CANONICAL DECOMPOSITION OF TUPLES OF OPERATORS CAUSED BY SYSTEMS OF OPERATOR INEQUALITIES

Takateru Okayasu and Yasunori Ueta

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Abstract. Let $\mathcal{B}(\mathcal{H})^n$ be the algebra of all n-tuples of bounded linear operators on a separable Hilbert space $\mathcal{H}$, and $\mathcal{G}$ a set of maps on $\mathcal{B}(\mathcal{H})^n$ belong to an appropriate class. Then any n-tuple $\boldsymbol{T}$ can be decomposed into the direct sum $\boldsymbol{T}_0 \oplus \boldsymbol{T}^{'}$ of the maximum $\mathcal{G}$-definite (respectively, $\mathcal{G}$-semidefinite) part $\boldsymbol{T}_0$ and the completely non $\mathcal{G}$-definite (resp., non $\mathcal{G}$-semidefinite) part $\boldsymbol{T}^{'}$. It follows that any bounded operator T has the maximum k-hyponormal part for any positive integer k, and so, it can be decomposed into the direct sum $T = T_0 \oplus T_1 \oplus T_2 \oplus \cdots \oplus T_s$ of the completely non hyponormal part T_0, the k-hyponormal but non $(k+1)$-hyponormal part T_k $(1 \leq k < \infty)$ and the maximum subnormal part T_s.

1 Introduction Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. It is known that any $T \in \mathcal{B}(\mathcal{H})$ has the maximum subspace which reduces T to a positive operator, and that for any pair of self-adjoint operators $A,\ B \in \mathcal{B}(\mathcal{H})$ there exists the maximum subspace $\mathcal{M}$ of $\mathcal{H}$ which reduces A and B and on which $A \leq B$ holds. These facts inspire the existence of maximum subspaces on which given operator inequalities hold. For a given family $\mathcal{G}$, however, of polynomials in two noncommuting variables, it is shown in [6] that any $T \in \mathcal{B}(\mathcal{H})$ has the maximum T-reducing subspace $\mathcal{M}$ on which T is $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite), i.e., $p(T|_{\mathcal{M}}, (T|_{\mathcal{M}})^*) = O$ (resp., $p(T|_{\mathcal{M}}, (T|_{\mathcal{M}})^*) \geq O$) holds for any $p \in \mathcal{G}$. We concerned in [8] with a larger family $\mathcal{G}$ than that of polynomials and made some considerations on the maximum subspaces on which given essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite) tuples of operators are $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite). In this paper, we will consider the maximum subspaces on which given systems of operator inequalities are satisfied.

2 Maximum $\mathcal{G}$-definite, and $\mathcal{G}$-semidefinite parts Let $\mathcal{B}(\mathcal{H})^n$ be the algebra of all n-tuples of operators in $\mathcal{B}(\mathcal{H})$, $\mathcal{S}$ a subset of $\mathcal{B}(\mathcal{H})$, and $\mathcal{S}^n$ the set of all n-tuples whose terms are in $\mathcal{S}$. For tuples $\boldsymbol{A} = (A_1,\ A_2,\ \dots,\ A_n)$, $\boldsymbol{B} = (B_1,\ B_2,\ \dots,\ B_n) \in \mathcal{B}(\mathcal{H})^n$, the map $\lambda_{\boldsymbol{A},\ \boldsymbol{B}}$ is defined by

$$\tau_j(\lambda_{\boldsymbol{A},\ \boldsymbol{B}}(\boldsymbol{T})) = A_j T_j B_j \quad (1 \leq j \leq n),$$

where $\tau_j(T_1,\ T_2,\ \dots,\ T_n) = T_j$ $(1 \leq j \leq n)$, and for $\boldsymbol{p} = (p_1,\ p_2,\ \dots,\ p_n) \in \mathcal{P}_n$, the set of all n-tuples of polynomials in $2n$ noncommuting variables $z_1,\ z_2,\ \dots,\ z_n,\ \bar{z}_1,\ \bar{z}_2,\ \dots,\ \bar{z}_n$, the map $\psi_{\boldsymbol{p}}$ is defined by

$$\tau_j(\psi_{\boldsymbol{p}}(\boldsymbol{T})) = p_j(T_1,\ T_2,\ \dots,\ T_n,\ T_1^*,\ T_2^*,\ \dots,\ T_n^*).$$

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Let $\mathcal{E}_{\mathcal{S}}$ be the pointwise norm closed subalgebra generated by the maps $\lambda_{\boldsymbol{A},\ \boldsymbol{B}}$, $\boldsymbol{A},\ \boldsymbol{B} \in \mathcal{S}^n$ and $\psi_{\boldsymbol{p}}$, $\boldsymbol{p} \in \mathcal{P}_n$. For a subset $\mathcal{G}$ of $\mathcal{E}_{\mathcal{S}}$, a tuple $\boldsymbol{T}$ is said to be $\mathcal{G}$-*definite* (resp., $\mathcal{G}$-*semidefinite*) if $\tau_j(\phi(\boldsymbol{T})) = O$ (resp., $\tau_j(\phi(\boldsymbol{T})) \geq O$) $(1 \leq j \leq n)$ hold for any $\phi \in \mathcal{G}$.

If a subspace $\mathcal{M}$ of $\mathcal{H}$ reduces $\boldsymbol{T}$ (i.e., $\mathcal{M}$ reduces each term of $\boldsymbol{T}$), put

$$\boldsymbol{T}|_{\mathcal{M}} = (T_1|_{\mathcal{M}},\ T_2|_{\mathcal{M}},\ \ldots,\ T_n|_{\mathcal{M}}).$$

If a subspace $\mathcal{M}$ reduces $\mathcal{S}$ (i.e., $\mathcal{M}$ reduces any operator in $\mathcal{S}$), then for any $\phi \in \mathcal{G}$, the map $\phi_{\mathcal{M}}$ of $\mathcal{B}(\mathcal{M})^n$ into itself can be defined by the canonical way, and when $\mathcal{M}$ reduces $\boldsymbol{T}$, the concepts of $\mathcal{G}_{\mathcal{M}}$-definiteness and $\mathcal{G}_{\mathcal{M}}$-semidefiniteness for $\boldsymbol{T}|_{\mathcal{M}}$ make sense, where $\mathcal{G}_{\mathcal{M}} = \{\phi_{\mathcal{M}} :\ \phi \in \mathcal{G}\}$.

We have the following:

Theorem 1. *Let $\mathcal{G}$ be a subset of $\mathcal{E}_{\mathcal{S}}$. Then any tuple $\boldsymbol{T} \in \mathcal{B}(\mathcal{H})^n$ has the maximum subspace $\mathcal{M}$ of $\mathcal{H}$ which reduces $\boldsymbol{T}$ and $\mathcal{S}$ such that $\boldsymbol{T}|_{\mathcal{M}}$ is $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite), and the maximum subspace $\mathcal{N}$ of $\mathcal{H}$ which reduces $\boldsymbol{T}$ and $\mathcal{S}$ such that $\boldsymbol{T}|_{\mathcal{N}'}$ is essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite) on any subspace $\mathcal{N}'$ of $\mathcal{N}$ which reduces $\boldsymbol{T}$ and $\mathcal{S}$, and on which the C^*-algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$ is irreducible.*

In the case, $\mathcal{M} \subseteq \mathcal{N}$ holds and the projection onto $\mathcal{M}$ is contained in the center of the von Neumann algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$.

Proof. First, we consider the $\mathcal{G}$-semidefinite case. $\boldsymbol{T}$ is $\mathcal{G}$-semidefinite on a subspace $\mathcal{M}$ of $\mathcal{H}$ which reduces $\boldsymbol{T}$ and $\mathcal{S}$ if and only if

$$\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))|\big)\xi = o \quad \text{for any } \xi \in \mathcal{M},\ \phi \in \mathcal{G},\ \text{and } 1 \leq j \leq n,$$

so, it suffices to show that the subspace

$$\mathcal{M} = \bigcap \Big\{ \mathrm{Ker}\Big(\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))|\big)A\Big) : \phi \in \mathcal{G}, A \in \mathcal{A}, 1 \leq j \leq n \Big\}$$

of $\mathcal{H}$, where $\mathcal{A}$ is the C^*-algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$, is the maximum subspace which reduces $\boldsymbol{T}$ and $\mathcal{S}$, and on which $\boldsymbol{T}$ is $\mathcal{G}$-semidefinite. Since $I \in \mathcal{A}$, it is clear that $\boldsymbol{T}|_{\mathcal{M}}$ is $\mathcal{G}_{\mathcal{M}}$-semidefinite. Let $\mathcal{M}'$ be arbitrary subspace of $\mathcal{H}$ which reduces $\boldsymbol{T}$ and $\mathcal{S}$, and on which $\boldsymbol{T}$ is $\mathcal{G}$-semidefinite. Then $\mathcal{M}'$ reduces $\mathcal{A}$ and $\tau_j(\phi_{\mathcal{M}'}(\boldsymbol{T}))$ for any $\phi \in \mathcal{G}$ and $1 \leq j \leq n$. Hence we have

$$\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))|\big)A\xi = o \quad \text{for any } \xi \in \mathcal{M}',\ \phi \in \mathcal{G},\ A \in \mathcal{A} \text{ and } 1 \leq j \leq n.$$

Therefore we have $\mathcal{M}' \subseteq \mathcal{M}$ and hence $\mathcal{M}$ is the maximum subspace. For $B \in \mathcal{A}'$, $\xi \in \mathcal{M}$, $\phi \in \mathcal{G}$, $A \in \mathcal{A}$ and $1 \leq j \leq n$, we see that

$$\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))|\big)AB\xi = B\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))|\big)A\xi = o$$

and hence $B\xi \in \mathcal{M}$. Thus $\mathcal{M}$ reduces B. Consequently, the projection onto $\mathcal{M}$ is contained in the center of the von Neumann algebra generated by $\mathcal{A}$.

In the $\mathcal{G}$-definite case, it turns out by the same way that

$$\mathcal{M} = \bigcap \Big\{ \mathrm{Ker}\tau_j(\phi(\boldsymbol{T}))A : \phi \in \mathcal{G}, A \in \mathcal{A}, 1 \leq j \leq n \Big\}$$

is nothing but the subspace of $\mathcal{H}$ stated in the theorem.

To prove the essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite) case, decompose $\mathcal{A}$ to the direct sum $\bigoplus \mathcal{A}_k$, where $\mathcal{H} = \bigoplus \mathcal{H}_k$, of irreducible algebras $\mathcal{A}_k$, and let $\mathcal{N}$ be the direct sum

$\bigoplus \mathcal{H}_{k'}$ of $\mathcal{H}_{k'}$ on which $\boldsymbol{T}$ is essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite). Then $\mathcal{N}$ is the subspace stated in the theorem. □

Theorem 1 has led us to the following:

Corollary 1. *If ϕ, ψ are in $\mathcal{E}_{\mathcal{S}}$, then any $\boldsymbol{T} \in \mathcal{B}(\mathcal{H})^n$ has the maximum subspaces $\mathcal{M}$ of $\mathcal{H}$ which reduces $\boldsymbol{T}$ and $\mathcal{S}$ such that $\tau_j(\phi(\boldsymbol{T})), \tau_j(\psi(\boldsymbol{T}))$ $(1 \leq j \leq n)$ are self-adjoint and $\tau_j(\phi(\boldsymbol{T})) \geq \tau_j(\psi(\boldsymbol{T}))$ $(1 \leq j \leq n)$ hold on $\mathcal{M}$, and the maximum subspace $\mathcal{N}$ of $\mathcal{H}$ which reduces $\boldsymbol{T}$ and $\mathcal{S}$ such that $\tau_j(\phi(\boldsymbol{T}))$, $\tau_j(\psi(\boldsymbol{T}))$ $(1 \leq j \leq n)$ are essentially self-adjoint and $\tau_j(\phi(\boldsymbol{T})) \geq \tau_j(\psi(\boldsymbol{T}))$ $(1 \leq j \leq n)$ hold essentially on any $\boldsymbol{T}$, $\mathcal{S}$-reducing subspace of $\mathcal{N}$ on which the C^*-algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$ is irreducible.*

In the case, one has $\mathcal{M} \subseteq \mathcal{N}$.

Proof. Apply Theorem 1 to the set $\mathcal{G} = \{\phi - \psi\}$. □

The preceding corollary is well illustrated by the following examples:

Example 1. It follows that, any pair of positive self-adjoint operators $A,\ B \in \mathcal{B}(\mathcal{H})$ has the maximum A, B-reducing subspace on which an indicated operator inequality, e.g., $e^A \leq e^B$, $\log A \leq \log B$, or $A^p \leq B^p (p > 0)$, holds, and the maximum A, B-reducing subspace on which the operator inequality essentially holds on any A, B-reducing subspace on which the C^*-algebra generated by $A,\ B$ is irreducible. To see this, consider the 2-tuple $\boldsymbol{T} = (A, B)$ and apply Corollary 1 to the set $\mathcal{G}$ of the maps suitably chosen. For the operator inequality stated above, we consider the sets $\mathcal{G}_1 = \{\phi_1,\ \psi_1\}$, $\mathcal{G}_2 = \{\phi_2,\ \psi_2\}$, $\mathcal{G}_3 = \{\phi_3,\ \psi_3\}$, correspondingly, where

$$\begin{array}{ll} \phi_1(T_1,\ T_2) = e^{T_1}, & \psi_1(T_1,\ T_2) = e^{T_2}; \\ \phi_2(T_1,\ T_2) = \log T_1, & \psi_2(T_1,\ T_2) = \log T_2; \\ \phi_3(T_1,\ T_2) = {T_1}^p, & \psi_2(T_1,\ T_2) = {T_2}^p. \end{array}$$

Example 2. It is known that an operator S is subnormal if and only if

$$\phi_{A_1,\ A_2,\ \ldots,\ A_n}(S) = \sum_{0 \leq j,k \leq n} A_j^* S^{*k} S^j A_k \geq O$$

for any $A_1,\ A_2,\ \ldots,\ A_n$ $(n \geq 1)$ in the C^*-algebra generated by S and the identity operator (see [1]). Then, applying the Theorem 1 to the set $\mathcal{G} = \{\phi_{A_1, A_2, \ldots, A_n}\}$, we see that any operator T has the maximum subnormal part T_s and the completely non subnormal part $T^{'}$ such that $T = T_s \oplus T^{'}$. Moreover, T has the maximum subspace $\mathcal{N}$ which reduces T such that T is essentially subnormal on any subspace of $\mathcal{N}$ which reduces T and on which T is irreducible.

3 Applications to operator matrices In this section, we intend to apply Theorem 1 to operators which satisfy given inequalities of operator matrices.

The next theorem is the operator matrix version of Theorem 1:

Theorem 2. *For $T \in \mathcal{B}(\mathcal{H})$ and a subset $\{\phi_{i,j} : 1 \leq i, j \leq N\}$ of $\mathcal{E}_{\mathcal{S}}$, there exists the maximum subspace $\mathcal{M}$ which reduces T and $\mathcal{S}$ such that $(\phi_{i,j}(T))|_{\mathcal{M}^N} \geq O$ holds on the direct sum $\mathcal{M}^N$ of N copies of $\mathcal{M}$.*

Proof. Put $M(T) = (\phi_{ij}(T))$ and $q_{ij}(T) = \sum_{k=1}^{N} \phi_{ki}(T)^* \phi_{kj}(T)$. Then $q_{ij} \in \mathcal{E}_{\mathcal{S}}$ and $M(T)^*M(T) = (q_{ij}(T))$. Choose a sequence $\{p_n\}$ of polynomials in single variable such that $||p_n(M(T)^*M(T)) - |M(T)||| \to 0$ as $n \to \infty$. Put $p_n(M(T)^*M(T)) = (p_{nij}(T))$ and $|M(T)| = (\psi_{ij}(T))$. Then $p_{nij} \in \mathcal{E}_{\mathcal{S}}$ and $||p_{nij}(T) - \psi_{ij}(T)|| \to 0$ as $n \to \infty$ for any $1 \leq i, j \leq N$. Therefore ψ_{ij} is the pointwise norm limit of $\{p_{nij}\}$. So we have $\psi_{ij} \in \mathcal{E}_{\mathcal{S}}$. Now apply Theorem 1 to $\mathcal{G} = \{\phi_{ij} - \psi_{ij} \ : \ 1 \leq i, j \leq N\}$, then we have the maximum subspace $\mathcal{M}$ which reduces T and $\mathcal{S}$, and on which T is $\mathcal{G}$-definite. Therefore $\mathcal{M}$ is the maximum subspace such that $(\phi_{i,j}(T))|_{\mathcal{M}^N} \geq O$ holds. □

The similar argument used in preceding proof together with the results on the essential $\mathcal{G}$-semidefiniteness showed in [8] leads us to the following:

Theorem 3. *Let $\{\phi_{i,j} : 1 \leq i, j \leq N\}$ be a subset of $\mathcal{E}_{\mathcal{S}}$. If $T \in \mathcal{B}(\mathcal{H})$ satisfies that $\pi((\phi_{ij}(T))) \geq O$, π is the Calkin map, then there exists an orthogonal family $\{\mathcal{H}_m : m \geq 0\}$ of subspaces of $\mathcal{H}$ which reduce T and $\mathcal{S}$, and satisfies the following statements*:

(i) $\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$, *and there is no nontrivial subspace of $\mathcal{H}_m$ which reduces T and $\mathcal{S}$ if $m \geq 1$.*

(ii) *$\mathcal{H}_0$ is the maximum subspace which reduces T and $\mathcal{S}$ such that $(\phi_{ij}(T))|_{\mathcal{H}_0^N} \geq O$ holds.*

Therefore, if $m \geq 1$, $(\phi_{ij}(T))|_{\mathcal{H}_m^N} \geq O$ holds on essentially, but there is no nontrivial subspace of $\mathcal{H}_m$ which reduces T and $\mathcal{S}$, and on which $(\phi_{ij}(T)) \geq O$ holds.

Proof. Put $M(T) = (\phi_{ij}(T))$ and $|M(T)| = (\psi_{ij}(T))$, $\{\psi_{ij}\} \subset \mathcal{E}_{\mathcal{S}}$. Since $\pi((\phi_{ij}(T))) = (\pi(\phi_{ij}(T)))$ and $\pi(|T|) = |\pi(T)|$, it follows that $\pi((\phi_{ij}(T))) \geq O$ if and only if T is essentially $\mathcal{G}$-definite, where $\mathcal{G} = \{\phi_{ij} - \psi_{ij}\}$. So, apply Theorem 1 in [8] to $\mathcal{G}$, we obtain the family $\{\mathcal{H}_m\}$ of subspaces of $\mathcal{H}$ stated in the theorem. □

Now we apply Theorem 2 and Theorem 3 to k-hyponormal operators:

Example 3. An operator $T \in \mathcal{B}(\mathcal{H})$ is called k-hyponormal $(1 \leq k \leq \infty)$ if the operator matrix

$$\begin{pmatrix} I & T^* & T^*2 & \cdots & T^{*k} \\ T & T^*T & T^{*2}T & \cdots & T^{*k}T \\ T^2 & T^*T^2 & T^{*2}T^2 & \cdots & T^{*k}T^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & T^{*2}T^k & \cdots & T^{*k}T^k \end{pmatrix}$$

is positive. The k-hyponormal operators are investigated by [2], [3], [4], [5], [7], and others. We apply the preceding theorems to this operator matrix, and conclude that any $T \in \mathcal{B}(\mathcal{H})$ has the maximum k-hyponormal part, and any essentially k-hyponormal operator T can be decomposed into the direct sum $T = T_0 \oplus T_1 \oplus T_2 \oplus \cdots$ of the maxmum k-hyponormal part T_0 and the irreducible essentially k-hyponormal, but non k-hyponormal parts $T_1,\ T_2,\ \ldots.$

Let $\boldsymbol{H}_k (1 \leq k \leq \infty)$ be the set of all k-hyponormal operators, then it is clear that $\boldsymbol{H}_{k+1} \subseteq \boldsymbol{H}_k (1 \leq k \leq \infty)$, while it is known that $\bigcap \boldsymbol{H}_k$ coincides with the set of all subnormal operators. It follows that any $T \in \mathcal{B}(\mathcal{H})$ can be decomposed into

$$T = T_0 \oplus T_1 \oplus T_2 \oplus \cdots \oplus T_s \quad \text{on} \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_s,$$

where T_0 is completely non hyponormal, T_k $(1 \le k < \infty)$ is k-hyponormal but non $(k+1)$-hyponormal and T_s is subnormal. To show this, first we decompose $T = T_s \oplus T^{'}$ where T_s is the maximum subnormal part (acting on the subspace $\mathcal{H}_s$) and $T^{'}$ is the completely non subnormal part (acting on $\mathcal{H}^{'}$). Next, decompose $T^{'} = T_0 \oplus T_1^{'}$ where T_0 is the completely non hyponormal part (acting on $\mathcal{H}_0$) and $T_1^{'}$ is the maximum hyponormal part (acting on $\mathcal{H}_1^{'}$) of $T^{'}$. Further, we decompose $T_1^{'} = T_1 \oplus T_2^{'}$ where T_1 is the completely non 2-hyponormal but hyponormal part and $T_2^{'}$ is the maximum 2-hyponormal part (acting on $\mathcal{H}_2^{'}$) of $T_1^{'}$. Recursively, if $T_k^{'}$ is the maximum k-hyponormal part (acting on $\mathcal{H}_k^{'}$) of $T_{k-1}^{'}$, then $T_k^{'}$ is decomposed into the direct sum $T_k^{'} = T_k \oplus T_{k+1}^{'}$ of k-hyponormal but non $(k+1)$-hyponormal operator T_k and the maximum $(k+1)$-hyponormal part $T_{k+1}^{'}$ (acting on $\mathcal{H}_{k+1}^{'}$) of $T_k^{'}$. Then, it is clear that $\mathcal{H}_1^{'} \supseteq \mathcal{H}_2^{'} \supseteq \mathcal{H}_2^{'} \supseteq \cdots$ and $\bigcap_{k=1}^{\infty} \mathcal{H}_k^{'}$ is a subspace of $\mathcal{H}^{'} (= \mathcal{H}_s^{\perp})$ which reduces T, and on which T is subnormal. Thus, by the maximality of $\mathcal{H}_s$, we have that $\bigcap_{k=1}^{\infty} \mathcal{H}_k^{'} = \{o\}$ and hence, putting $\mathcal{H}_k = \mathcal{H}_k^{'} \ominus \mathcal{H}_{k+1}{'} (k = 1,\ 2,\ \ldots)$, we have $\mathcal{H}_1^{'} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$ and thus

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1^{'} \oplus \mathcal{H}_s = \left(\bigoplus_{k=0}^{\infty} \mathcal{H}_k \right) \oplus \mathcal{H}_s \quad \text{and} \quad T = T_0 \oplus \left(\bigoplus_{k=0}^{\infty} T_k \right) \oplus T_s.$$

This is the aimed decomposition.

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Department of Mathematical Sciences
Faculty of Science
Yamagata University
Yamagata 990-8560, Japan