DISCRETENESS CRITERIA FOR ISOMETRY GROUPS OF NEGATIVE CURVATURE

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Abstract. In this paper, we study the discreteness of nonelementary isometry group of negative curvature and obtain a sufficient and necessary condition for a nonelementary subgroup to be discrete.

1.Introduction. A Hadamard manifold H is a complete simply connected Riemannian manifold with nonpositive curvature. A pinched Hadamard manifold X is a Hadamard manifold of pinched negative curvature; that is, all of the sectional curvatures $K(X)$ satisfy

$$-1 \leq K(X) \leq -a^2,$$

where the constant $a \neq 0$. The n-dimensional hyperbolic space H^n is a pinched Hadamard manifold with constant curvature $K = -1$. We write $Isom(X)$ for the group of all isometrics on a pinched Hadamard manifold X.

Throughout this paper, we adopt the same notations and definitions as in ([2], [5], [9]) such as X_c, X_I, *discrete groups*, *elementary subgroups* and so on. For example, we define *elementary groups* as following:

Definition 1.1. *A subgroup G of $Isom(X)$ is elementary either if $fix(G) \neq \emptyset$, or else if G preserves setwise some bi-infinite geodesic in X_c. Otherwise G is nonelementary.*

Let G be a subgroup of $Isom(X)$. The limit set $L(G)$ is defined as following:

$$L(G) = \{x \in X_I |\ g_m \in G \ with \ \lim_{m\to\infty} g_m(p) \to x \ for \ some \ point \ p \in X\}$$

It is clear that the limit set $L(G)$ is closed in X_I and invariant under G. The limit set $L(G)$ is defined independently of the choice of the point $p \in X$ (see [5; p246]).

For $x \in X, z_1, z_2 \in X_c, x \neq z_1, x \neq z_2$ we have [8]

$$\sphericalangle_x(z_1, z_2) := \sphericalangle(\dot{c}_1(0), \dot{c}_2(0))$$

where $c_i(i = 1, 2)$ is the geodesics from x to z_i and $c_i(0) = x$. For $x \in X,\ z \in X_I, \epsilon > 0$, let

$$C_x(z, \epsilon) = \{y \in X_c | y \neq x, \quad \sphericalangle_x(z, y) < \epsilon\}.$$

The set $C_x(z, \varepsilon)$ is called the *cone* of vertex x and angle ϵ.

For $g \in Isom(X)$, we define the *rotation* of g in $x \in X$ as following [2]:

$$r_g(x) := \max_{w \in T_x X} \sphericalangle(w, P_{g(x),x} \circ g_{*_x} w)$$

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where, $g_{*_x} : T_xX \to T_{g(x)}X$ is the differential and $P_{g(x),x} : T_{g(x)}X \to T_xX$ is the parallel transport along the unique geodesic from $g(x)$ to x. We then define the *norm* of g at x as following:

$$N_g(x) := \max\{(r_g(x), \quad 8d_g(x)\}.$$

For the general theory of pinched Hadamard manifolds, see ([2], [4], [5], [7], [8]).

It is well known that the discreteness of subgroups of pinched Hadamard manifolds is a fundamental problem and has been investigated by many authors (see [1], [9], [11], [12], [13], [15]). In 1976, Jørgensen ([Jø]) gave the following famous criterion of discreteness for subgroups of $SL(2, \mathbf{C})$:

Theorem A. *A nonelementary subgroup of $SL(2, \mathbf{C})$ is discrete if and only if each subgroup generated by two elements is discrete.*

For the study of the discreteness criterion of any nonelementary subgroup we must add some conditions by the Example of Abikoff and Hass [1]. In 1989 and 1993, Martin ([12], [13]) introduced the condition of uniformly bounded torsion and established Theorem A for nonelementary subgroups of $M(\bar{R}^n)$ and negatively curved groups under the condition of uniformly bounded torsion: *A nonelementary subgroup G of $M(\bar{R}^n)$(or negatively curved groups) with the condition of uniformly bounded torsion is discrete if and only if every two generator subgroup is discrete.*

Let G be any non-elementary subgroup of $Isom(X)$ and $G_h = \{f \in G : f$ stabilizes pointwise the set of fixed points of $h\}$ for any non-elliptic element h. Let $G^* = \cap G_h$ for all non-elliptic ellement h of G. We generalities Theorem A to negatively curved groups.

Theorem 1.2. *Let G be a nonelementary subgroup of $Isom(X)$. Then G is discrete if and only*

(1) G^* is a finite group;

(2) every two-generator subgroup of G is discrete.

Corollary 1.3. *Let G be a non-elementary subgroup of $Isom(X)$. Then G is discrete if and only*

(1) G^* has uniformly bounded torsion;

(2) every two-generator subgroup of G is discrete.

Remark 1.4. Especially, let G denote a nonelementary subgroup of $SL(2, \mathbf{C})$ in Theorem 1.2. Then $G^* = \{I\}$. Thus Theorem 1.2 coincides with Theorem A.

Since nonelementary subgroups of $Isom(X)$ are more complicated than nonelementary subgroups of $M(\bar{R}^n)$, to investigation of the discreteness of any nonelementary subgroup of $Isom(X)$, we have to face some difficulty and our methods of proof are different from those of Martin's [13] and Jørgensen's [11].

2. The proof of Theorem 1.2.

In order to prove Theorem 1.2, we need the following Lemmas.

Firstly we need the following lemma on limit sets of subgroups of $Isom(X)$ which extends a Chen and Greenberg's result [6; Lemma 4.3.5] on limit sets of subgroups in complex hyperbolic space to subgroups of $Isom(X)$:

Lemma 2.1. *Suppose that one of the following conditions is satisfied: (1) $L(G) = \emptyset$, or (2) G has more than two fixed points in X_I. Then G has a fixed point in X. The set of all fixed points in X is either a single point or a totally geodesic submanifold.*

Proof. (1) By [7; Proposition1.9.6], G has a fixed point in X.

(2) Since G has more than two fixed points in X_I, G is a pure elliptic group. For any two fixed points x_0, y_0 in X_I, G leaves the geodesic $[x_0, y_0]$ pointwise fixed. So G has a fixed point in X.

Since the set of fixed points of an elliptic element is either a single point or a totally geodesic submanifold and the intersection of totally geodesic submanifolds is totally geodesic, the last statement of the lemma follows.

Secondly, we need the following Martin and Skora's Definition and Lemma on discrete subgroups of $Isom(X)$:

Definition 2.2. *A discrete subgroup $G \subset Isom(X)$ is called a discrete convergence group if every sequence $\{g_j\}$ of distinct elements of G contains a subsequence $\{g_{j_k}\}$ for which there are x_0 and y_0 in X_I such that*

$$g_{j_k} \to x_0 \ \ locally\ uniformly\ in\ X_c \backslash \{y_0\}$$

and

$$g_{j_k}{}^{-1} \to y_0\ locally\ uniformly\ in\ X_c \backslash \{x_0\}.$$

Lemma 2.3 [14; Theorem5.6]. *Let X be a Hadamard manifold, such that $K(X) \leq A < 0$. If $G \subset Hom(X_c)$ is a discrete group that acts as isometrics on X, then G is a discrete convergence group.*

In Definition 2.2, Martin and Skora generalized the Gehring and Martin's concept of discrete quasiconformal convergence group on $\bar{R^n}$ [10] to Hadamard manifold X_c.

Now we generalize several Gehring and Martin's results [10] by Lemmas 2.1 and 2.3.

Lemma 2.4. *If G is discrete and $Card(L(G)) \geq 2$, then G contains a loxodromic element.*

Proof. Similar to the proof of [10], we can prove this lemma.

Corollary 2.5. *An infinite pure elliptic group is not discrete.*

Corollary 2.6. *Every elliptic element of a discrete subgroup of $Isom(G)$ is of finite order.*

By Lemmas 2.1, 2.4 and Corollary 2.5, we can obtain the following Lemma:

Lemma 2.7. *Let G be a discrete subgroup of $Isom(X)$. We have*

(1) $L(G)$ is empty if and only if G is a finite group of elliptic elements.

(2) $L(G)$ contains exactly one point x_0 if and only if G is an infinite group which contains only elliptic and parabolic elements, $fix(G) = x_0$ and G definitely contains a parabolic isometry.

(3) $L(G)$ contains exactly two points x_0 and y_0 if and only if G is an infinite group which contains only loxodromic elements which fix x_0 and y_0, and elliptic elements which either fix or interchange x_0 and y_0 and G at least contains a loxodromic isometry.

We can further obtain:

Lemma 2.8. *Suppose that G is a discrete subgroup of $Isom(X)$. Then the following statements are equivalent.*
(1) G is elementary.
(2) G has a finite orbit in X_c.
(3) Every two non-elliptic elements of G have a common fixed point.
(4) $Card(L(G)) \leq 2$.

Proof. (1)$\Longrightarrow$(2).
Since G is elementary, we can separate G into three mutually exclusive classes by [5; p244]:
Case(i) $fix(G)$ is a nonempty subspace of X_c.
Case(ii) $fix(G)$ consists of a single point of X_I.
Case(iii) G has no fixed point in X, and G preserves setwise a unique bi-infinite geodesic in X.
Thus G has a finite orbit in X_c.
(2)$\Longrightarrow$(3).
Suppose that G contains two non-elliptic elements f and g such that $fix(f) \cap fix(g) = \emptyset$. We have the following three cases:
(i) f and g are both loxodromic. Let $fix(f) = \{x_0, y_0\}$ and $fix(g) = \{z_0, w_0\}$, where x_0, y_0, z_0, w_0 are distinct points. By [5; p244], for all $x \in X_c \backslash \{x_0, y_0\}$, we have $f^n x \to x_0$ and $f^{-n} x \to y_0$ and for all $x \in X_c \backslash \{z_0, w_0\}$, we have $g^n x \to z_0$ and $g^{-n} x \to w_0$. Therefore G has no finite orbit in X_c.
(ii) f and g are both parabolic. Let $fix(f) = x_0 \neq y_0 = fix(g)$. By [2; Lemma 6.3 (1)], for all point $x \in X$ the orbits $A_x = \{f^n(x)\}$ and $B_x = \{g^m(x)\}$ have both accumulation points in X_I. From [8; Lemma 6.2] x_0 is the unique accumulation point in X_I of the set A_x and y_0 is the unique accumulation point in X_I of the set B_x.
By [7; Proposition 1.93] we have

$$f^{n_k}(x) \to x_0 \quad and \quad f^{n_k - 1}(x) \to x_0$$

for some $x \in X$.
For all other $y \in X$ we also have

$$f^{n_k}(y) \to x_0 \quad and \quad f^{n_k - 1}(y) \to x_0$$

If not, let $y \neq x \in X$, the sequence $\{f^{n_k}(y)\}$ does not converge to x_0, then there exists a subsequence which converges to a point $x^* \in X_C$, $x^* \neq x_0$. This is a contradiction.
For all $x^{'} \in X_I$, $x^{'} \neq x_0$ there is a unique geodesic γ which joins $x^{'}$ and x_0 such that $x^{'} = \gamma(\infty)$ and $x_0 = \gamma(-\infty)$. By [7; Proposition 1.9.13(3)], $f^{n_k}(x^{'}) \to x_0$ as $n_k \to \infty$. So

$$f^{n_k}(x) \to x_0$$

for any point $x \in X_c \backslash \{x_0\}$
Similarly, we can obtain

$$g^{m_l}(x) \to y_0$$

for any point $x \in X_c \backslash \{y_0\}$

Hence it is enough to see that G has no finite orbit in X_c.

(iii) One of f and g is loxodromic, the other is parabolic. Similar to cases (i) and (ii), we can prove that G has no finite orbit in X_c.

Therefore every two non-elliptic elements of G have a common fixed point.

(3)$\Longrightarrow$(4).

If G contains no loxodromic element, then $Card(L(G)) \leq 1$ by Lemma 2.4.

Suppose that G contains a loxodromic element g with fixed points x_0 and y_0. If G contains a parabolic element f, then f and g have a common fixed point. By [8; Proposition 6.8], the two fixed points of g is also fixed by f, this contradicts the fact that f has only one fixed point. Thus G contains only loxodromic and elliptic elements. In the following we prove that $Card(L(G)) = \{x_0, y_0\}$.

Firstly, for any other loxodromic element h of G, we have $fix(g) \cap fix(h) \neq \emptyset$. Without loss of generality, we assume that $h(x_0) = x_0$ and another fixed point of h is z_0. By [8; proposition 6.8], $z_0 = y_0$. So every loxodromic element of G has the same fixed points x_0 and y_0.

Secondly, for any elliptic element h of G, if h and g have a common fixed point, then the other fixed point of g is also fixed by h [8; proposition 6.8]. If $fix(h) \cap fix(g) = \emptyset$, we will show that $g(x_0) = y_0$ and $g(y_0) = x_0$; since otherwise we have the following three cases:

(i) $h(\{x_0, y_0\}) \cap \{x_0, y_0\} = \emptyset$;

(ii) $h(x_0) = y_0$ and $h(y_0) \neq x_0$;

(iii) $h(x_0) \neq y_0$ and $h(y_0) = y_0$.

In case (i), let $f = h^{-1}gh$, then f is loxodromic.Thus G contains two loxodromic isometries g and f which share no common fixed points. This contradicts the hypothesis that every two non-elliptic elements of G have a common fixed point.

In case (ii), we can obtain that either $hgh(\{x_0, y_0\}) \cap \{x_0, y_0\} = \emptyset$ or $hg^2h(\{x_0, y_0\} \cap \{x_0, y_0\} = \emptyset$. Replacing h in case (i) by hgh or hg^2h, we can prove that case (ii) still leads to a contradiction by using the same method in case (i).

In case (iii), it is easy to obtain a contradiction by using the similar method in case (ii).

By above-mentioned argument, we deduce that G contains only loxodromic elements which fix x_0 and y_0 and elliptic elements which either fix or interchange x_0 and y_0. Hence $L(G) = \{x_0, y_0\}$ by Lemma 2.7.

(4)$\Longrightarrow$(5).

It is easy to prove.

In this paper, the key tool to prove Theorem 1.2 is the following famous *Margulis Lemma* which can be found in ([2; 8.3] , [4; p565]) :

Margulis Lemma. *Given $n \in \mathbf{N}$ there are constants $\mu = \mu(n) > 0$ and $I(n) \in \mathbf{N}$ with the following property: Let X be an n-dimensional Hadamard manifold which satisfies the curvature condition $-1 \leq K \leq 0$ and let Γ be a discrete group of isometrics acting on X. For $x \in X$ let $\Gamma_\mu(x) :=< \{\gamma \in \Gamma | d_\gamma(x) \leq \mu\} >$ be the subgroup generated by the elements γ with $d_\gamma(x) \leq \mu$. Then $\Gamma_\mu(x)$ is almost nilpotent, thus it contains a nilpotent subgroup of finite index. The index is bounded by $I(n)$.*

Now we prove the discreteness criterion

2.9. The proof of Theorem 1.2 The necessity is obviously. In the following we prove the sufficiency.

Suppose that G is not discrete. Then there is a sequence $\{g_i\}$ in G such that $g_i \to I$ uniformly in X_c as $i \to \infty$. We will show that this leads to a contradiction.

For every non-elliptic element $h \in G$, from hypothesis, $< g_i, h >$ is discrete. As $\lim_{i\to\infty} g_i = I$, it follows that $\lim_{i\to\infty} h^j g_i h^{-j} = I$ for any integer j. Let $\mu = \mu(n)$ be a Margulis constant. For sufficiently large i and a fixed point $x \in X$ we have

$$d_{g_i} + d_{hg_ih^{-1}} + \cdots + d_{h^jg_ih^{-j}} + \cdots + d_{h^{p+1}g_ih^{-(p+1)}} < \mu$$

and

$$N_{g_i}(x) < \pi/2$$

where $j = 0, 1, \ldots, p+1$ and $p = dim(fix(g_i))$ if g_i is elliptic or $p = 0$ if g_i is parabolic or loxodromic. By Margulis Lemma, $G_i =< h^j g_i h^{-j} | i = 0, 1, \ldots, p+1 >$ and $G_{i,1} =< g_i, hg_ih^{-1} >$ are virtually nilpotent. By [5; Proposition 3.1.1], G_i and $G_{i,1}$ are elementary. As $< g_i, h >$ is discrete, G_i and $G_{i,1}$ are both discrete. If g_i is parabolic or loxodromic, then $< g_i, h >$ is elementary by [13; Lemma 2.2]. If g_i is elliptic, then $< g_i, h >$ is elementary by [13; Lemma 2.3] and [2; §12.3]. Thus $< g_i, h >$ is discrete and elementary for sufficiently large i. By Lemma 2.8, $Card(L(< g_i, h >)) < 3$. We have the following two cases:

(i) h is loxodromic with fixed points $\{x_0, y_0\}$. Since $Card(L(< g_i, h >)) < 3$, we obtain $L(< g_i, h >) = \{x_0, y_0\}$ by Lemma 2.7. If g_i is loxodromic, then g_i stabilizes pointwise the set of fixed points of h; if g_i is elliptic, then g_i stabilizes pointwise the set of fixed points of h or interchanges the two fixed points of h.

If g_i is elliptic, we can prove that there are at most finitely many g_i interchanging the two fixed points of h. If not, $\{g_i\}$ has a subsequence $\{g_{i_k}\}$ such that $\lim_{k\to\infty} g_{i_k} = I$ and g_{i_k} interchanges the two fixed points of h, i.e., $g_{i_k}(x_0) = y_0$ and $g_{i_k}(y_0) = x_0$. In the following we prove that this can lead to a contradiction.

Since $\lim_{k\to\infty} g_{i_k} = I$, for any $\epsilon > 0$ and $x \in X_c$, we have $\measuredangle_p(g_{i_k}(x), x) < \epsilon$ for some $p \in X$ and sufficiently large k. So $\measuredangle_p(g_{i_k}(x_0), x_0) < \epsilon$. Thus $\lim_{k\to\infty} g_{i_k}(x_0) = x_0$. This contradicts the fact that

$$\lim_{k\to\infty} g_{i_k}(x_0) = \lim_{k\to\infty} y_0 = y_0$$

Hence there are at most finitely many g_i interchanging the two fixed points of h. Thus $g_i(x_0) = x_0$ and $g_i(y_0) = y_0$.

(ii) h is parabolic with fixed point x_0. Since $Card(L(< g_i, h >) < 3$, we obtain $L(< g_i, h >) = \{x_0\}$ by Lemma 2.7. Thus g_i are elliptic or parabolic and $g_i(x_0) = x_0$.

From (i) and (ii), we know that $g_i \in G^*$ for sufficiently large i. This is a contradiction.

The fact that G is discrete is a consequence of the above argument.

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