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GRAND FURUTA INEQUALITY OF THREE VARIABLES

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Abstract. Uchiyama gave a generalization of the grand Furuta inequality and Furuta discussed it based on his previous result. Motivated by such discussions, we consider grand Furuta type operator inequalities of 3 variables, whose hidden key is the chaotic order, i.e., $\log A \geq \log B$ for positive invertible operators A and B. Among others, Uchiyama's theorem and Furuta's theorem are appeared as follows: For $A \geq B \geq C > 0$ and $0 \leq t \leq 1 \leq p$

$$B \geq C \geq (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \geq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

and

$$B \geq C \geq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{p}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

hold for $\beta \geq p$ and $r \geq t$.
As a complement to our preceding inequality, we give an inequality for $A \gg B \gg C$ and $t \geq 0,\ 0 \leq p \leq \beta \leq 2p$,

$$C^{-t} \sharp_{\frac{p+t}{\beta+t}} B^{\beta} \geq B^p \geq (C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)^{\frac{p}{\beta}} \geq A^{-t} \sharp_{\frac{p+t}{\beta+t}} (C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p).$$

1. Introduction. Throughout this note, A and B are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. invertible) operator. The α-power mean of A and B introduced by Kubo-Ando [19] is given by

$$A \sharp_{\alpha} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

The Furuta inequality [7] can be written by the form of α-power mean as follows ([2],[3],[13],[14],[15]).

Furuta inequality: *If $A \geq B \geq 0$, then*

$$\text{(F)} \qquad A^u \sharp_{\frac{1-u}{p-u}} B^p \leq A \quad \text{and} \quad B \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

holds for $u \leq 0$ and $1 \leq p$.

It is a marvelous extension of the Löwner-Heinz inequality:

$$\text{(LH)} \qquad \textit{If} \quad A \geq B \geq 0, \quad \text{then} \quad A^{\alpha} \geq B^{\alpha} \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

As shown in [13](cf.[8]), we can arrange (F) in one line as a satellite theorem of the the Furuta inequality as follows:

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If $A \geq B \geq 0$, then

$$A^u \natural_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \natural_{\frac{1-u}{p-u}} A^p \tag{SF}$$

holds for all $u \leq 0$ and $p \geq 1$.

For $A,\ B > 0$, we denote by $A \gg B$ if $\log A \geq \log B$ and call it the chaotic order ([3],[17],[18]). The next characterization of the chaotic order we obtained in [3] is usefull and starting point of our following discussions about the chaotic order, so we call it chaotic Furuta inequality.

If $A \gg B$, then

$$A^u \natural_{\frac{-u}{p-u}} B^p \leq I \leq B^u \natural_{\frac{-u}{p-u}} A^p \tag{CF}$$

for any $p \geq 0$ and $u \leq 0$.

A satellite theorem (SF) of the Furuta inequality (F) illustrates the difference between the usual order $A \geq B$ and the chaotic order $A \gg B$. As a matter of fact, in [17] and [18] we have shown the following:

If $A \gg B$, then

$$A^u \natural_{\frac{1-u}{p-u}} B^p \leq B \ll A \leq B^u \natural_{\frac{1-u}{p-u}} A^p \tag{SCF}$$

holds for any $p \geq 1$ and $u \leq 0$.

Further generalizations of (CF) and (SCF) have been given as follows [17]:

Theorem A. *For $A,\ B > 0$, if $A \gg B$, then the following (1) and (2) hold.*

$$A^u \natural_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \quad \text{and} \quad A^\delta \leq B^u \natural_{\frac{\delta-u}{p-u}} A^p \quad \text{for} \quad u \leq 0 \quad \text{and} \quad 0 \leq \delta \leq p \tag{1}$$

$$A^u \natural_{\frac{\alpha-u}{p-u}} B^p \leq A^\alpha \text{ and } B^\alpha \leq B^u \natural_{\frac{\alpha-u}{p-u}} A^p \quad \text{for} \quad u \leq \alpha \leq 0 \quad \text{and} \quad 0 \leq p. \tag{2}$$

These are main tools of our discussions below.

2. Grand Furuta inequality. As a generalization of the Furuta inequality, Furuta [9] had given an inequality which we called the grand Furuta inequality in [4],[5] and [16]. It interpolates the Furuta inequality and the Ando-Hiai inequality [1] which is equivalent to the main result of log majorization. We here cite it in terms of operator mean:

The grand Furuta inequality: *If $A \geq B \geq 0$ and A is invertible, then for each $1 \leq p$ and $0 \leq t \leq 1$,*

$$A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \quad \text{and} \quad B \leq B^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (B^t \natural_s A^p) \tag{GF}$$

holds for $t \leq r$ and $1 \leq s$.

The best possibility of the power $\frac{1-t+r}{(p-t)s+r}$ is shown in [20]. The notation $\natural_s$ is defined by $A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}$ for $s \notin [0,\ 1]$. Replacing s in (GF) with $\frac{\beta-t}{p-t}$ for $1 \leq p \leq \beta$, we can state this theorem by the satellite form as follows [16]:

If $A \geq B > 0$, then the following $(*)$ holds for $0 \leq t \leq 1 \leq p \leq \beta$ and $u \leq 0$.

$(*)$

$$A^u \sharp_{\frac{1-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}} \leq B^u \sharp_{\frac{1-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The middle part in $(*)$ had been obtained in [4], [5] and this is the most important improvement of (GF). Now we prepare a similar one which is needed below:

Theorem 1. *Let $A \geq B > 0$ and $0 \leq t \leq 1 \leq p$. Then*

$$H(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$$

is a decreasing function with $\beta \geq p$ and in particular $H(\beta) \leq B^p$ for $\beta \geq p$.

Proof. First of all, suppose that $1 \leq \frac{\beta-t}{p-t} \leq 2$. Then

$$A^t \natural_{\frac{\beta-t}{p-t}} B^p = B^p \natural_{\frac{p-\beta}{p-t}} A^t = B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} A^{-t})B^p \leq B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} B^{-t})B^p = B^\beta$$

By (LH), we have $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p$.

Since $p \geq 1$, we have $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A$. Next if we take β_1 with $1 \leq \frac{\beta_1-t}{\beta-t} \leq 2$, then the preceding argument ensures that

$$A^t \natural_{\frac{\beta_1-t}{p-t}} B^p = A^t \natural_{\frac{\beta_1-t}{\beta-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) = A^t \natural_{\frac{\beta_1-t}{\beta-t}} B_1^\beta \leq B_1^{\beta_1},$$

that is, $A^t \natural_{\frac{\beta_1-t}{p-t}} B^p \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}}$. So

$$(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}} \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p$$

follows from (LH), which shows the monotonicity of $H(\beta)$ and $H(\beta) \leq B^p$ for $\beta \geq p$.

3. Furuta's generalization of Uchiyama's theorem.

Recently, Uchiyama [21] has shown the following inequality as an extension of (GF).

If $A \geq B \geq C > 0$, then for each $0 \leq t \leq 1 \leq p$

$$\text{(U)} \qquad A^{1-t} \geq A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

holds for $r \geq t$ and $s \geq 1$.

Furuta [10] extended this result to the following two directions.
For $A \geq B \geq C > 0$ and $0 \leq t \leq 1 \leq p$, $t \leq r$, $1 \leq s$, the followings hold.

$$(1) \quad B^{1-t} \geq B^{-\frac{t}{2}} C B^{-\frac{t}{2}} \geq A^{-t} \sharp_{\frac{1}{(p-t)s+t}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s \geq A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

$$(2) \quad B^{1-t} \geq B^{-\frac{t}{2}} C B^{-\frac{t}{2}} \geq A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}} \geq A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

The proofs of these in [10] are depending on the results in [12] and a little complicated. So we prove them directly in our own methods. Our expressions of Furta's results (1) and (2) are given as follows by replacing s with $\frac{\beta-t}{p-t}$ for $\beta \geq p$ and we can obtain a little precise forms.

Theorm 2. *If $A \geq B \geq C > 0$ and $0 \leq t \leq 1 \leq p$, then*
(1)
$B \geq C \geq (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \geq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

and

(2) $B \geq C \geq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{p}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

hold for $\beta \geq p$ and $r \geq t$.

Proof. (1) First of all, the assumption $B \geq C > 0$ ensures $(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \leq C$ by $(*)$. Let $D = (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{1}{p-t}}$, then

$A^{-t} \sharp_{\frac{t}{\beta}} D^{\beta-t} \leq B^{-t} \sharp_{\frac{t}{\beta}} D^{\beta-t} = B^{-\frac{t}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{t}{\beta}} B^{-\frac{t}{2}} \leq B^{-\frac{t}{2}} C^t B^{-\frac{t}{2}} \leq B^{-\frac{t}{2}} B^t B^{-\frac{t}{2}} = I,$

that is,

$$(\dagger) \qquad (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{\beta}})^{\frac{t}{\beta}} \leq A^t$$

and we have $(A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{1}{\beta}} \ll A$.
Therefore by (SCF), $A^{-r+t} \sharp_{\frac{1-(t-r)}{\beta-(t-r)}} \{(A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{\beta}})^{\frac{1}{\beta}}\}^{\beta} \leq (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{1}{\beta}}$, namely,

$$A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} \leq A^{-t} \sharp_{\frac{1}{\beta}} D^{\beta-t}.$$

Since $B^{\frac{t}{2}} D^{\beta-t} B^{\frac{t}{2}} = B^t \natural_{\frac{\beta-t}{p-t}} C^p$, we have

$$\begin{aligned} & B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \leq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \\ \leq \quad & B^{\frac{t}{2}} B^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) = (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \leq C \leq B. \end{aligned}$$

(2) is also shown as follows: Since $A \gg (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{1}{\beta}}$ as in above, Theorem A (1) implies that

$$A^{-r+t} \sharp_{\frac{p-t+r}{\beta-t+r}} A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}} \leq (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{p}{\beta}}.$$

Multiplying $A^{-\frac{t}{2}}$ from the both sides of the above, we have

$A^{-r} \sharp_{\frac{p+r-t}{\beta+r-t}} D^{\beta-t} \leq A^{-t} \sharp_{\frac{p}{\beta}} D^{\beta-t} \leq B^{-t} \sharp_{\frac{p}{\beta}} D^{\beta-t} = B^{-\frac{t}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{p}{\beta}} B^{-\frac{t}{2}} \leq B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}},$

where the final inequality follows from Theorem 1. Again multiplying $B^{\frac{t}{2}}$ to each sides of this formula, we have

$$B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \leq C^p.$$

Hence it follows that

$$\begin{aligned} & B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \\ = \quad & B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} \{B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)\} \\ \leq \quad & B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p. \end{aligned}$$

The rest inequalities follow from (1) as the case $\beta = p$.

A key point of this theorem is to attain the condition (†). Several conditions are cosiderable to attain (†) but the condition $A \gg D = (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{1}{p-t}}$ is playing an essential role in our proofs. So we reconstruct our discussions under this condition.

Theorem 3. *If* $A,\ B,\ C > 0$ *satisfy* $A \gg D = (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{1}{p-t}}$ *for some* $0 \le t \le 1 \le p$, *the the following* (1) *and* (2) *hold for* $\beta \ge p$ *and* $r \ge t$.

(1) $$B^t \ \sharp_{\frac{1-t}{p-t}} \ C^p \ge B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \ \sharp_{\frac{1}{\beta}} \ (B^t \ \natural_{\frac{\beta-t}{p-t}} \ C^p) \ge B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \ \sharp_{\frac{1-t+r}{\beta-t+r}} \ (B^t \ \natural_{\frac{\beta-t}{p-t}} \ C^p)$$

(2)
$$B^t \ \sharp_{\frac{1-t}{p-t}} \ C^p \ge B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \ \sharp_{\frac{1}{p}} \ C^p \ge B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \ \sharp_{\frac{1-t+r}{p-t+r}} \ C^p \ge B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \ \sharp_{\frac{1-t+r}{\beta-t+r}} \ (B^t \ \natural_{\frac{\beta-t}{p-t}} \ C^p)$$

Proof. (1) follows from Theorem A.

$$\begin{aligned} & A^{-r} \ \sharp_{\frac{1-t+r}{\beta-t+r}} \ D^{\beta-t} = D^{\beta-t} \ \sharp_{\frac{\beta-1}{\beta-t+r}} \ A^{-r} = D^{\beta-t} \ \sharp_{\frac{\beta-1}{\beta}} \ (D^{\beta-t} \ \sharp_{\frac{\beta}{\beta-t+r}} \ A^{-r}) \\ = \ & D^{\beta-t} \ \sharp_{\frac{\beta-1}{\beta}} \ (A^{-r} \ \sharp_{\frac{-t+r}{\beta-t+r}} \ D^{\beta-t}) \le D^{\beta-t} \ \sharp_{\frac{\beta-1}{\beta}} \ A^{-t} = A^{-t} \ \sharp_{\frac{1}{\beta}} \ D^{\beta-t} \\ = \ & A^{-t} \ \sharp_{\frac{1-t+t}{\beta-t+t}} \ (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \le (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{1-t}{p-t}} \end{aligned}$$

The first inequality is assured by (2) of Theorem A and the second one by (1) of theorem A. Multiplying $B^{\frac{t}{2}}$ both sides of each term, we have the conclusion.
Most parts of (2) are obtained from (1) by putting $\beta = p$ in (1) except the final inequality, which is also owing to (2) of Theorem A as follows:

$$A^{-r} \ \sharp_{\frac{1-t+r}{\beta-t+r}} \ D^{\beta-t} = A^{-r} \ \sharp_{\frac{1-t+r}{p-t+r}} \ \{A^{-r} \ \sharp_{\frac{p-t+r}{\beta-t+r}} \ D^{\beta-t}\} \le A^{-r} \ \sharp_{\frac{1-t+r}{p-t+r}} \ D^{p-t}$$

Multiplying $B^{\frac{t}{2}}$ to each term from both sides, we have the conclusion.

4. A variant of Theorem 2 under the chaotic order.

Recently, we proposed in [5] the following inequality because (U) seems to be a skewed form of (SGF) from our view point.

Theorem B. *If* $A,\ B,\ C > 0$ *satisfy* $A \gg B$ *and* $B \ge C$, *then for each* $0 \le t \le 1$

$$B \ge C \ge (B^t \ \natural_s \ C^p)^{\frac{1}{(p-t)s+t}} \ge A^{-r+t} \ \sharp_{\frac{1+r-t}{(p-t)s+r}} \ (B^t \ \natural_s \ C^p)$$

holds for all $p \ge 1,\ s \ge 1$ *and* $r \ge t$.

In this inequality, if $A \ge B = C$, then we have (F) and if $A = B \ge C$, then (GF) is obtained. But the assumption $A \gg B \ge C$ is unbalanced, so we study its variant under $A \gg B \gg C$.

Theorem 4. *If* $A,\ B,\ C > 0$ *satisfy* $A \gg B \gg C$, *then for* $t \ge 0$ *and* $0 \le p \le \beta \le 2p$

$$C^{-t} \ \sharp_{\frac{p+t}{\beta+t}} \ B^{\beta} \ge B^p \ge (C^{-t} \ \natural_{\frac{\beta+t}{p+t}} \ B^p)^{\frac{p}{\beta}} \ge A^{-t} \ \sharp_{\frac{p+t}{\beta+t}} \ (C^{-t} \ \natural_{\frac{\beta+t}{p+t}} \ B^p).$$

Proof. Since the first inequality is obtained by Theorem A (1), we show the rest inequalities.

$$\begin{aligned} & C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p = B^p \natural_{\frac{p-\beta}{p+t}} C^{-t} = B^p(B^{-p} \sharp_{\frac{\beta-p}{p+t}} C^t)B^p \\ = \ & B^p(C^{-t} \sharp_{\frac{2p-\beta+t}{p+t}} B^p)^{-1}B^p \leq B^p B^{-2p+\beta} B^p = B^\beta \end{aligned}$$

We have $(C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)^{\frac{p}{\beta}} \leq B^p$ by (LH) and $(C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)^{\frac{1}{\beta}} \ll B$, see [6;Theorem 1](cf.[18;Lemma 4]). So $A \gg (C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)^{\frac{1}{\beta}}$ is obtained and by Theorem A (1) we have

$$A^{-t} \sharp_{\frac{p+t}{\beta+t}} (C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p) \leq (C^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)^{\frac{p}{\beta}}.$$

Remark. If we put $B = C$ in Thorem 4, then $A^{-t} \sharp_{\frac{p+t}{\beta+t}} B^p$ is obtained. That is, Theorem 4 is a generalization of Theorem A (1).

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