1+1}$ but $t'_{\delta_2} \in T[f_{\delta_1}]$ so $t'_{\delta_2} \in N_{\delta_1+1}$ hence $\mathcal{F}_{t'_{\delta_2}} \in N_{\delta_1+1}$ hence there is $f' \in \mathcal{F}_{t'_{\delta_2}} \cap N_{\delta_1+1}$ hence $\mathrm{Rang}(f') \subseteq N_{\delta_1+1}$ contradicting the demand $\alpha(t_{\delta_2}) = \delta_2$. So in Definition 4.1(2) only "$S$ is a stationary subset of $\lambda$" may fail, but something has to fail. So $S$ is not stationary.

Let $E$ be a club of $\lambda$ disjoint to $S$ and we can find $\bar{N}' = \langle N'_\alpha : \alpha < \lambda \rangle$ like $\langle N_\alpha : \alpha < \lambda \rangle$ such that $E, \bar{N} \in N'_0$ so for $\bar{N}', S = \emptyset$. Recall that by the assumption of 4.5, $T$ has cardinality $\leq \lambda$ hence $T \subseteq \cup\{N_\alpha : \alpha < \lambda\}$. So for $\alpha \in (0, \lambda)$ for some $\delta \in E$ we have $N'_\alpha \cap \lambda = \delta$ so $N'_\alpha \cap T = N_\delta \cap T, N'_\alpha \cap \bigcup\limits_t \mathcal{F}_t \cap \bigcup\limits_\beta N_\beta = N_\delta \cap \bigcup\limits_t \mathcal{F}_t$. Now for each $\alpha$ we use the induction hypothesis on $X_\alpha =: X \cap N'_{\alpha+1} \backslash N'_\alpha$ and $\langle \bar{\mathcal{F}}^{\langle \alpha \rangle}_t : t \in T^{\langle \alpha \rangle} \rangle$ where $T^{\langle \alpha \rangle} = \{t \in T : t \in N'_{\alpha+1}, t \notin N'_\alpha$ and, moreover, for every $f \in \mathcal{F}_t$ the set $\{i \in \mathrm{Dom}(f) : f(i) \in N'_\alpha\}$ belongs to $J\}$ and $\mathcal{F}^{\langle \alpha \rangle}_t = \{f \restriction \mathcal{U} : \mathcal{U}$ is $\{i \in \mathrm{Dom}(f) : f(i) \in X_\alpha\}$ and $f \in \mathcal{F}_t \cap N'_{\alpha+1} \backslash N'_\alpha\}$, and $x_\alpha = \langle x^*, \alpha, \bar{N}' \rangle$ so by it we get $\langle N^\alpha_\eta : \eta \in S^\alpha \rangle$ and we let $S = \{<>\} \cup \{< \alpha > \hat{} \nu : \nu \in S_\alpha, \alpha < \lambda\}$ and $N_{<\alpha>\hat{}\nu} = N^\alpha_\nu, N_{<>} = N'_0, N_{<>^+} = \bigcup\limits_\alpha N^0_\alpha$.

Note that if $\lambda > \mu^+, |T| > \lambda$, we can still manage[14] but not needed. Also if $\sigma = \mathrm{cf}(\sigma) \leq \mu$, we can guarantee clause (h) of 4.12(2); similarly to Case 3.

Case 3: $\lambda$ singular $> \mu$.

Let $\lambda = \sum\limits_{i < \mathrm{cf}(\lambda)} \lambda_i$ and $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ increasing continuous, $\lambda_0 > \mu^+ + \mathrm{cf}(\lambda)$. We choose by induction on $\zeta < \mu^+$ a sequence $\langle N^\zeta_i : i < \mathrm{cf}(\lambda) \rangle$ such that:

- $(a)$ $N^\zeta_i$ is $\prec$-increasing continuous in $i$
- $(b)$ $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle, X, \lambda, \mu, \bar{\mathcal{F}}$ all belong to $N^\zeta_0$
- $(c)$ $\lambda_i \subseteq N^\zeta_i$ and $\|N^\zeta_i\| = \lambda_i$ except that $\|N^0_0\| = \mu, \mu \subseteq N^0_0$
- $(d)$ for each $i, \langle N^\zeta_i : \zeta \leq \mu^+ \rangle$ is $\prec$-increasing continuous
- $(e)$ $\langle \langle N^\varepsilon_i : i < \mathrm{cf}(\lambda) \rangle : \varepsilon \leq \zeta \rangle \in N^{\zeta+1}_i$.

For each $i < \mathrm{cf}(\lambda)$ and $\zeta < \mu^+$ and $(I, J) \in \mathcal{Y}$ let $\mathcal{F}^{\zeta,i}_{(I,J)}$ be a maximal family of functions $f \in \{f \restriction \mathcal{U} : \mathcal{U} \in J^+, f \in \bigcup\limits_{t \in T} \mathcal{F}_t, \mathcal{U} \subseteq \mathrm{Dom}(f)\}$ such that $\mathrm{Rang}(f) \subseteq X \cap N^\zeta_i$ and $f \neq g \in \mathcal{F}^{\zeta,i}_{(I,J)} \Rightarrow \{i : i \in \mathrm{Dom}(f), i \in \mathrm{Dom}(g)$ and $f(i) \neq g(i)\} \in I$. Without loss of generality $\mathcal{F}^{\zeta,i}_{(I,J)} \in N^{\zeta+1}_0$ and by 4.2(4) and assumption 4.5 clause $(c)'$ we know $|\mathcal{F}^{\zeta,i}_{(I,J)}| \leq \lambda_i$, so a list of it of length $\leq \lambda_i$ belongs to $N^{\zeta+1}_i$ hence $\mathcal{F}^{\zeta,i}_{(I,J)} \subseteq N^{\zeta+1}_i$. So if $t \in T$ and we define $\alpha(t)$ as in Case 2 for $\langle N^{\mu^+}_\alpha : \alpha \leq \mathrm{cf}(\mu) \rangle$, we get that $\alpha(t)$ is necessarily nonlimit.

[14]we should strengthen the induction hypothesis: instead $X$ we have $X_0 \subseteq X_1$ such that $|X_0| = \lambda$, and continues naturally

Then let $N_\alpha = N^{\mu^+}_\alpha$ if $\alpha \in (0, \mathrm{cf}(\lambda))$ and $N_0 = N^0_0$ and proceed as there (recalling that in Definition 4.12 we have not demanded that $N_\eta \in N^+_\eta$). $\square_{4.14}$

*Proof of 4.5.*
Fix $\sigma = \mathrm{cf}(\sigma) \le \mu$ and by 4.14 we can find a full $(\lambda, \mu, \sigma)$-decomposition of $X$ for $\chi, x$ which is good for $(X, \mathcal{Y}, \bar{F})$. Let $N = \langle N_\eta : \eta \in S\rangle$ and note that $<_{\ell x}$ linearly order $S$ in an order of order type $\lambda$, so let $\langle \eta_\alpha : \alpha < \lambda\rangle$ list $S$ in $<_{\ell x}$-increasing order. Choose $M_\alpha = \cap\{N^+_{\eta_\alpha \restriction \ell} : \ell < \ell g(\eta)\} \cap N_{\eta_\alpha}$ and check that $\langle M_\alpha : \alpha < \lambda\rangle$ is as required in 4.5 (reading Definition 4.13).
This completes the proof of 4.4 above, too. $\square_{4.4,4.5}$

*Proof of Lemma 3.7.*
Just[15] by 4.14 above and 4.7, we do not elaborate as 3.7, 3.8 are not used in other proofs.

*Proof of 3.13.* We use 4.4 + 4.7(1) above.
1) Without loss of generality let $\lambda$ be the set of points of $X$ where $X, \mu$ are given in 3.13, $I = \{A \subseteq {}^\omega 2 : \text{the closure of } A \text{ is countable}\}$, $J$ the following ideal on ${}^\omega 2$

$$\{\mathcal{U} \subseteq {}^\omega 2 : |\mathcal{U}| < 2^{\aleph_0}\}$$

and

$$\mathcal{Y} = \{(I, J)\}.$$

So the conclusion $(*)$ of 3.13 just means "for some regular $\kappa \in (\mu, \lambda]$ we have $\mathrm{NFr}_2(\kappa, \mathcal{Y})$" and toward contradiction assume it fails. Clearly $\chi \ge \mu$. Let $c\ell : [\lambda]^{\le \mu} \to [\lambda]^{\le \mu}$ be

$$\begin{aligned} c\ell(A) = \big\{\alpha : &\alpha \in A \text{ or for some countable } B \subseteq A, \alpha \text{ belongs} \\ &\text{to the closure of } B \text{ in the topological space } X \text{ and} \\ &c\ell(B) \text{ has cardinality } \le 2^{\aleph_0}\big\} \end{aligned}$$

(if we like to have $c\ell(A) = c\ell(c\ell(A))$, iterate this $\omega_1$-times).
Let us consider the assumptions of 4.4 and 4.7. Now clause (a) holds by the explicit choice of $\mathcal{Y}$ above, as for clause (b), we have $|\mathcal{Y}| = 1, \kappa(\mathcal{Y}) = 2^{\aleph_0}$ which is $\le \mu$ by the assumption of 3.13. Clause (c) is the assumption toward contradiction above, clause (d) (on $c\ell$) holds as clearly $A \in [\lambda]^{\le \mu}$ implies $c\ell(A) = \cup\{c\ell(B) : B \in [A]^{\le \aleph_0} \text{ and } |c\ell(B)| \le 2^{\aleph_0}\}$ and $[A]^{\le \aleph_0}$ has cardinality $\le \mu^{\aleph_0} = \mu$ and each countable $B$ contribute at most $2^{\aleph_0}$ points. Clause (e) holds by the properties of closure. Lastly, for clause (f) including subclause $(\gamma)$ which was added in 4.7 we define

$$\begin{aligned} \mathcal{P} = \{A : &A \subseteq \lambda \text{ is a subset of } A, \\ &\text{has cardinality continuum and } X \restriction A \text{ is homeomorphic to } {}^\omega 2\}. \end{aligned}$$

So for $A \in \mathcal{P}$ let $f = f_A$ be a homeomorphism from the topological space ${}^\omega 2$ onto the space $X \restriction A$ and $\mathcal{U}_A = {}^\omega 2$; we shall show that they are as required in (f) of 4.4. Now for

[15] recall that 3.7 is not used in the proof of 4.18

$A \in \mathcal{P}, X \restriction A$ is a compact space and $X$ is Hausdorff, hence $A$ is a closed subset of $X$. If $\mathcal{U}' \subseteq {}^{\omega}2, \mathcal{U}' \notin I$, i.e. $\mathcal{U}'$ is not scattered letting $\mathcal{U}'' = \{\nu \in \mathcal{U}'$: for no open nb of $\nu$ in ${}^{\omega}2$ is $\mathcal{U}'$ scattered$\}$ and $f = f_A$, <u>then</u> we have

$$A' = c\ell\{f(i) : i \in \mathcal{U}''\} = \{f(i) : i \in c\ell_{^{\omega}2}(\mathcal{U}'')\} \subseteq A$$

is homeomorphic to ${}^{\omega}2$ hence $\in \mathcal{P}$, and this proves clause $(\alpha)$.
Let $\langle \mathcal{U}'_\alpha : \alpha < 2^{\aleph_0}\rangle$ list the countable nonscattered subsets of ${}^{\omega}2$, it clearly exemplifies clause $(\beta)$ (of (f) of 4.4).

Lastly, clause $(\gamma)$ (which does not appear in 4.4 but in 4.7(1)), any perfect subsets of ${}^{\omega}2$ contain $2^{\aleph_0}$ many pairwise disjoint perfect subsets so any member of $J$ is disjoint to all but $< 2^{\aleph_0}$ of them.
So as all the assumptions of 4.4 hold so we can apply 4.7(1). There for our $\theta = \mu$, if $(A)_\theta$ of 4.7(1) holds, then we get $(B)_\theta$ which says $X \nrightarrow [{}^{\omega}2]^1_\theta$ contradicting an assumption. But $(A)_\theta$ of 4.7 holds as we have assumed in 3.13: for every subspace $X^*$ of $X$ with $\leq \mu$ points, $X^* \nrightarrow ({}^{\omega}2)^1_\theta$ and as for $(\delta)$ there it was checked above.
2) For $\rightarrow_w$ we observe it is the same proof. For $\mathbb{R}$ we just should be more accurate about closure; note that the topological closure of a countable set may have cardinality bigger than $2^{\aleph_0}$. For $A \subseteq X$ let $c\ell(A) = c\ell(A, X) = \cup\{\mathrm{Rang}(f) : f$ a one to one mapping from $\mathbb{R}$ to $X$ which is a homeomorphism onto $X \restriction \mathrm{Rang}(f)$ and such that $Y_f = \{x \in \mathbb{R} : f(x) \in A\}$ is a dense subset of $\mathbb{R}$. But for any such $f_1, f_2$, if some $Y \subseteq Y_{f_1} \cap Y_{f_2}$ is countable dense and $[x \in Y \Rightarrow f_1(y) = f_2(y)]$ then $f_1 = f_2$, so the proof is similar. Alternatively replaced $\mathbb{R}$ by $[0,1]_{\mathbb{R}}$. $\square_{3.13}$

As should be clear from the previous part of the paper, $\mathrm{NFr}_2(\lambda, \mathcal{Y})$ is closely connected to pcf theory. In particular, on the one hand, §1 uses essentially the cases of $\mathrm{NFr}_1$ whose consistency is not clear (i.e. hopefully it will be proved that they are impossible). On the other hand, §2 uses a case of $\mathrm{NFr}_2$, say for $I = [\omega_1]^{<\aleph_0}$. So let us explicate the obvious relation (and the connection to [Sh 460, 3.9]).
The reader may wonder why not finer properties complimentary to the existence of large almost disjoint families were used, as in [Sh 430]; the answer is that here assumption like $\mu^\sigma = \sigma$ are natural (and limitations on time).

**4.15 Claim.** *1) If* $\mathrm{NFr}_1(\lambda^*, \lambda, \mathcal{Y})$ *and $I^*$ is an ideal on $\kappa = \kappa(\mathcal{Y})$, satisfying $(*)$ below, <u>then</u> there is $\mathcal{F} \subseteq {}^{\kappa}\lambda$ such that $f \neq g \in \mathcal{F} = \{i < \kappa : f(i) = g(i)\} \in I^*$ and $|\mathcal{F}| = \lambda^*$ where*

- $(*)$ *if $(I, J) \in \mathcal{Y}$ and $A \in J^+$ <u>then</u> for some one-to-one function $h$ from $\kappa$ into $A$ we have $\mathrm{Rang}(h) \notin J$ and for every $B \subseteq \kappa$ we have $[\{h(\alpha) : \alpha \in B\} \in I \Rightarrow B \in I^*]$.*

*2) If $NFr_1(\lambda^*, \lambda, \{(I^*, J^*)\})$ or $(\lambda^*, \lambda, \mathcal{Y}, I^*)$ is as in part (1), and $2^\kappa \leq \lambda$ <u>then</u> for some sequence $\bar{\theta} = \langle \theta_i : i < \kappa\rangle$ of regular cardinals $\in [2^\kappa, \lambda]$ we have $\prod_{i<\kappa} \theta_i / I^*$ has true cofinality which is $\geq \lambda^*$.*
*3) Assume*

- $(a)$ *$NFr_2(\lambda, \mathcal{Y})$, so $\lambda$ regular $> |\mathcal{Y}|$*
- $(b)$ *$\mathcal{Y}'$ a family of pairs $(I, J)$ satisfying $\kappa(\mathcal{Y}') \leq \kappa$ and: if $(I, J) \in \mathcal{Y}, h$ is a function from $\mathrm{Dom}(I, J)$ into a limit ordinal $\delta$, <u>then</u> for some $A \in J^+, h''(A)$ is bounded in $\delta$ and $(I \restriction A, J \restriction B) \in \mathcal{Y}'$.*

*<u>Then</u> for some $\lambda' < \lambda$, we have* $\mathrm{NFr}_1(\lambda, \lambda', \mathcal{Y}')$.

*Proof.* Straight.

*4.16 Conclusion.* If $\mu$ is a limit cardinal satisfying $\otimes_\mu$ below, <u>then</u> $\lambda = \mathrm{cf}(\lambda) > \mu > \kappa$ implies $\mathrm{Fr}_2(\lambda, ([\mu]^{<\mu}, [\mu]^{<\kappa}))$ where

$\otimes_\mu$ for every $\lambda > \mu$ for some $\theta < \mu$ we have:
if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \backslash \mu$ and $|\mathfrak{a}| < \mu$ then $\mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a}) \subseteq \lambda$.

*Proof.* Easy by 4.15.

*4.17 Concluding Remark.* 1) Of course, we may replace in 3.2 the space ${}^\omega 2$ by many others, e.g. $\mathbb{R}$, or any Hausdorff $Y^*$ space with $2^{\aleph_0}$ points such that for any uncountable $A \subseteq Y^*$, for some countable $B \subseteq A, |c\ell_{Y^*}(B)| = 2^{\aleph_0}$ moreover if $Z \subseteq Y^*, |Z| < 2^{\aleph_0}$ for some uncountable $B' \subseteq c\ell_{Y^*}(B)$ we have $c\ell_{Y^*}(B')$ is disjoint to $Z$.

We can also add variants with $\rightarrow_w$ replacing $\rightarrow$. As long as the space has $\leq 2^{\aleph_0}$ points, the only place we should be concerned is the proof of 3.13, we reconsider the choice of $c\ell$ in the proof. In all cases for an embedding $f$ from $Y \subseteq Y^*$ to $X$, let $c\ell(\mathrm{Rang}(f)) = \{x \in X : \text{for some } y \in Y^*, f \cup \{\langle y, x\rangle\} \text{ is an embedding of } Y^* \restriction (\mathcal{U} \cup \{x\}) \text{ to } X \restriction ((\mathrm{Rang}(f)) \cup \{y\})\}$ and $f^+ = f \cup \{\langle y, x\rangle : x, y \text{ as above}\}$. The point is that for this choice of $c\ell$, if $Y_1 \subseteq Y_2 \subseteq Y^*, Y_2 \subseteq c\ell_{Y^*}(X_1)$ and $f$ embeds $Y_2$ into $X$ with $\mathrm{Rang}(f)$ not necessarily close, then $(f \restriction X_1)^+$ is a function from some $Y_3 \subseteq Y^*$ into $X$ extending $f$.
2) We may like to add to 3.2 the case with continuum many colours that is let $(B_m)_{<\mu}[{}^\omega 2]$ and $(B_m)^+_{<\mu}$ be defined like $(B)[{}^\omega 2], (B)^+$, replacing $)^1_{<\,\mathrm{cf}(2^{\aleph_0})}$ by $)^1_{<\mu}$ and we add $(B_m)_{<\beth_2^+}[{}^\omega 2], (B)^+_{<\beth_2^+}$ to the list of equivalent statements. Similarly for $(A)$. More is proved, that is $X \rightarrow ({}^\omega 2)^1_{<\lambda}$ where $X$ has $\lambda$ points (or we get $\lambda$ when we ask for compact $X$). The main point is adopting 1.2 (and 1.7).

For this we add also $(C_m)_{\beth_2, \beth_2, \aleph_2}$ where for $\kappa \geq \theta \geq \sigma$ we let

$(C_m)_{\kappa,\theta,\sigma}$ there are $\lambda, S, \bar f$ such that

- $(a)$ $S \subseteq \lambda$ is stationary $> \kappa^+, \kappa > \theta \geq \sigma$
- $(b)$ $\bar f = \langle f_\delta : \delta \in S\rangle$
- $(c)$ $\mathrm{Dom}(f_\delta) = \theta$, each $f_\delta(i)$ is a subset of $\delta \backslash i$ of cardinality $\leq \kappa$ and $\langle \min(f_\delta(i)) : i < \theta\rangle$ is increasing with limit $\delta$ (can ask $i < j < \theta \Rightarrow f(\sup(f_\delta(i)) < \min(f_\delta(j))$)
- $(d)$ if $\delta_1 < \delta_2$ are in $S$ then $\{i < \theta : f_{\delta_2}(i) \cap \bigcup\limits_{j<\theta} f_{\delta_1}(j) \neq \emptyset\}$ has cardinality $< \sigma$
- $(e)$ if $F_\ell : \lambda \rightarrow [\lambda]^{\leq \kappa}$ for $\ell = 0, 1$ and $F_0(\alpha) \in [\lambda \backslash \alpha]^{\leq \kappa}$, <u>then</u> for some $\delta \in S$ we have:
  - $(\alpha)$ $f_\delta$ is $(F_0, F_1)$-free which means:
    for $i \neq j < \theta$, the set $F_1(f_\delta(i))$ is disjoint to $F_0(f_\delta(j))$
  - $(\beta)$ there are $\langle \alpha_i : i < \theta\rangle$ such that $f_\delta(i) = F_0(\alpha_i)$ and $\sup[\bigcup\limits_{j<i} f_\delta(i)] < \alpha_i$.

Similarly for (D). Why is this O.K.? See below, noting that we get more.
3) As before, $(B_m)^+ \Rightarrow (B_m)[{}^\omega 2] \Rightarrow (A_m)[{}^\omega 2]$ and $(B_m)^+ \Rightarrow (A_m)^+ \Rightarrow (A_m)[{}^\omega 2]$, also easily $(C) \Rightarrow (C)^+_{\beth_2, \beth_2, \aleph_2}; (B_m)^+ \Rightarrow (B)^+, (A_m)^+ \Rightarrow (A)^+, (B_m)[{}^\omega 2] \Rightarrow (B)[{}^\omega 2]$ and $(A_m)[{}^\omega 2] \Rightarrow (A)[{}^\omega 2]$

- $(f)$ if $(F_0, F_1)$ is a pair of functions with domain $\lambda$ and $F_0(i) \in [\lambda \backslash i]^{\leq \kappa}$

3A) The forcing in 2.8, with the role of $A_\zeta$ being replaced by $\bigcup\limits_{i<\theta} f_\zeta(i)$ and $A^p_\zeta \subseteq \bigcup\limits_{i<\theta} f_\delta(i)$ such that $i<\theta \Rightarrow |A^p_\zeta \cap f_\delta(i)| \le 1$ works.

4) Also

$\boxtimes_4$ $(D)_{\beth_2,\beth_2,\aleph_2}$ implies the consistency of $(B_m)^+_{<\beth_2^+}$.

As before without loss of generality for some $\kappa = \kappa^{<\kappa} \ge \theta = 2^{\aleph_0}, \sigma$ are such that $(C)_{\kappa,\theta,\sigma}$ hold. Now we just need to repeat the proof of 1.2. The asymmetry in clause (d) does not hurt as if $\delta_2 \ne \delta_2, A^{p_1}_{\delta_1}, A^{p_2}_{\delta_2}$ are well defined, then it follows that $|A^{p_1}_{\delta_1} \cap A^{p_2}_{\delta_2}| < \sigma$.

In the crucial point we let $p^* \Vdash$ "$\underset{\sim}{c} : \lambda \to \underset{\sim}{\mu}$ for some $\underset{\sim}{\mu} < \lambda$". Really less is enough: let $p^* \Vdash$ "$\underset{\sim}{Z} \subseteq \lambda$ is unbounded" and we shall find $q$ and $\delta \in S$ such that $p^* \le q \in \mathbb{P}$ and $q \Vdash$ "$\underset{\sim}{X}^* \restriction A^p_\delta$ is a copy of the space $Y$ (e.g. ${}^\omega 2$) and $A^p_\delta \subseteq Y$". How? We define $F_0(\alpha) = \{\beta : \beta \in [\alpha,\lambda)$ and $p^* \nVdash \beta \ne \operatorname{Min}(\underset{\sim}{Z} \backslash \alpha)\}$.

$F_1(\alpha) = \cup\{u^{p_{\alpha,i}} : i<\kappa\}$ where $\langle p_{\alpha,i} : i<\kappa\rangle$ is a maximal antichain above $p^*$ such that $p_{\alpha,i}$ forces $\alpha \in \underset{\sim}{Z}$ or forces $\alpha \notin \underset{\sim}{Z}$.

Now we repeat the proof of 1.2, but instead deciding the colour we decide the right member of $\underset{\sim}{Z}$.

5) Lastly, we get $(C)^+_{\kappa,\theta,\nu}$ from $(C)_{\kappa,\theta,\nu}$. So assume $\lambda > \kappa^+, \kappa > \theta \ge \sigma$ and $\langle A_\delta : \delta \in S\rangle$ are as in $(C)$ and as before (by forcing) without loss of generality $\diamondsuit_S$. Now we can actually prove $(C)_{\kappa,\theta,\sigma}$ for $\lambda$. So we prove

$\boxtimes_5$ if

- $(\alpha)$ $\lambda > \kappa^+, \kappa > \theta \ge \sigma, \kappa^\sigma < \lambda$
- $(\beta)$ $J$ an ideal on $\theta$ such that $(\forall A \in J^+)(\exists a \in J^+)(a \subseteq A)$
- $(\gamma)$ $S \subseteq \lambda$ is stationary, $\bar f = \langle f_\delta : \delta \in S\rangle, f_\delta : \theta \to \theta$ increasing, $\delta_1 < \delta_2 \Rightarrow \{i<\theta : f_{\delta_1}(i) = f_{\delta_2}(i)\} \in J^+$
- $(\delta)$ $\diamondsuit_S$.

Then $(C)_{\kappa,\theta,\sigma}$ as witnessed by $\lambda$.

So let $\langle (F^\delta_0, F^\delta_1) : \delta \in S\rangle$ be such that $F^\delta_\ell : \delta \to [\delta]^{<\kappa}$ for $\ell = 0,1$ be such that: if $F_\ell : \lambda \to [\lambda]^{\le\kappa}$ for $\ell = 0,1$ then $S_{(F_0,F_1)} = \{\delta \in S : F_0 \restriction \delta = F^\delta_0$ and $F_1 \restriction \delta = F^\delta_1\}$ is stationary. We now choose by induction on $\delta \in S$ a function $f_\delta$ such that:

- $(a)$ if there is a function $f$ with domain $\theta$ satisfying the conditions below then $f_\delta$ is such a function, otherwise $f_\delta$ is constantly $\emptyset$
  - $(\alpha)$ $f(i) \in [\delta]^{\le\kappa} \backslash \{\emptyset\}$
  - $(\beta)$ $i<j \Rightarrow \sup(f_\delta(i)) < \min(f_\delta(j))$
  - $(\gamma)$ for each $i<\theta$ for some $\alpha_i < \delta$ we have $F^\delta_0(\alpha_i) = f_\delta(i)$ and $\sup(\bigcup\limits_{j<i} f(j)] < \alpha_i \le \min f(i))$
  - $(\delta)$ $\langle \min(f(i)) : i<\theta\rangle$ converge to $\delta$
  - $(\varepsilon)$ for $i \ne j < \theta$ the set $F^\delta_1(f(i))$ and $F^\delta_6(f(j))$ are disjoint
  - $(\zeta)$ if $\delta_1 \in \delta \cap S$ then $\{i<\delta : f(i) \cap \bigcup\limits_{j<\theta} f_{\delta_1}(j) \ne \emptyset\}$ has cardinality $<\sigma$.

Let $S^- = \{\delta \in S : f_\delta$ is not constantly $\emptyset\}$ and we suffice to prove that $\bar f = \langle f_\delta : \delta \in S^-\rangle$ is as required. Most clauses hold by the definition and we should check clause (e), so let

$F_0, F_1$ be as there. Let $S_{F_0,F_1} = \{\delta \in S : F_0 \restriction \delta = F_0^\delta$ and $F_1 \restriction \delta = F_1^\delta\}$, so this set is stationary.
For every $\alpha \in S^* = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa^+\}$ let $g(\alpha) = \sup(\alpha \cap F_1(\alpha)) < \alpha$ so $g$ is constantly $\alpha(*)$ on some stationary $S^{**} \subseteq S$.

$E_0 = \{\delta < \lambda : \mathrm{otp}(S^{**} \cap \delta) = \delta$ and $\alpha < \delta \Rightarrow \sup(F_0(\alpha)) < \delta$ and $\alpha < \delta \Rightarrow \sup(F_1(\alpha)) < \delta\}$.

Let $E_1^* = \{\delta < \lambda : \mathrm{otp}(E_0 \cap \delta) = \delta\}$ and for $\delta \in E_1 \cap S_{F_0,F_1}$ let $A'_\delta = \{\alpha \in E_0 : \mathrm{otp}(\alpha \cap E_0) \in A_\delta\}$, so $A_\delta \subseteq \delta = \sup(A_\delta)$, $\mathrm{otp}(A_\delta) = \theta$ and $\delta_1 \neq \delta_2 \in E_1 \cap S_{F_0,F_1} \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \sigma$.

Let $A_\delta = \{\alpha'_{\delta,i} : i < \theta\}$ increasingly and let $\alpha_{\delta,i} =$ $\mathrm{Min}(S^{**} \backslash (\alpha'_{\delta,i} + 1))$ so $\alpha_{\delta,i} < \alpha''_{\delta,i+1}$ (even $\alpha_{\delta,i} <$ $\mathrm{Min}(E_1 \backslash (\alpha'_{\delta,i} + 1)$ and choose $f'_\delta$ a function with domain $\theta$ by

$$f'_\delta(i) = F_0(\alpha_{\delta,i}) = F_0^\delta(\alpha'_i)$$

(the last equality as $F_\ell \restriction \delta = F_\ell^\delta$ as $\delta \in S_{F_0,F_1}$).
Clearly $f'_\delta(i) = F_0(\alpha_i) \subseteq$ $\mathrm{Min}(E_1 \backslash (\alpha'_\ell + 1))$ and

$$\gamma \in f'_\delta(i) \Rightarrow F(\gamma) \subseteq \ \mathrm{Min}(E_1 \backslash (\alpha'_{\delta,i} + 1)) \leq \alpha'_{\delta,i+1} < \alpha'_{\delta,i+1}$$

$$\gamma \in f'_\delta(i) \Rightarrow F(\gamma) \cap \alpha_{\delta,i} \subseteq \alpha(*) < \alpha_0$$

Now $f'_\delta$ satisfies almost all the requirements on $f_\delta$ and if $f'_\delta = f_\delta$ for stationarily many $\delta \in E_1 \cap S_{F_0,F_1}$ we are done. Let $W = \{\delta \in E_1 \cap S_{F_0,F_1} : f'_\delta \neq f_\delta\}$, we shall prove that $W$ is not stationary - this is more than enough.

So for $\delta \in W$ necessarily for some $h(\delta) \in \delta \cap S$ we have

$$w_\delta = \{i < \theta : f'_\delta(i) \cap \bigcup_{j<\theta} f_{(\delta)}(j) \neq 0\}$$

has cardinality $\geq \sigma$, so by Fodor's lemma for some $\delta(*)$ we have $W_1 = \{\delta \in W : h(\delta) = \delta(*)\}$ is stationary.
Similarly as $\theta^\sigma < \lambda =$ $\mathrm{cf}(\lambda)$ for some $w^* \in [\theta]^\sigma, w_2 = \{\delta \in w_1 : w^* \subseteq w_\delta\}$ is stationary. As ${}^\sigma[\bigcup_{j<\theta} f_{\delta(*)}(j)]^\sigma$ has cardinality $\kappa^\sigma$ which is $< \lambda$ without loss of generality for some $h^* : w^* \to \bigcup_{j<\delta} f_{\delta(*)}(j)$ the set

$$W_3 = \{\delta \in W_2 : (\forall i \in w^*)(h^*(i) \in f'_\delta(i) \cap \bigcup_{j<\theta} f_{\delta(*)}(j))\}$$

is stationary. So if $\delta_1 < \delta_2$ are in $w_3$ the set $\{i < \theta : f'_{\delta_1}(i) = f'_{\delta_2}(i)\}$ include $w^*$. But $f'_{\delta_1}(i) = f'_{\delta_2}(i)$ implies that $\alpha_i^{\delta_1} = \alpha_i^{\delta_2}$, hence $A_{\delta_1} \cap A_{\delta_2}$ has cardinality $\geq \sigma$ continuously.
6) $W$ has a $\boxtimes$ clause $(\delta)$, we add: $\mathrm{Rang}(f_\delta)$ is bound in $\delta$?
This is equivalent to: for some fixed $\mu < \lambda, (\forall \delta)(\mathrm{Rang}(f_\delta) \subseteq \mu)$. Repeating the proof and replacing club of $C \in [\mu]^\mu$ we get clause $(C)_{\kappa,\theta,\sigma}$ witnessing $\lambda$ with $\mathrm{Rang}(f_\delta) \subseteq \mu$. We then get versions of the $(A)$'s and $(B)$'s with $\mu$ points.
(Note one special point: we should rephrase the "weak $\Delta$-system argument, by using it on a tree with two levels.
7) Note that by part (5) we get a stronger version of the topological statements: for any $\lambda$ (or $\mu$ in (6)) points there is a close copy of ${}^\omega 2$ (or the space $Y$) included in it. Of course, if we

like the space to be compact this refers only to any set of $\lambda$ (or $\mu$) points among the original ones. Note the Boolean Algebra of clopen sets (when $Y$ has such a basis) satisfies the c.c.c. (remember in the cases only $u^p_{\zeta,2i} \cap u^p_{\zeta,2i+1} = \emptyset$ is demanded, the Boolean Algebra is free) so we cannot control the set of ultrafilters (= points), but if we allow more disjointness demand we may, but we have not considered it.

**4.18 Claim.** *If $\mu = \mu^{<\mu}$. Then there is a $\mu$-complete $\mu^+$-c.c. forcing notion $\mathbb{Q}$ of cardinality $2^\mu$ such that*

$$\Vdash_{\mathbb{Q}} \text{“ there is a function } h : {}^\mu\mu \to \mu \text{ such that}$$

$$\begin{aligned}
&(\alpha) \quad \text{if } C \in \mathbf{V} \text{ is a closed subset of } {}^\mu\mu \text{ of cardinality } \le \mu \\
&\qquad \underline{\text{then}}\ \alpha < \mu \Rightarrow |C \cap h^{-1}\{\alpha\}| < \mu \\
&(\beta) \quad \text{if } A \in \mathbf{V} \text{ is a subset of } {}^\mu\mu \text{ of cardinality } > \mu \\
&\qquad \underline{\text{then}}\ \alpha < \mu \Rightarrow |A \cap h^{-1}\{\alpha\}| = |A|\text{”}.
\end{aligned}$$

*Proof.* As in the proof of 3.14, it suffices to prove:

- $(*)$ Assume that $i^*, j^* < \mu$ and $\eta_{\alpha,i} \in {}^\mu\mu$ for $\alpha < \mu^+, i < i^*$ is with no repetitions and $C_{\alpha,j} \subseteq {}^\mu\mu$ is closed with $\le \mu$ points for $\alpha < \mu^+, j < j^*$. Find $\alpha < \beta$ such that $i < i^*\ \&\ j < j^* \Rightarrow \eta_{\alpha,i} \notin C_{\beta,j}$.

Why $(*)$ holds? Assume not. First choose $\delta^* < \mu^+$ such that:

- $(**)$ if $\beta < \mu^+$ and $\zeta < \mu$ then for some $\alpha < \delta^*$ we have $i < i^* \Rightarrow \eta_{\alpha,i} \restriction \zeta = \eta_{\beta,i} \restriction \zeta$.

  We can find $\beta$ such that $\delta^* < \beta < \mu^+$ and $\{\eta_{\beta,i} : i < i^*\}$ is disjoint to $\bigcup\limits_{j<j^*} C_{\delta^*,j}$, noting that $\beta$ exists as $|\bigcup\limits_{j<j^*} C_{\delta^*,j}| \le \mu$. Let $\zeta^* < \mu$ be large enough such that $i < i^*\ \&\ j < j^* \Rightarrow \neg(\exists \nu)(\eta_{\beta,i} \restriction \zeta \triangleleft \nu \in C_{\delta^*,j})$. Lastly, choose $\alpha < \delta^*$ such that $i < i^* \Rightarrow \eta_{\alpha,i} \restriction \zeta = \eta_{\beta,i} \restriction \zeta$.

Now the pair $(\alpha, \delta^*)$ can serve as $(\alpha, \beta)$ above. $\square_{4.18}$

## Glossary

§1 <u>General spaces</u>: Consistency from strong assumptions.

<u>Definition 1.1</u> $X^* \to (Y^*)^n_\theta$, (having a closed copy of $Y$), monochromatic for a colouring of $n$-tuples by $\theta$ colours, $X^* \to_w (Y^*)^n_\theta$ (not necessarily a closed copy).

<u>Theorem 1.2</u> A sufficient condition for a forcing adding a space $X^*$ such that $X^* \to (Y^*)^n_{<\mathrm{cf}(\theta)}$, consisting on conditions on the cardinals
$((A), (B)_1, (B)_2, (C))$ and on the space $Y^*$
$((D), (E))$ [Saharon: copy and revise to be a proof of 1.5].

<u>Claim 1.4</u>: Sufficient pcf conditions for the set theoretic hypothesis of 1.2.

<u>Observation 1.5</u>: on beautifying nice scales.

<u>Claim 1.7</u>: A variant of 1.2.

<u>Comments 1.8</u>: We deal with some variants (e.g. regular spaces $X^*, Y^*$).

<u>Concluding Remarks 1.8</u>: Mainly on $T_3$ spaces.

§2 <u>Consistency from supercompacts</u>

<u>Observation 2.1</u>: How to deduce $(C)$ from $(C)^+$, a new condition.

<u>Claim 2.2</u>: Quoting a "consistency by a supercompact".

<u>Claim 2.3</u>: Sufficient condition of the set theoretic assumption of 1.2.

<u>Conclusion 2.6</u>: Getting from a supercompact a universe with CH + there is a Hausdorff space $X$ with clopen basis such that $X \to (\text{Cantor discontinuum})^1_{\aleph_0}$.

<u>Claim 2.7</u>: Upgrading by a small forcing a stationary $S \notin I[\mu^+]$ included in $\{\delta < \mu^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)^+\}$ to $\bar{A} = \langle A_\delta : \delta \in S' \subseteq S \rangle, S'$ stationary, $A_\delta \subseteq \delta = \sup A_0, \delta_1 \neq \delta_2 \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \mathrm{cf}(\mu)$.

<u>Claim 2.8</u>: Upgrading $\underline{A}$ as in 2.7 to $\underline{A} \restriction S'$ which is $\kappa$-free by a $\mathrm{cf}(\mu)^+$-c.c., $(< \mathrm{cf}(\mu))$-complete forcing notion.

<u>Observation 2.9</u>: By forcing we can partition $S$ to nonreflectiving subsets.

<u>Conclusion 2.10</u>: Getting the necessary assumptions from non trivial $I[\lambda]$.

§3 <u>Equi-consistency</u>

<u>Problem 3.1</u>: What if we assume G.C.H.?

<u>Theorem 3.2</u>: Equi-consistency of several related statements, some are versions of "there is $X \to ({}^\omega 2)^1_2$", and some relate to pcf statement (and relative to $I[\lambda]$ non trivial).

Question 3.4: Phrase such theorems for other spaces.

Definition 3.5: Is $(\kappa, I_0, I_1, \theta)$-approximate.

Example 3.6: On the Cantor discontinuum.

Lemma 3.7: Sufficient conditions for the existence of a $[K, \sigma]$-colouring of $\lambda$.

Conclusion 3.8: A sufficient condition on "$\lambda$ has approximation" for $X \rightarrow [\text{Cantor set}]^1_{2^{\aleph_0}}$.

Claim 3.9: The forcing notions of §1 satisfies a strong $\kappa^+$-c.c.

Definition 3.10: A strong $\mu^+$-c.c. called $*^\varepsilon_D$.

Lemma 3.11: "$\mathbb{Q}$ is $(< \mu)$-strategically complete and has $*^\varepsilon_\mu$" is preserved by $(< \mu)$-support iteration.

Definition 3.12: $X^* \rightarrow [Y^*]^n_\theta$.

Claim 3.13: From $X \nrightarrow [{}^\omega 2]^1_{2^{\aleph_0}}$ to $\langle f_\alpha : \alpha \in S \rangle$, to help 3.2.

Proof of 3.2:

Observation 3.14: Existence of forcing replacing "countable scattered" by finite.

Claim 3.15: The old claim on $I[\lambda]$ non trivial from a strongly compact.

§4 Helping equi-consitency

Definition 4.1: $NFr_\ell(\lambda, \mathcal{Y})$, variant of almost free not free.

Fact 4.2: Basic properties of $\mathrm{NFr}_\ell$.

Claim 4.3: Improving examples for NFr by forcing (toward freeness).

The Decomposition Claim 4.4: Analyzing NFr.

Claim 4.5: A variant of the previous claim 4.4.

Observation 4.6: Improving 4.4.

Claim 4.7: Getting a colouring from decomposition.

Definition 4.9: Defining $(S, \bar{\lambda})$ a full $(\lambda, \mu)$-set.

Observation 4.10: On $\lambda$-set ($\bar{\lambda}$ is computable from $S$).

Fact/Definition 4.11: Analyzing full sets.

Definition 4.12: $\bar{N}$ is a $\mu$-decomposition of $X$ for $\mathcal{H}(\chi), x$.

Definition 4.13: $\bar{N}$, a full $\mu$-decomposition is good for $(X, \mathcal{Y}, \bar{\mathcal{F}})$.

Claim 4.14: In 4.5, there is a good decomposition.

Proof of 3.13:

Claim 4.15: On $NF_1$

Conclusion 4.16: On $Fr(\lambda, [\mu]^{<\mu}, [\mu]^{<\kappa})$

Concluding Remarks 4.17:

Claim 4.18: Properties of NFr.

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