A NEW PROOF OF KAPLANSKY-JACOBSON THEOREM ON ONE-SIDED INVERSES

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Received August 25, 2003; revised September 11, 2003

Abstract. We give a new proof of the well known Kaplansky-Jacobson Theorem on one-sided inverses for rings with identity. We also discuss whether we can extend this interesting result to monoids with no ring structure.

1 Monoids admitting ring structure The following interesting theorem was proved by N. Jacobson in 1950 [2].

Theorem 1.1. *If u is an element of a ring R with identity 1 such that u has more than one right inverses, then u has infinitely many right inverses.*

The proof provided by Jacobson for this theorem is a constructive proof. He applied the supposition that u has more than one right inverses, and then he constructed infinitely many right inverses of u as follows:

$$w_k = v_0 + (1 - v_0 u)u^k, \quad k = 1, 2, ...,$$

where v_0 is a fixed right inverse of u. He also remarked in [2] that this result was proved firstly by Kaplansky (oral communication) using structure theory.

We now call the above theorem the Kaplansky-Jacobson Theorem. In the literature, this theorem has also been reproved again by M. Osima [3] and C. W. Bitzer [1]. The proof of Bitzer is noteworthy because he used an elementary method which does not involve ring structure. In fact, he considered the set $S = \{x | ux = 1\}$ and its proper subset $T = \{xu - 1 + s | x \in S\}$ (when $|S| \geq 2$), for some fixed $s \in S$. By showing that the mapping of S onto T given by $x \longmapsto xu - 1 + s$ is injective, he proved that S is infinite, because there does not exist any bijection between a finite set and any of its proper subsets.

In this note, we give a new version and a new proof of Kaplansky-Jacobson Theorem in monoids with ring structure. Our method of proof is different from Bitzer [1], although the method of Bitzer and us are both nonconstructive by using the basic property of finite sets.

Theorem 1.2. *(Kaplansky-Jacobson). If an element of a (multiplicative) monoid admitting ring structure has more than one right inverse, then it has infinitely many.*

Proof. Let e be the identity element of the monoid M. For any $a \in M$, consider the set $S_a = \{b \in M | ab = e\}$. We first claim that if $|S_a| \geq 2$, then $ba \neq e$ for all $b \in S_a$. Suppose if possible that $ba = e$ for some $b \in S_a$, then we have $b = be = b(ac) = (ba)c = ec = c$ for any $c \in S_a$. This contradicts to $|S_a| \geq 2$. Hence our claim is established.

We next show that if $|S_a| \geq 2$, then $|S_a| \not< \infty$. To prove this result, we use the addition of a ring structure admitted by M. In fact, we can easily observe that S_a is the solution set

2000 *Mathematics Subject Classification.* 16A99.
Key words and phrases. Kaplansky-Jacobson Theorem; right inverses; monoids.
*This research is partially supported by a UGC(HK) grant #2160187.
†This research is partially supported by a NSF(China) grant #10071068

of the linear equation $ax = e$ in M. Let the solution set of the corresponding homogenerous equation $ax = 0$ of the equation $ax = e$ be T_a. Then, it is clear that $S_a = T_a + b_0$, where b_0 is a particular solution of $ax = e$. Thus we immediately see that $|S_a| \not< \infty$ if and only if $|T_a| \not< \infty$. We now show that $|T_a| \not< \infty$. For this purpose, we consider the transformation $f : T_a \longrightarrow T_a$ defined by $x \overset{f}{\longmapsto} xa$, for all $x \in T_a$.

If $xa = ya$ for some $x, y \in T_a$, then we have $x = xe = x(ab_0) = (xa)b_0 = (ya)b_0 = y(ab_0) = ye = y$. This shows that f is an injective mapping.

Suppose that f is surjective. Then we have $Imf = T_a$. Since $|S_a| \geq 2$, by our claim above, we have $0 \neq b_0a - e$ and $a(b_0a - e) = (ab_0)a - ae = ea - ae = 0$. Hence $0 \neq b_0a - e \in T_a$ so that there is some $x_0 \in T_a$ such that

$$0 \neq b_0a - e = f(x_0) = x_0a.$$

Thereby, we deduce that

$$\begin{aligned} x_0 &= x_0e = x_0(ab_0) \\ &= (x_0a)b_0 = (b_0a - e)b_0 \\ &= b_0(ab_0) - eb_0 = b_0 - b_0 \\ &= 0. \end{aligned}$$

Because f is injective, we have $b_0a - e = 0$, a contradiction. Hence, we have shown that f is not surjective.

Recall that the fact for any set X, $|X| < \infty$ if and only if for all $f \in \mathcal{T}(X)$, "f is injective $\Longleftrightarrow$ f is surjective." Since $f \in \mathcal{T}(T_a)$ is injective, but not surjective, $|T_a| \not< \infty$. Consequently, we have $|S_a| \not< \infty$. The Kaplansky-Jacobson Theorem is hence proved. □

2 General Monoids We first give an example to show that Kaplansky-Jacobson Theorem fails to be true for some monoids having no ring structure.

Example 2.1. Let $B_n = < a, b | a^n b = 1 >$, for $n \in \mathbf{N}$. B. J. Yu and Q. F. Jiang called such semigroup B_n a generalized bi-cyclic semigroup with identity 1 in [4]. With respect to the Green's relations on B_n, we can see that B_n is $\mathcal{D}$-simple, that is, bi-simple, and is right inverse, that is, every $\mathcal{R}$-class of B_n contains an unique idempotent.

Let Y^* be the free monoid over $Y = \{b, ab, ..., a^{n-1}b\}$. Then, it was proved by B. J. Yu and Q. F. Jiang that the unique $\mathcal{D}$-class of B_n can be determined by the following table(see [4]).

	L_1					
R_1	1	a	a^2	$\cdots$	a^k	$\cdots$
	ω_1	$\omega_1 a$	$\omega_1 a^2$	$\cdots$	$\omega_1 a^k$	$\cdots$
	ω_2	$\omega_2 a$	$\omega_2 a^2$	$\cdots$	$\omega_2 a^k$	$\cdots$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	ω_k	$\omega_k a$	$\omega_k a^2$	$\cdots$	$\omega_k a^k$	$\cdots$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Where $a^0 = w_0 = 1, w_i \in Y^*$.

It was then shown by B. J. Yu and Q. F. Jiang [4] that

$$|\{x' \in B_n | x' \text{ is a right inverse of } a^k\}| = 2^{k-1}, k \in \mathbf{N}.$$

¿From the above result, it is clear that Kaplansky-Jacobson Theorem on one-sided inverses does not hold for the monoid B_n, for $n \geq 2$. Therefore, the semigroup $B_n (n \geq 2)$, regarded as a monoid, does not admit ring structure!

However, there are also monoids admitting no ring structure such that Kaplansky-Jacobson Theorem on one-sided inverses still holds, in particular, every free monoid ($\{1\}= H_1 = D_1$) is such an example. But such an example is trivial.

In closing this paper, we propose the following problem.

Problem. Can we find a monoid M admitting no ring structure with $H_1 \subsetneq D_1$ such that the Kaplansky-Jacobson Theorem still holds, and there really exists a in such a monoid M, with $|S_a| \not< \infty$?

Remark. Let M be a monoid with identity 1 and denote the set of idempotents of M by E. Consider $S_m = \{m' \in M | mm' = 1\}$, for $m \in M$. Then, we can easily observe the following facts:

(i). $L_1 \cap E = \{1\}$;

(ii). $S_m \neq \emptyset$ if and only if $m\mathcal{R}1$; where $m\mathcal{R}1$ means that the principal right ideal generated by m is equal to the right principal ideal generated by 1 in the monoid M.

(iii). $S_m \subseteq L_1 \cap V(m)$, where $V(m)$ is the set of all inverse of m;

(iv). The mapping $\phi : S_m \longrightarrow L_m \cap E$ by $m' \longmapsto m'm$ is bijective.

Thus, we can also formulate our problem as follows:

(i). Can we find a monoid M admitting no ring structure with $L_1 \subsetneq D_1$ such that the Kaplansky-Jacobson Theorem still holds?

(ii). Also, does there exist a monoid M as above with at least one $\mathcal{L}$-class L in D_1 such that $|L \cap E| \not< \infty$?

For concepts and notations not given in this paper, such that Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{D}$, the readers are referred to Howie [5].

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